Guide for the Qualifying Exams Doctoral Program in Mathematics

Department of Mathematics

Instituto Superior Técnico

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Overview

The Doctoral Program in Mathematics at IST aims at a balanced training in mathematics and its applications in the broadest sense. The main purpose of the qualifying exams is to guarantee that the candidate has sufficient mathematical knowledge and maturity to start this training.

Each student is required to take two written qualifying examinations covering two distinct areas of mathematics, to be chosen from the following list:

- Algebra
- Geometry and Topology
- Mathematical Analysis
- Numerical Analysis
- Logic and Computation

The qualifying exams are offered once per semester, in March and September. Students must pass their qualifying exams within three semesters after enrollment in the Program. By default, the exams are offered in english, altough answers in portuguese are also accepted.

The syllabi for the qualifying exams and a list of preparatory courses for each exam are given below.

Copies of previous qualifying exams are available online at:

https://fenix.ist.utl.pt/cursos/dmat/qualifying-exams

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The following webpages contain some relevant information, likely to be usefull for a student pursuing a Doctoral Degree in Mathematics:

- Doctoral Degree in Mathematics:
  https://fenix.ist.utl.pt/cursos/dmat/
- Doctoral Degree in Statistics and Stochastic Processes:
  https://fenix.ist.utl.pt/cursos/depe
- Master Degree in Mathematics and Applications:
  https://fenix.ist.utl.pt/cursos/mma/
- Bachelor Degree in Applied Mathematics and Computation:
  https://fenix.ist.utl.pt/cursos/lmac/
- Department of Mathematics:
  http://www.math.ist.utl.pt/

For questions and request for further information please contact the Program Coordinator by e-mail to phdstudies@math.ist.utl.pt .

Qualifying Exam in Algebra

Syllabus

Groups: Subgroups, homomorphisms, quotient groups. Lagrange's theorem. Groups of transformations and Cayley's theorem. Cyclic groups, permutation groups. Sylow theorems and p-groups. Solvable and nilpotent groups. [Artin, chapters 2 and 6] and [Hungerford, chapters 1 and 2].

Rings: Homomorphisms, ideals and quotient rings. Unique factorization domains, principal ideal domains and Euclidean domains. Fields of fractions and localization. Prime and maximal ideals. Polynomial rings. Polynomials with coefficients over unique factorization domains, Gauss lemma. [Hungerford, chapter 3]

Modules: Submodules, homomorphisms, quotient modules. Direct sums and direct products. Free modules. Exact sequences. Projective and injective modules. Tensor products. Algebras. Finitely generated modules over a principal ideal domain. Noetherian and Artinian modules. Hilbert basis theorem. Hilbert Nullstellensatz. [Hungerford, chapter 4 and chapter 8, sections 1 and 4]

Linear Algebra: Linear transformations. Jordan canonical form. Inner products. Hermitian matrices and the spectral theorem. [Artin, chapter 4, sections 1--6] for linear transformations. [Artin, chapter 12, section 7] and [Hungerford, chapter 7, section 4] for the Jordan canonical form. [Artin, chapter 7] for the other topics.

Fields and Galois Theory: Field extensions. The fundamental theorem. Splitting fields, algebraic closure and normality. The Galois group of a polynomial. Finite fields. Separability. Cyclic extensions. Cyclotomic extensions. Radical extensions. [Hungerford, chapter 5]

Categories and functors: Functors. Natural transformations. Adjoint functors. Morphisms. [Hungerford, chapter 10]

Recommended Texts

- M. Artin, Algebra, Prentice-Hall, 1991.
- T. Hungerford, *Algebra*, Springer, 1974.

Other references:

- M. Atiyah and I. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.
- N. Jacobson, *Basic Algebra I*, Freeman, 1985.

Courses covering the exam material

- Foundations of Algebra (Master in Mathematics and Applications)
- Complements of Algebra (Master in Mathematics and Applications)

Note: The exam material includes also the basic concepts of Algebra, covered in the course *Introduction to Algebra* (Bachelor in Applied Mathematics and Computation).

Qualifying Exam in Mathematical Analysis

Syllabus

Measure and Integration: measure spaces; outer measures and measurable sets; measures on topological spaces and representation theory; Lebesgue measure; measurable functions; Lebesgue integral; convergence theorems; absolute continuity and Radon-Nikodym theorem; product spaces; Lp spaces and representation theory.

Functional Analysis: Banach spaces; bounded linear operators; strong and weak convergence; uniform boundedness principle; open mapping, closed-graph, and Hahn-Banach theorems; duals and reflexive spaces; adjoint operators; conjugates of Lp and C[0,1]; compact operators and spectral properties; Hilbert spaces; projections; self-adjoint operators and spectral theory.

Complex Analysis: holomorphic functions and Cauchy-Riemann equations; integration and Cauchy's formula; power series representation; open mapping theorem; residues and integrals; maximum modulus principle; Riemann mapping theorem; zeros of holomorphic functions and infinite products; analytic continuation.

Recommended Texts

- S. Lang, *Real and Functional Analysis*, 2ª ed. ,Springer-Verlag, 1993 (chapters 1, 2, 3, 6, 7).
- J. B. Conway, *A Course in Functional Analysis*, 2ª ed., Springer-Verlag, 1990 (chapters 1, 2, 3, 6, 7).
- L.V. Ahlfors, *Complex Analysis*, 3ªed. McGraw-Hill, 1979 (chapters 1, 2, 3, 4, 5.1, 5.2, 6.1).

Courses covering the exam material

- Foundations of Topology and Real Analysis (Master in Mathematics and Applications)
- Functional Analysis (Master in Mathematics and Applications)
- Complements of Complex Analysis (Master in Mathematics and Applications)

Qualifying Exam in Geometry and Topology

Syllabus

Topology: open sets, compact sets, conected sets, metric spaces, Hausdorff spaces, normal spaces, countable bases.

Algebraic Topology: singular homology, CW complexes, Euler characteristic, fundamental group, covering spaces.

Differential Topology: manifolds, submanifolds, differential forms, Frobenius theorem, Stokes theorem, de Rham cohomology.

Differential Geometry: manifolds, vector fields, Riemannian metrics, connections, geodesics, curvature.

Examples: projective spaces, Grassmannians, classical Lie groups.

Recommended Texts

- R. Bott, L. W. Tu, *Differential forms in algebraic topology*. Springer-Verlag, New York, 1982. (chapter 1)
- M. do Carmo, *Differential Forms and Applications*, Springer-Verlag, New York, 1994 (chapters 1-6).
- M. do Carmo, *Riemannian Geometry*, Birkhauser, 1992 (chapters 0, 1, 2, 3, 4).
- A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002. http://www.math.cornell.edu/~hatcher/AT/ATpage.html (chapters 1, 2)
- J. Munkres, *Topology -A First Course*, Prentice-Hall, 1975 (chapters 2, 3, 4, 7, 8).
- J. Vick, *Homology Theory - An Introduction to Algebraic Topology*, Springer-Verlag, New York-Berlin, 1994 (chapters 1, 2).
- F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*. Graduate Springer-Verlag, New York-Berlin, 1983. (chapter 1, 2, 3, 4)

Courses covering the exam material

- Algebraic Topology (Master in Mathematics and Applications)
- Riemannian Geometry (Master in Mathematics and Applications)
- Differential Geometry (Master in Mathematics and Applications)

Qualifying Exam in Numerical Analysis

Syllabus

Fundamental concepts of number representation, error propagation and conditioning of numerical problems. Direct and iterative methods for solving linear systems and eigenvalue problems. Numerical solution of nonlinear equations and nonlinear systems. Interpolation, least squares and orthogonal polynomials in approximation theory. Numerical integration.

Numerical methods for the solution of initial and boundary value problems for ordinary differential equations. Numerical solution of linear partial differential equations by the finite difference method. Finite element approximation. Galerkin formulation and the finite element method forelliptic boundary value problems.

Recommended Texts

- K.E. Atkinson, *An Introduction to Numerical Analysis*, 2nd edition, Wiley, 1989. (chapters 1-9)
- A. Quarteroni, R. Sacco and F. Saleri, *Numerical Mathematics*, 2nd Ed., Springer Texts in Applied Mathematics, Springer, 2007. (chapters 11,12).
- C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, 1988. (chapters 1-5)
- J.W. Thomas, *Numerical Partial Differential Equations: Finite Difference Methods*, Springer, 1998. (chapters 1-5)
- Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer Series in Computational Mathematics, Springer, 1994. (chapters 3,6)

Courses covering the exam material

- Numerical Analysis (Master in Mathematics and Applications)
- Numerical Analysis of Partial Differential Equations (Master in Mathematics and Applications)

Note: The exam material includes also the basic concepts of Numerical Analysis, covered in the course *Computational Mathematics* (Bachelor in Applied Mathematics and Computation).

Qualifying Exam in Logic and Computation

Syllabus

Computability: computable functions, decidable sets, computably enumerable sets, decision problems, universal functions, Rice's theorem, Rice{Shapiro theorem, Rogers' theorem, Myhill-Shepherdson theorem, Kleene's least fixed point theorem, recursion theorem, reducibilities, Turing machines, recursive functions, Ackermann's map, Kleene's normalization theorem, recursion elimination.

Complexity: classes of space and time complexity, polynomial reducibility, NPcompleteness, Cook's theorem.

First-order logic: Hilbert calculus, metatheorem of deduction, theories and presentations, Craig's theorem, Tarskian truth semantics, soundness, completeness via Henkin's construction, equality, compactness theorem, model theoretic results on quantifier elimination, sequent calculus, cut elimination.

Arithmetic and incompleteness: standard semantics, theories of arithmetic, representability of computable maps, Church's theorem, Gödel's first incompleteness theorem, Tarski's theorem, pseudo representability of derivability, Hilbert-Bernays-Löb conditions, fixed point theorem, Löb's theorem, Gödel's second incompleteness theorem.

Set theory: Russell's paradox, Zermelo{Fraenkel theory with the axiom of choice, ordinals, cardinals, continuum hypothesis, interpretations of set theory, constructibility, overview of independence and consistency results.

Recommended Texts

- S. Homer and A. L. Selman, *Computability and Complexity Theory*, Springer, 2001 (chapters 2-6).
- D. Marker, *Model Theory: An Introduction*, Springer, 2002 (chapters 2-4).
- A. Sernadas and C. Sernadas, *Foundations of Logic and Theory of Computation*, College Publications, 2008 (chapters 4-14).
- A. Shen and N. K. Vereshchagin, *Computable Functions*, Student Mathematical Library 19, AMS, 2003 (chapters 1-11).
- J. R. Shoenfield, *Mathematical Logic*, ASL, 2001 (chapters 2-9).

Courses covering the exam material

- Mathematical Logic (Bachelor in Applied Mathematics and Computation)
- Introduction to Computability and Complexity (Bachelor in Applied Mathematics and Computation)
- Foundations of Logic and Theory of Computation (Master in Mathematics and Applications).