RESEARCH STATEMENT: HAMILTONIAN GROUP ACTIONS IN SYMPLECTIC AND CONTACT GEOMETRY

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1. Introduction

My research is on symplectic and contact manifolds and Hamiltonian group actions. A manifold M^{2n} is **symplectic** if M has a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$. The classical example, which arises quite naturally in physics, is the cotangent bundle T^*X of a manifold X. The manifold X can be thought of as the possible positions of particles in a physical system, and the cotangent bundle T^*X is the phase space: all the possible positions and momenta. For any symplectic manifold M let $Symp(M)$ denote the group of diffeomorphisms of M that preserve ω. An injective homomorphism $G \to Symp(M)$, from a Lie group G, is a **Hamiltonian action** if there exists a G-equivariant function $\mu : M \to \mathfrak{g}^*$, called **momentum map**, from M to the dual of the Lie algebra of G, such that for each $\xi \in \mathfrak{g}$, the component $\mu^\xi : M \to \mathbb{R}$ defined by $\mu^\xi(p) = \mu(p)\,(\xi)$ satisfies $d\mu^\xi = \iota_\xi(\omega)$. If $G = T^n$, where $2n$ is the dimension of M, then the action is called a **toric action**. An important class of examples of symplectic manifolds is given by the orbits of the coadjoint action of a Lie group G on $\mathfrak{g}^*$, the dual of its Lie algebra. Each such orbit $\mathcal{O}$ is naturally equipped with the Kostant-Kirillov symplectic form. The action of G on an orbit $\mathcal{O}$ is Hamiltonian, and the momentum map is just inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$. For example, when $G = U(n)$ the group of (complex) unitary matrices, a coadjoint orbit can be identified with the set of Hermitian matrices with a fixed set of eigenvalues. Apart from few exceptions, the dimension of maximal torus of G acting effectively on $\mathcal{O}$ is smaller than the complex dimension of $\mathcal{O}$, thus its action is not toric.

A manifold V^{2n+1} is called a (co-oriented) **contact manifold** if it is equipped with a maximally non-integrable $2n$-plane field $\xi = \ker(\alpha)$. Non-integrability implies that $\alpha \wedge (d\alpha)^n$ is a nowhere vanishing top-dimensional form. A diffeomorphism ϕ of V is called a contactomorphism if it preserves the distribution ξ (does not need to preserve α). Natural examples are provided by the co-sphere bundles of compact manifolds and prequantization spaces of symplectic manifolds with integral symplectic form.

2. Current Research

I am currently involved in the following research projects in the topics of symplectic and contact geometry and topology.

November 2, 2014.

2.1. **Project on non-displaceable Lagrangians and pre-Lagrangians in symplectic and contact toric manifolds.** Joint with Aleksandra Marinković (a PhD student at IST that I am co-advising together with prof. Miguel Abreu).

In symplectic geometry a question of great importance is whether a (Lagrangian) submanifold is displaceable, that is, if it can be made disjoint from itself by the means of a Hamiltonian isotopy. Motivated by the search of non-displaceable Lagrangians we reprove results of Fukaya-Oh-Otha-Ono and Gonzales-Woodward that every symplectic toric manifold (even orbifold) contains a non-displaceable Lagrangian toric fiber, [MP14s]. The tools we use include symplectic reduction and properties of weighted projective spaces. On the way we prove that every compact toric symplectic orbifold is a centered reduction of a product of projective spaces (possibly weighted). This result is interesting by itself and could potentially have an application also to the construction of new quasimorphisms (see [B13, Theorem 1.1]).
We have also analyzed displaceability of pre-Lagrangian toric fibers in the contact toric manifolds. A **contact toric manifold** is a co-oriented contact manifold (V^{2d-1}, ξ) with an effective action of the torus T^d, that preserves the contact structure ξ. A submanifold $L \subset V$ is a **pre-Lagrangian** if it is a diffeomorphic image of some Lagrangian submanifold $\widetilde{L} \subset SV$ of the symplectization of V under the projection $\pi \colon SV \to V$. Any generic T^d-orbit in a toric contact manifold is a pre-Lagrangian. Similarly to the symplectic case, one can ask whether a given pre-Lagrangian in a contact manifold can be displaced from itself by the means of a contact isotopy (in contact setting any contact isotopy is Hamiltonian). If a contact toric manifold admits a monotone quasimorphism (i.e. homomorphism, up to a bounded error, to $(\mathbb{R}, +)$) with a vanishing property, then it contains at least one non-displaceable pre-Lagrangian ([BZ13]), as in the symplectic case. (One can prove the above statement by repeating the argument of Entov and Polterovich, [EnP06, Theorem 2.1], from the symplectic setting.) However this property does not hold for all contact toric manifolds. For example, the standard contact sphere S^{2n-1}, $n \geq 2$, does not admit such a quasimorphism and we proved that all pre-Lagrangian toric fibers (of the standard toric action) are displaceable, [MP14c]. This seems to be linked to orderability. A contact manifold (V, ξ) is called **orderable** if the relation on the universal cover of $Cont_0(V, \xi)$ (identity component of the group of contactomorphisms) introduced by Eliashberg and Polterovich in [ElP00], gives a genuine partial order. The existence of a monotone quasimorphism with a vanishing property implies orderability ([BZ13, Theorem 1.28], see also [ElP00, Criterion 1.2.C]). The existence of a stably non-displaceable pair of a Legendrian and a pre-Lagrangian implies orderability ([ElP00, Theorem 2.3.A]). The standard sphere S^{2n-1}, as well as $S^1 \times S^{2n}$, are not orderable for $n \geq 2$. We managed to displace a large collection of pre-Lagrangian toric fibers of $S^1 \times S^{2n}$ and now seek for methods to displace the remaining fibers.
One can also ask about the existence of a displaceable fiber. In the symplectic toric case there are always infinitely many toric fibers that can be displaced by McDuff's method of probes ([MD11]). In the contact world this is not necessarily true. We prove that the co-sphere bundles of tori, with the natural contact toric action, have all pre-Lagrangian toric fibers non-displaceable ([MP14c]). We expect this result to be related to the freeness of the action. The toric action on co-sphere bundles is free, while in symplectic setting any Hamiltonian circle action on any compact manifold must have

fixed points.
We now continue to investigate the connection between displaceability, orderability, existence of quasi-morphisms, and fillability.

2.2. **Constructing a non-linear Maslov index on lens spaces and other contact toric manifolds.** This is a joint project with Sheila Sandon (Univ. de Strasbourg) and Yael Karshon (Univ. of Toronto).

Givental in [Giv90] constructed a non-linear generalization of the Maslov index for $\mathbb{R}P^{2n-1}$ and $\mathbb{C}P^n$. This index gives rise to quasimorphisms on $\widetilde{Cont_0}(\mathbb{R}P^{2n-1})$ and $\widetilde{Symp_0}(\mathbb{C}P^n)$. Quasimorphism is a "homomorphism up to a bounded error" to the group $(\mathbb{R}, +)$. (The groups above are often perfect and do not admit any non-trivial homomorphism.) Givental used his quasimorphism to give Morse-type lower bounds for the number of fixed points of a symplectomorphism of $\mathbb{C}P^n$, i.e. to prove a version of the Arnold Conjecture. As observed by Sandon in [S04], this quasimorhpism can also be used to give Morse-type lower bounds for the number of translated points of contactomorphisms of $\mathbb{R}P^{2n-1}$, proving the contact version of the Arnold Conjecture for $\mathbb{R}P^{2n-1}$. A point $p \in V$ is called a translated point of a contactomorphism Φ of $(V, \xi = \ker\alpha)$ if p and $\Phi(p)$ lie on the same Reeb orbit and $\Phi^*(\alpha)_{|p} = \alpha_{|p}$. Contactomorphisms in $Cont_0(V, \xi)$ (i.e. contact isotopic to identity) can easily have no fixed points, but, it is conjectured, that they must have translated points. From other applications of quasimorphisms we mention that the existence of a quasimorphism on $\widetilde{Cont_0}(\mathbb{R}P^{2n-1})$ proves also that $\mathbb{R}P^{2n-1}$ is orderable, contains non-displaceable pre-Lagrangian and has unbounded discriminant metric (defined by Collin and Sandon in [CS12]).

We mimic the construction by Givental of a non-linear Maslov index on the real projective space (quotient of a sphere by $\mathbb{Z}_2$ action) and adapt it to lens spaces (quotients of spheres by $\mathbb{Z}_p$ action). As such non-linear Maslov index is a quasimorphism, its existence reproves orderability of lens spaces, gives lower bounds on the number of translated points of any contactomorphism of lens space (though not as strong as we expected) and proves unboundness of discriminant metric on lens space. This is a very recent result and the paper is still in preparation. The difficulty of passing from $\mathbb{Z}_2$ to $\mathbb{Z}_p$ comes from the fact that $H^*(B\mathbb{Z}_p)$ is not a PID if the prime $p > 2$ and thus certain results from [Giv90] do not hold for lens spaces.

The next step in this project is to generalize the non-linear Maslov index to contact toric manifolds obtained as prequantizations of symplectic toric manifolds. Givental in [Giv00] proved a version of the Arnold Conjecture for symplectic toric manifolds, using their presentation as symplectic reductions of $\mathbb{C}^d$ and the technique of generating functions. He did not construct any quasimorphism but we believe that the background required for such construction is contained in his work. The situation is much more complicated than in the case of the projective space. Each Hamiltonian function on the projective space can be uniquely lifted to an invariant function on a sphere, and then uniquely extended to an $\mathbb{R}_+$ equivariant function on $\mathbb{C}^n$ (continuous on $\mathbb{C}^n$, C^∞ on $\mathbb{C}^n \setminus \{0\}$). The gain of moving to $\mathbb{C}^n$ is that one can use the generating functions techniques (such functions are guaranteed to exists there). In the case of a Hamiltonian on a general symplectic toric manifold, there is no unique way of lifting it to $\mathbb{C}^n$. It is possible to define a lift with a help of a "bump" function, but then one needs to incorporate Lagrange multipliers to the whole construction in order to "pick out" only the relevant data. We are currently working on extracting from ([Giv00]) a quasimorphism, and generalizing the whole construction to the contact toric manifolds which are prequantizations of symplectic toric manifolds. Hopefully in the future we will be able to extend it even further, to all contact toric manifolds.

2.3. **Constructing a canonical basis for the equivariant K-Theory ring of symplectic toric manifolds.** This is a joint project with Silvia Sabatini (Univ. of Cologne).

In the setting of equivariant cohomology of a toric manifold a theorem of Kirwan guarantees existence of a basis, consisting of Kirwan classes, satisfying certain nice properties. In general this basis is not unique. Goldin and Tolman showed that when the moment map for a generic subcircle is index increasing one can make the choice of the Kirwan classes unique by requiring one extra condition. The Theorem of Kirwan mentioned above has its analogue in the equivariant K-theory setting: there always exists a (non-unique) basis for the K-theory ring. Guillemin and Kogan in [GK04] introduced an invariant called a local index and used it to characterize uniquely one possible choice of a basis. Their construction works not only for toric manifolds but also for more general GKM spaces, but it does not give explicit formulas even for well-understood manifolds. Also, this basis does not include the K-theory class corresponding to the trivial line bundle. We propose a slightly different definition of a local index (explicit and easy for computational purposes) and use it to uniquely characterize a different choice of a basis. We give explicit formulas for the elements of this basis in the case when moment map is index increasing and inductive formulas if it is not. For the projective space the elements of our basis correspond to the products of prequantization line bundle. We are currently working on generalizing these ideas to GKM manifolds. We expect that the explicit formulas for our canonical classes in the GKM case will turn out to be computationally challenging. To make our classes useful in practice, we probably will need to develop some method of simplifying the computation. Our hope is to construct such a method by extending the ideas from the effective algorithm of Sabatini and Tolman, [ST], for calculating a basis for the equivariant cohomology ring.

2.4. **Gromov width of polygon spaces.** This is a joint project with Alessia Mandini (Univ. of Pavia).

Polygon space $\mathcal{M}(r_1, \dots, r_n)$ is the moduli space of polygons in $\mathbb{R}^3$ with n edges (n-gons) of lengths $(r_1, ..., r_n)$. Under some genericity assumptions on the lengths r_i, the polygon space is a symplectic manifold. In fact it is a symplectic reduction of the Grassmannian manifold of 2-planes in $\mathbb{C}^n$. Moreover an open dense subset of a (smooth) polygon space can be equipped, in various ways, with a toric Hamiltonian action (see for example [NU14] for a nice collection of such actions). Using toric actions we construct explicit symplectic embeddings of balls and therefore establish lower bounds for the Gromov width of all smooth spaces of 5- and 6-gons, i.e. the capacity of the biggest ball that symplectically embeds into the manifold (see Section 3.2 for precise definition). The fact that the moduli spaces of 5- and 6-gons are often toric Fano, or a blow up of a toric Fano, allows us to use the results of Lu [Lu06, Theorems 1.2 and 6.2] and establish upper bounds for the Gromov width of these spaces, (equal to our lower bounds). This way we establish the Gromov width of almost all 5-gons and a collection of 6-gons. With the help of Moser trick we extend our result to cover all smooth spaces of 5-gons. We are currently searching for tools to extend our result to all 6-gons, and to higher dimensional polygon spaces.

2.5. **Symplectomorphisms among 3-stage Bott towers.** This is a joint project with Susan Tolman (Univ. of Illinois at Urbana-Champaign)

We use the toric degeneration construction of Harada and Kaveh [HK12] to construct symplectomorphisms between certain 3-stage Bott towers. The goal is to show that if the cohomology rings over $\mathbb{Z}$, and one simple numerical invariant agree, then the manifolds are symplectomorphic. It was already proved by S. Choi, M. Masuda and D. Suh in [CMS10, Theorem 1.3] that they are diffeomorphic.

3. Research contained in PhD thesis

3.1. **Equivariant Cohomology.** One of the fundamental invariants for manifolds with group actions is the equivariant cohomology ring, $H^*_T(M)$. In the case of Hamiltonian torus actions, this invariant can be presented in two different ways. One is as a subring of $H^*_T(M^T)$, namely the image of an injective map $H^*_T(M) \to H^*_T(M^T)$ induced by the inclusion of the fixed points ([Ki84]). If there are d isolated fixed points then $H^*_T(M^T;R) = \oplus_{j=1}^d R[x_1,\dots,x_n]$, where n is the dimension of the torus acting, thus $H^*_T(M)$ is a subring of a well-understood ring. This description is especially nice for GKM actions ([GKM98]). Another presentation is also provided by Kirwan who, using Morse theory for the components of the momentum map, constructed equivariant cohomology classes forming a basis for the integral equivariant cohomology ring of M.

I used the Kirwan's generating classes to give necessary and sufficient conditions for $f = (f_1,\dots,f_d) \in \oplus_{j=1}^d \mathbb{Q}[x_1,\dots,x_n] = H^*_T(M^T;\mathbb{Q})$ to be in the image of the inclusion $H^*_T(M;\mathbb{Q}) \hookrightarrow H^*_T(M^T;\mathbb{Q})$, i.e. to represent an equivariant cohomology class of M ([P14]). In the case of circle actions, this result is also valid for integral coefficients.

This algorithm is extremely useful if we are given a GKM action of a torus T and a subtorus $K \hookrightarrow T$ which acts not in a GKM fashion. The relations describing the image of $H^*_T(M)$ in $H^*_T(M^T)$ (i.e. the GKM relations) will not induce all the relations needed to describe the image of $H^*_K(M)$ in $H^*_K(M^K)$. On the other hand, necessary and sufficient conditions for f to be in $H^*_K(M)$ coming from my algorithm can be easily obtained from conditions for $H^*_T(M)$.

3.2. **Gromov width of coadjoint orbits.** The **Gromov width** of (M,ω) is defined to be the supremum of the set of a's such that a ball of capacity a:

$$B^{2n}_a = \left\{ z \in \mathbb{C}^n \;\middle|\; \pi \sum_{i=1}^n |z_i|^2 < a \right\}$$

can be symplectically embedded in (M^{2n},ω).

Existence of Hamiltonian (not necessarily toric) torus action on M allows one to construct explicitly symplectic embeddings balls (use the flow of vector fields induced by the action; [KT05]). This use of equivariant techniques leads to establishing lower bounds on the Gromov width. Holomorphic techniques provide upper bounds. In my thesis, I studied the Gelfand-Tsetlin integrable system and the toric action it induces on the open dense subset of the coadjoint orbits ([GS83],[K00]). I proved a lower bound for the Gromov width for regular coadjoint orbits of $U(m)$ and of $SO(m)$ which is equal to the conjectured Gromov width. Later I extended this result to almost all coadjoint orbits of $U(m)$ or $SO(m)$.

Theorem 3.1. [P14] *Let G be $U(m)$ or $SO(m)$ with the maximal torus T. Let $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ be an orbit of the coadjoint G action through a point λ in the positive Weyl chamber. Then the Gromov width of $\mathcal{O}_\lambda$ is at least the minimum*

$$\min\{\, |\langle \alpha^\vee, \lambda \rangle| \ ; \ \alpha^\vee \text{ a coroot}\}$$

for any λ in $U(m)$ case, and for "almost all" λ in the $SO(m)$ case (see [P14] *for precise formulation.)*

This particular lower bound is important for the following reasons:
- The above value is the Gromov width of complex Grassmannians ([KT05]).
- The above value is an upper bound for Gromov width of regular, indecomposable coadjoint orbits of other simple compact Lie groups (Zoghi [Z10]).
- The above result of Zoghi was recently generalized by Caviedes to cover all coadjoint orbits of any compact Lie group ([C14]). Therefore
- The Gromov width of all $U(m)$ and almost all $SO(m)$ coadjoint orbits is exactly equal to the above value.

The result for $SO(2m+1)$ orbits is especially important. Applying the method of Karshon and Tolman, [KT05], to the standard, not toric, action of the maximal torus of $SO(2m+1)$ gives a lower bound that is smaller than the one claimed in Theorem 3.1. (This is related to the fact that the root system for $SO(2m+1)$ is non-simply laced.)

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