A NOTE ON STRICT $\mathbb{C}$-CONVEXITY

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Abstract. We establish a relation between strict $\mathbb{C}$-convexity of a real hypersurface of $\mathbb{C}^n$ and the behavior of its complex Gauss map. In that way we recover -with an improvement on the regularity- the known results about the topology of these hypersurfaces by using elementary differential geometric arguments. Our approach can be though of as being a complex analog of the description of strictly convex hypersurfaces in Euclidean space via Morse functions associated to pencils of hyperplanes.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open subset. Convexity of Ω is an affine notion and two approaches are possible: a global or synthetic one in which the intersection of $\overline{\Omega}$ with any affine line is asked to be either empty of connected, and an infinitesimal or analytical one which assumes $\partial\Omega$ to be a C^2-hypersurface, and requires its Euclidean shape operator to be definite positive at every point.

Convexity can be generalized to the complex setting in two different ways, according to whether we want it to be a complex analytic or a complex affine property. In the first case the appropriate notion is that of (Levi) pseudoconvexity. We say that $\Omega \subset \mathbb{C}^N$ an open bounded domain with C^2-boundary is pseudoconvex if at each point of $\partial\Omega$ the Levi form is definite positive. Recent work of Krantz [4] shows that a semilocal approach to pseudoconvexity is possible: if all holomorphic closed disks of small enough diameter whose boundary lie in $\partial\Omega$ lie entirely in Ω, then the domain is pseudoconvex.

A bounded connected open subset $\Omega \subset \mathbb{C}^N$ is $\mathbb{C}$-convex if the intersection of $\overline{\Omega}$ with any complex affine line is either empty or 1-connected. $\mathbb{C}$-convexity is a complex affine notion. It also admits an infinitesimal reformulation: for each $x \in \partial\Omega$ let

$$D_x := T_x\partial\Omega \cap JT_x\partial\Omega,$$

where $J\colon \mathbb{C}^N \to \mathbb{C}^N$ denotes complex multiplication. We use the sign convention which identifies the Euclidean shape operator for $\partial\Omega$, with the differential of the Gauss map $\partial\Omega \to S^{2N-1}$ which sends a point $x \in \partial\Omega$ to the outward pointing unit vector normal to $T_x\partial\Omega$; in that way the Euclidean shape operator for the sphere is strictly positive definite.

Definition 1. *A bounded connected domain $\Omega \subset \mathbb{C}^N$ with boundary of class C^h, $h \geq 2$, is (strictly) $\mathbb{C}$-convex if for any point $x \in \partial\Omega$ the restriction of the Euclidean shape operator to D_x is (strictly) positive definite.*

Convexity implies $\mathbb{C}$-convexity, and $\mathbb{C}$-convexity implies pseudoconvexity. In particular strictly $\mathbb{C}$-convex domains have contact boundary.

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Topologically, $\mathbb{C}$-convex domains are quite elementary since they are known to be homeomorphic to balls (theorem 2.4.2 in [1] or theorem 4.6.12 in [2]). This is proved using the global or synthetic approach to $\mathbb{C}$-convexity, and with very non-trivial tools such as a parametric version of the Riemann mapping theorem. For a strict $\mathbb{C}$-convex hypersurface of class C^∞, very deep work of Lempert [5] gives a diffeomorphism from $\partial\Omega$ to the sphere (via the Lempert uniformization maps). In this note we want to exploit the infinitesimal point of view to prove

Theorem 1. *A strict $\mathbb{C}$-convex hypersurface of class C^h, $h \geq 2$, is C^{h-1}-diffeomorphic to the sphere.*

The relevance of theorem 1 is not quite the outcome -the improvement of regularity- but rather the new approach used. We will show that basic constructions for strictly convex hypersurfaces in Euclidean space admit a complex analog for strictly $\mathbb{C}$-convex hypersurfaces. Namely, for a strictly convex hypersurface $\partial\Omega$ a pencil L of (real) hyperplanes whose base does not intersect Ω, determines a Morse function $\phi_L \colon \partial\Omega \to \mathbb{R}$ of the simplest kind: it has just two critical points, a global minimum and a global maximum, and hence Morse theory already implies that $\partial\Omega$ is homeomorphic to the sphere (theorem 4.1. in [8]). Any hypersurface of $\mathbb{C}^N$ carries a canonical CR structure. If $\partial\Omega$ is strictly $\mathbb{C}$-convex, a pencil of complex hyperplanes whose base B does not intersect Ω determines a function $\phi_L \colon \partial\Omega \to \mathbb{C}$, which can be naturally understood as a Morse function in the CR setting. Moreover, it will be seen to be of the simplest kind -in the sense that the critical set and critical values are as elementary as possible- and basic tools from Morse theory and homotopy theory will allow us to prove that $\partial\Omega$ is homeomorphic to a sphere, and ultimately C^{h-1}-diffeomorphic to it.

Strictly $\mathbb{C}$-convex domains are very important from the point of view of hyperbolic geometry: the aforementioned work of Lempert shows that in the smooth case each point and direction determines a unique extremal holomorphic disk, which can be extended to the boundary. This disk gives the Kobayashi distance between any two of its points; dually, each extremal disk is a holomorphic retract (determines a holomorphic projection) which gives the Caratheodory distance between any two of its points [6]. When Ω is a ball, a maximal disk is exactly the intersection of Ω with a complex line; the dual projection is the map associated to a pencil of hyperplanes; if the ball is centered at the origin, then for a disk through the origin the projection is the Hermitian orthogonal one, and thus the pencil has base a codimension 2 complex subspace contained in the hyperplane at infinity. When Ω is not a ball, extremal disks and their dual retractions are not linear anymore. Our work shows that the linear dual projections -which are not Caratheodory extremal- are still very efficient in keeping track of the differential geometry of the boundary of the domain.

The structure of this note is as follows: In section 2 we notice a connection between strict $\mathbb{C}$-convexity of $\partial\Omega$ and its complex dual map. In section 3 we introduce a natural analog of Morse functions in the CR setting, and show how certain pencils of hyperplanes provide such functions when restricted to hypersurfaces whose complex dual map is an immersion. The proof of theorem 1 is completed in section 4 by showing that strict $\mathbb{C}$-hypersurfaces admit the most elementary CR Morse functions. The techniques are a combination of standard facts in dual geometry, homotopy theory, differential topology and topology of the plane.

The notion of $\mathbb{C}$-convexity -and thus also the one of strict $\mathbb{C}$-convexity- is relevant from the point of view of analytic function theory (see [7] where it was introduced under the name of strong linear convexity, and also [1, 2]). Perhaps this new approach to strictly $\mathbb{C}$-convex domains may reveal interesting features of their analytic function theory.

2. Strictly $\mathbb{C}$-convex hypersurfaces and dual geometry

All our manifolds and maps among them will be of class C^h, $h \geq 2$.

Let M be a hypersurface inside $\mathbb{CP}^N$. In this section we want to make precise the relation between the shape operator along the canonical CR distribution D_x, $x \in M$, and the way these complex hyperplanes vary in dual projective space. This relation -which is the content of lemma 1- is known to experts, but we have not found it explicitly stated in the literature.

The *complex dual map* is

$$\begin{array}{rcl} \nu\colon M & \longrightarrow & \mathbb{CP}^{N*} \\ x & \longmapsto & D_xM, \end{array} \tag{1}$$

where D_xM is the complex projective hyperplane through x tangent to D_x. The image of M by the complex dual map is M^*, the set of complex hyperplanes which are not transverse to M.

Let $Z_0, \dots, Z_N$ be homogeneous coordinates in $\mathbb{CP}^N$. Let $\mathbb{C}^N$ be the domain of the affine chart with affine coordinates $z_1, \dots, z_N$, $z_j = Z_j/Z_0$, $z_j = x_j + iy_j$. For any $x \in M$, we can find a projective transformation sending (i) x to the origin of the affine chart z_0, (ii) D_x to the complex hyperplane D_0 with equation $z_N = 0$, and (iii) T_xM to the real hyperplane with equation $y_N = 0$.

The parametrization

$$\begin{array}{rcl} \psi\colon U \subset \mathbb{C}^{N-1} \times \mathbb{R} & \longrightarrow & M \cap \mathbb{C}^N \\ (w_1, \dots, w_{N-1}, t) & \longmapsto & (w_1, \dots, w_{N-1}, t + i\varphi(w,t)) \end{array} \tag{2}$$

is obtained by inverting the orthogonal projection $\pi\colon M \cap \mathbb{C}^N \to T_xM$.

Since D_x is $T_xM \cap JT_xM$, about z_0 we have

$$D_{\psi(w,t)}M \equiv \sum_{j=1}^{N-1} \left(\frac{\partial \varphi}{\partial v_j} + i \frac{\partial \varphi}{\partial u_j} \right) z_j + \left(-1 + i \frac{\partial \varphi}{\partial t} \right) z_N = 0, \tag{3}$$

where $w_j = u_j + iv_j$. Equivalently,

$$D_{\psi(w,t)}M \equiv \sum_{j=1}^{N-1} 2i \frac{\partial \varphi}{\partial w_j} z_j + \left(-1 + i \frac{\partial \varphi}{\partial t} \right) z_N = 0.$$

Definition 2. *Let $\psi\colon U \subset \mathbb{C}^{N-1} \times \mathbb{R} \to M \cap \mathbb{C}^N$ be a local parametrization of M constructed as in (2). The complex Gauss map $G\colon U \to \mathbb{CP}^{N-1*}$ is defined to be the map sending a point x to the linear hyperplane in $\mathbb{C}^N$ parallel to D_x. Using our fixed coordinates in U and $z_N = 1$, its components are*

$$G(w,t)_j = \frac{\left(-1 - i\frac{\partial \varphi}{\partial t}\right)}{1 + \left(\frac{\partial \varphi}{\partial t}\right)^2} \left(\frac{\partial \varphi}{\partial v_j} + i \frac{\partial \varphi}{\partial u_j} \right) = \frac{2\left(\frac{\partial \varphi}{\partial t}\right) - 2i}{1 + \left(\frac{\partial \varphi}{\partial t}\right)^2} \frac{\partial \varphi}{\partial w_j},\ j = 1, \dots, N-1.$$

By construction the complex dual map $\nu\colon U \to \mathbb{C}^N \subset \mathbb{CP}^{N*}$ is the complex Gauss map together with the component

$$\nu_N(w,t) = \sum_{j=1}^{N-1} \frac{\left(-2\frac{\partial \varphi}{\partial t} + 2i\right)}{1 + \left(\frac{\partial \varphi}{\partial t}\right)^2} \frac{\partial \varphi}{\partial w_j} w_j - (t + i\varphi).$$

One checks $d\nu_N(0) = \frac{\partial}{\partial t}\nu_N(0) = -1$. Hence injectivity of the differential of ν is equivalent to $dG(0)$ being an isomorphism when restricted to the hyperplane $t = 0$.

Definition 3. *A hypersurface in $\mathbb{CP}^N$ is said to have immersed dual set if the (complex) dual map (equation (1)) is a local embedding.*

Lemma 1. *If $\partial\Omega \subset \mathbb{C}^N$ is a strictly $\mathbb{C}$-convex hypersurface then the dual map is a local embedding.*

Proof. Using the parametrizations introduced above and the description of the dual map and complex Gauss map, the non-degeneracy of the differential of the dual map is equivalent to the non-degeneracy of the of the Hessian of $\varphi(u, v, 0)$ at 0 in the coordinates u, v. By the construction of the charts, the corresponding matrix is the matrix of the Euclidean shape operator along D_0 in the basis $\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_{N-1}, \partial/\partial y_{N-1}$. Note that the Euclidean metric we use is the one furnished by the chosen affine coordinates. The signature of the Euclidean shape operator of a hypersurface at any point is an affine invariant, meaning that it does not change if we compute the shape operator with respect to any metric in the orbit of the Euclidean metric by the affine group, and thus the lemma is proved. $\square$

Remark 1. *In this note we insist on not using properties of the global or synthetic definition of strict $\mathbb{C}$-convex hypersurfaces, and just working with the infinitesimal definition. A consequence of strict $\mathbb{C}$-convexity is that a tangent hyperplane only intersects $\partial\Omega$ at a tangency point (i.e. the dual map is an embedding). At this point we note for further use that this is seen to hold semilocally from the coordinate description above (see also corollary 2).*

3. CR Morse functions

In this section we introduce the notion of CR-Morse function. We show that for projective hypersurfaces with immersed dual set, the Morse condition for CR functions coming from pencils of hyperplanes is equivalent to the obvious transversality condition in the dual projective space.

A real function is Morse if its differential is transverse to the zero section of the cotangent bundle. The generalization to the CR setting is as follows: let (M, D, J) be a CR manifold of hypersurface type and $f\colon M \to \mathbb{C}$ a complex valued C^2-map. We can write the restriction to D of the differential uniquely as a sum of a complex linear and and complex anti-linear part

$$d_D f = \partial_D f + \bar{\partial}_D f.$$

The function is CR if and only if $\bar{\partial}_D f = 0$, so its differential restricted to D is a section of

$$T^{*1,0}D := \{h \in \mathrm{Hom}_{\mathbb{C}}(D_x, \mathbb{C}), \,|\, x \in M\}.$$

Let $q\colon T^{*1,0}D \to M$ be the projection onto the base and let $\underline{0}$ denote the zero section.

Definition 4. *A section $s \in \Gamma(T^{*1,0}D)$ is transverse to $\underline{0}$ along D if for all x with $s(x) \in \underline{0}$ we have*

$$(dq)^{-1}_{s(x)}(D_x) = (T_{s(x)}\underline{0} \cap (dq)^{-1}_{s(x)}(D_x)) + ds(D_x).$$

Transversality along D is stronger than usual transversality. If (M, D, J) is Levi-Flat, then transversality along D is simply foliated transversality. Definition 4 is the natural extension of the latter notion to the non-integrable setting.

Definition 5. *A CR function $f\colon M \to \mathbb{C}$ is Morse if $\partial_D f$ is transverse to $\underline{0}$ along D.*

Very much as critical points of (real) Morse functions are isolated, elementary transversality theory gives the following:

Lemma 2. *Let $f\colon M \to \mathbb{C}$ be a Morse CR function. Then the set of singular points*

$$\Sigma_D f := \partial_D f^{-1}(\underline{0})$$

is a 1-dimensional submanifold transverse to D.

Remark 2. *More generally the Morse condition is defined for CR functions taking values on Riemann surfaces, so that when composed with any holomorphic chart it yields a CR Morse function as in definition 5.*

If M is a real hypersurface of a complex manifold, then the restriction of any holomorphic function to M is a CR function w.r.t. the canonical induced CR structure (of hypersurface type). Now let M be a real hypersurface of $\mathbb{CP}^N$.

Let $L \subset \mathbb{CP}^{N*}$ be a pencil of hyperplanes, $L \equiv \lambda H_0 + \mu H_1$, $\lambda, \mu \in \mathbb{C}$, $H_0, H_1 \in L$, and consider

$$\begin{array}{rcl} \phi_L\colon M\backslash B & \longrightarrow & \mathbb{CP}^1 \\ x & \longmapsto & [H_0(x) : H_1(x)] \end{array} \tag{4}$$

its associated CR map, where $B = M \cap H_0 \cap H_1$ are the base points.

Let M have immersed dual set M^*. We say that a pencil of hyperplanes $L \subset \mathbb{CP}^{N*}$ intersects M^* transversely if for any $x^* \in L \cap M^*$, L has transverse intersection with all the branches of M^* through x^*.

Definition 6. *A pencil $L \subset \mathbb{CP}^{N*}$ is a Lefschetz pencil for M if M^* is immersed and L intersects M^* transversely.*

It is an elementary duality result that $H_0 \cap H_1$ is transverse to M if and only if L is transverse to M^*, so in particular for a Lefschetz pencil B is either empty or a real codimension four CR submanifold of M. As the following result, Lefschetz pencils should be an important tool for the study of real hypersurfaces of $\mathbb{CP}^N$ with immersed dual set.

Proposition 1. *Let $M \subset \mathbb{CP}^N$ be a real hypersurface with immersed dual set. Let L be a pencil of hyperplanes. Then the associated map $\phi_L\colon M\backslash B \to \mathbb{C}$ of equation (4) is CR Morse.*

Proof. Let $L \equiv \lambda H_0 + \mu H_1$ be a pencil for M. We want to check the assertion about $\partial_D \phi_L$ being transverse to $\underline{0}$ along D. We assume w.l.o.g. by composing with an affine chart of $\mathbb{CP}^1$ that ϕ_L takes values in $\mathbb{C}$. Consider the composition

$$\begin{array}{rcl} g := \phi_L \circ \psi\colon U \subset \mathbb{C}^n \times \mathbb{R} & \longrightarrow & \mathbb{C} \\ (w,t) & \longmapsto & \dfrac{t + i\varphi(w,t)}{H_1(\psi(w,t))}. \end{array}$$

In $\mathbb{C}^{N-1} \times \mathbb{R}$ we have the standard CR structure D_h given by the hyperplanes $\mathbb{C}^{N-1} \times \{\cdot\}$; the (1,0) part of the complexified cotangent bundle is

$$D_h^{*1,0} := \mathbb{C}^{N-1*} \times (\mathbb{C}^{N-1} \times \mathbb{R}).$$

The inverse of the parametrization ψ is defined by the projection

$$\pi\colon \mathbb{C}^N = \mathbb{C}^{N-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{N-1} \times \mathbb{R}.$$

The complex projection

$$\pi_{\mathbb{C}}\colon \mathbb{C}^N = \mathbb{C}^{N-1} \times \mathbb{C} \to \mathbb{C}^{N-1}$$

induces a bundle map $\pi_{\mathbb{C}*}\colon D \to D_h$ which sends the complex hyperplane over z to its projection by $\pi_{\mathbb{C}}$ over $\pi(z)$. It is a map of complex vector bundles and so is its dual map

$$\pi^*\colon D_h^{*1,0} \to D^{*1,0}.$$

As a result we have the following commutative diagram:

$$\begin{array}{ccc} D_h^{*1,0} & \xrightarrow{\pi^*} & D^{*1,0} \\ \Big\downarrow & & \Big\downarrow \\ U \subset \mathbb{C}^{N-1} \times \mathbb{R} & \xrightarrow{\psi} & M \end{array} \tag{5}$$

Usual transversality of a submanifold to the zero section of a vector bundle is preserved by an isomorphism of vector bundles. It is also true that transversality along D at $z_0 = \psi(0)$ is equivalent to transversality along D_h at 0, because those subspaces are preserved by the map $\pi = \psi^{-1}$ between the base spaces of the bundles. Therefore, $\partial_D \phi_L \in \Gamma(D^{*1,0})$ is transverse along D to $\underline{0}$ if and only if

$$\pi_* \partial_D \phi_L := (\pi^*)^{-1} \circ \partial_D \phi_L \circ \psi$$

is transverse along D_h to the zero section of $D_h^{*1,0}$.

Taking into account equation (3),

$$\pi_*^{-1}\left(\frac{\partial}{\partial u_j}\right) = \frac{\partial}{\partial x_j} + \frac{-\frac{\partial \varphi}{\partial u_j}\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial v_j}}{1 + \left(\frac{\partial \varphi}{\partial t}\right)^2}\frac{\partial}{\partial x_N} + \frac{\frac{\partial \varphi}{\partial v_j}\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial u_j}}{1 + \left(\frac{\partial \varphi}{\partial t}\right)^2}\frac{\partial}{\partial y_N}. \tag{6}$$

The pencil L induces a map $\Phi\colon \mathbb{CP}^N \backslash H_0 \cap H_1 \to \mathbb{C}$ which in our charts has the formula

$$(z_1, \ldots, z_N) \mapsto \frac{z_N}{H_1(z_1, \ldots, z_N)}.$$

Its differential is

$$\frac{dz_N}{H_1(z)} - \frac{z_N dH_1(z)}{H_1^2(z)}.$$

Thus, up to terms of order 2, and only along the hyperplane $t = 0$, equation (6) implies

$$\begin{aligned} \pi_* \partial_D \phi_L(u_1, v_1, \ldots, u_{N-1}, v_{N-1}) &= \sum_{j=1}^{N-1}\left(\sum_{p=1}^{N-1} \frac{\partial^2 \varphi}{\partial v_j \partial u_p} u_p + \frac{\partial^2 \varphi}{\partial v_j \partial v_p} v_p\right) du_j \quad (7)\\ &\quad + i\left(\sum_{p=1}^{N-1} \frac{\partial^2 \varphi}{\partial u_j \partial u_p} u_p + \frac{\partial^2 \varphi}{\partial u_j \partial v_p} v_p\right) du_j. \quad (8) \end{aligned}$$

Transversality of the above section to the zero section at 0, is equivalent to the linear independence *over the reals* of the following 2n vectors in $\mathbb{C}^N = \mathbb{R}^{2N}$:

$$\frac{\partial}{\partial u_1}\pi_* \partial_D \phi(0), \frac{\partial}{\partial v_1}\pi_* \partial_D \phi(0), \ldots, \frac{\partial}{\partial u_{N-1}}\pi_* \partial_D \phi(0), \frac{\partial}{\partial v_{N-1}}\pi_* \partial_D \phi(0). \tag{9}$$

According to equation (9) the corresponding matrix is

$$\begin{pmatrix}
\frac{\partial^2\varphi}{\partial v_1\partial u_1} & \frac{\partial^2\varphi}{\partial u_1\partial u_1} & \frac{\partial^2\varphi}{\partial v_2\partial u_1} & \frac{\partial^2\varphi}{\partial u_2\partial u_1} & \cdots & \frac{\partial^2\varphi}{\partial v_{N-1}\partial u_1} & \frac{\partial^2\varphi}{\partial u_{N-1}\partial u_1} \\
\frac{\partial^2\varphi}{\partial v_1\partial v_1} & \frac{\partial^2\varphi}{\partial u_1\partial v_1} & \frac{\partial^2\varphi}{\partial v_2\partial v_1} & \frac{\partial^2\varphi}{\partial u_2\partial v_1} & \cdots & \frac{\partial^2\varphi}{\partial v_{N-1}\partial v_1} & \frac{\partial^2\varphi}{\partial u_{N-1}\partial v_1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial^2\varphi}{\partial v_1\partial u_{N-1}} & \frac{\partial^2\varphi}{\partial u_1\partial u_{N-1}} & \frac{\partial^2\varphi}{\partial v_2\partial u_{N-1}} & \frac{\partial^2\varphi}{\partial u_2\partial u_{N-1}} & \cdots & \frac{\partial^2\varphi}{\partial v_{N-1}\partial u_{N-1}} & \frac{\partial^2\varphi}{\partial u_{N-1}\partial u_{N-1}} \\
\frac{\partial^2\varphi}{\partial v_1\partial v_{N-1}} & \frac{\partial^2\varphi}{\partial u_1\partial v_{N-1}} & \frac{\partial^2\varphi}{\partial v_2\partial v_{N-1}} & \frac{\partial^2\varphi}{\partial u_2\partial v_{N-1}} & \cdots & \frac{\partial^2\varphi}{\partial v_{N-1}\partial v_{N-1}} & \frac{\partial^2\varphi}{\partial u_{N-1}\partial v_{N-1}}
\end{pmatrix}$$

It coincides up to permutation of columns with the Hessian of $\varphi(w, 0)$ at 0. Therefore $\partial_D\phi L$ is transverse along D to $\underline{0}$ at z_0 if and only if the dual map is an immersion at z_0. □

Remark 3. *We observe as well that $\phi_L(\Sigma_D\phi_L)$ is an immersed curve. The Morse CR function ϕ_L is the restriction of a holomorphic submersion $\Phi\colon \mathbb{CP}^n\backslash H_0\cap H_1 \to \mathbb{CP}^1$. A point $x \in \Sigma_D\phi_L$ is by definition a point where $d\Phi_x$ vanishes along D_x. By lemma 2 the tangent space of $\Sigma_D\phi_L$ at x is transverse to D_x. If ϕ_L fails to immerse $\Sigma_D\phi_L$ at x, then the kernel of $d\Phi_x$ would contain the real hyperplane T_xM. But since $d\Phi_x$ is complex linear, we would have $d\Phi_x = 0$, this contradicting that Φ is a submersion.*

Corollary 1. *If $\partial\Omega \subset \mathbb{C}^N$ is a strictly $\mathbb{C}$-convex hypersurface, then for any pencil of hyperplanes L which contains a hyperplane which does not intersect $\partial\Omega$ (for example those with the base contained in the $\mathbb{CP}^{N-1}$ at infinity), $\phi_L\colon \partial\Omega \to \mathbb{C}$ is a CR Morse function.*

Proof. The dual statement to the base locus not intersecting $\partial\Omega$ is that $L \subset \mathbb{CP}^{N*}$ does not intersect $\partial\Omega^*$. Thus, L is a (Lefschetz) pencil for $\partial\Omega$ with empty base locus; because a hyperplane in the pencil misses $\partial\Omega$, the associated function takes values in $\mathbb{C}$. By proposition 1 it is CR Morse. □

4. Proof of theorem 1

We let $\partial\Omega$ be our strict $\mathbb{C}$-convex hypersurface. We fix a pencil L which contains a hyperplane which does not intersect $\partial\Omega$. By corollary 1 we have an everywhere defined associated function

$$\phi_L\colon \partial\Omega \to \mathbb{C}$$

which is CR Morse. We let $\Delta := \Sigma_D\phi_L$ be the singular subset and $K = \phi_L(\Delta)$ be the singular values. According to lemma 2, Δ is a collection of embedded circles transverse to D. Because $\partial\Omega$ is oriented, D is co-oriented and thus Δ inherits an orientation. By remark 3, K is a collection of immersed (closed) curves. We orient them so that $\phi_L\colon \Delta \to K$ is orientation preserving.

Our strategy is to prove that K is just one embedded curve. That will allow us to immediately construct a (real) Morse function $f\colon \partial\Omega \to \mathbb{R}$ with just two critical points, a maximum and a minimum, and hence to deduce that $\partial\Omega$ is homeomorphic to S^{2N-1}. Turning our attention back to the complex Gauss map, we will interpret it as endowing $\partial\Omega$ with a principal S^1-bundle structure, and the computation of its Euler class will produce a diffeomorphism between $\partial\Omega$ and S^{2N-1} of the required regularity.

The proof that K is just one embedded curve will be broken in several steps. Our fundamental technical result will be showing that (i) any two regular fibers are cobordant by a sequence of elementary cobordisms which amount to adding either a 0-handle or a 2N-2 handle. That will imply that (ii) a regular fiber is a collection of spheres of dimension 2N-3. Using connectivity of $\partial\Omega$ we will deduce that (iii) the regular fiber amounts to just a copy of S^{2N-3}. This last condition will be important to show that (iv) K is a collection of embedded circles bounding the set of regular values, and furthermore that (v) K has just one connected component.

Let us define $V = \phi_L(\partial\Omega\backslash\Delta)$. This is an open connected subset since it is the image by a submersion of an open connected manifold. Let $a, b \in V\backslash K$ be two regular values. Because ϕ_L is everywhere defined and $\partial\Omega$ is compact, the fibers W_a, W_b are compact, and we want to compare them. Because V is connected we can join a, b by a smooth path $\gamma \subset V$. Because K is immersed, by general position we can assume that γ is transverse to K (at a self intersection point we ask for transversality to all branches). To prove that W_a and W_b are cobordant by a cobordism which amounts to adding a 2N-2 handle or a 0-handle we follow the approach introduced in [9]. The submanifold γ is seen to be transverse to ϕ_L, and thus $\phi_L^{-1}(\gamma)$ provides a cobordism between the fibers. At a critical point x we use the coordinates of proposition 1. The origin 0 in $\mathbb{C}$ is a critical value and K at 0 is tangent to the real axis. We can assume w.l.o.g. that γ is a piece of the imaginary axis. Then it is easy to check that $\mathrm{Im}\phi_L$ restricts to $\phi_L^{-1}(\gamma)$ to a Morse function with just a critical point at x. The Hessian matrix at it is nothing but the Hessian of $\varphi(\cdot, 0)$ at the origin in $\mathbb{C}^{N-1}$. This means that if we move from a value with negative imaginary coordinate to one with positive imaginary coordinate the cobordism amounts to adding a 0 handle, so in the fiber over the latter point an S^{2N-3} appears. More invariantly, and recalling that K is oriented, if we orient γ so that at the intersection point with K the tangent vectors to K and γ are a negative oriented basis of the plane, then the cobordism as we move along γ through the critical value amounts to adding a 0-handle creating thus a new S^{2N-3} in the fiber. Going in the opposite direction, the cobordism caps off a sphere of the fiber, and this proves item (i). We can take $c \in \mathbb{C}\backslash\bar{V}$ and apply the above reasoning for the pair a, c, where a is any regular value in V. Then Morse theory implies that W_a is a disjoint union of manifolds diffeomorphic to S^{2N-3}, and therefore item (ii) holds. Note as well that a singular fiber is a collection of spheres of dimension 2N-3 and points, the latter corresponding to the intersection with Δ.

To show that the regular fiber is just one S^{2N-3}, we consider the restriction map

$$\phi_L \colon \partial\Omega\backslash\Delta \to V. \tag{10}$$

This is a submersion, and by the description of the singular fibers it is a proper map. Therefore, it is a locally trivial fibration. Since $\partial\Omega\backslash\Delta$ is connected, so is the fiber of (10) and in particular the regular fibers of ϕ_L must be connected, this proving item (iii).

We want to use the above information about the cobordisms while crossing K to show that $K = \partial\bar{V}$, and furthermore that Δ has just one connected component mapped by ϕ_L diffeomorphically to $\partial\bar{V}$. If that where the case, then we are justified to say that we found a CR Morse function of the simplest kind. For in the real case such a Morse function has image $\bar{D}^1 \subset \mathbb{R}$ with critical values identified with $\partial\bar{D}^1$. In the CR case we complexify the previous situation so that the image is $\bar{D}^2 \subset \mathbb{C}$ and the critical values are identified with $\partial\bar{D}^2$.

Because $\partial\Omega$ is compact, it is clear that $\partial\bar{V} \subset K$. To prove that $K \cap V$ is empty we argue by contradiction. If r is a critical value in V, we take a path γ joining regular values a, b and intersecting K at r transversely. The inverse image of γ by the restriction map in (10) defines a trivial cobordism W. By assumption there is at least a critical point over r, and item (i) implies that for each such critical point there is an S^{2N-3} either in the fiber over a or in the fiber over b and not in ∂W. This is a contradiction. Next we look at the behavior of the restriction map $\phi_L \colon \Delta \to K = \partial\bar{V}$. If we have a critical value $r \in K = \partial\bar{V}$, we consider all branches of K containing r (here we count branches with multiplicity, that is, all images of small arcs in Δ containing a point in $\phi_L^{-1}(r)$). For a small disk about r, each branch splits it in a positive side and a negative side, where the definition is such that crossing from positive to negative amounts to adding a 0-handle in the cobordism associated to that branch. A point c close to r in the complement of $\bar{V}$ belongs to all positive sides. It is easy to find an arc γ starting at c, cutting all branches transversely just at r, and ending in a value $a \in V$ in all negative sectors. Thus the fiber over a is the union of as many spheres of dimension 2N-3 as branches of K through r. Because we also know that the fiber over a is just one sphere, we conclude that r is contained in just one branch of K. Therefore, the restriction map $\phi_L \colon \Delta \to K = \partial\bar{V}$ is a bijection. Because by lemma 2 this restriction is a local embedding, item (iv) holds.

By the smooth Schoenflies theorem, we conclude that $\bar{V}$ is diffeomorphic to $\bar{D}^2_\Lambda$ a closed disk $\bar{D}^2$ with Λ small disks removed, where $\Lambda + 1 = \#\pi_0(\Delta)$.

We will prove theorem 1 under the assumption that $\Lambda = 0$. Again, this is done in two steps, the first one providing a homeomorphism $\partial\Omega \to S^{2n-1}$.

We take the pencil as above whose image is assumed to be the unit disk. We let $\pi_2 \colon \mathbb{R}^2 \to \mathbb{R}$ be the second projection and define

$$f = \pi_2 \circ \phi_L \colon \partial\Omega \to [-1, 1].$$

We claim that this is a Morse function with just 2 critical points: indeed, because $\phi_L \colon \partial\Omega\backslash\Delta \to \mathbb{C}$ and π_2 are submersions, the critical points are contained in the circle Δ. Because $\phi_L \colon \Delta \to \partial D^2$ is an embedding, the critical points are identified with the critical points of $\pi_2 \colon \partial D^2 \to [-1, 1]$, which are obviously the points $(-1, 0)$ and $(1, 0)$. Thus by theorem 4.1. in [8] $\partial\Omega$ is homeomorphic to the sphere.

Now we want to get a similar result but without losing all regularity. To that end we turn our attention back to the complex Gauss map,

$$G \colon \partial\Omega \to \mathbb{CP}^{N-1*},$$

which in our case is everywhere defined because $\partial\Omega$ lies in an affine chart. By lemma 1 it is a C^{h-1} submersion. By compactness of $\partial\Omega$ and connectedness of $\mathbb{CP}^{N-1^*}$ the map must be surjective, so it is a fiber bundle over $\mathbb{CP}^{N-1*}$ with fiber F a collection of circles. We consider now the following part of the long exact sequence of homotopy groups:

$$0 \to \mathbb{Z} \xrightarrow{e} \mathbb{Z} \to 0 \to 0 \to \pi_0(F) \to 0, \tag{11}$$

where the generators we choose use the complex orientation of $\mathbb{CP}^{1*} \subset \mathbb{CP}^{N-1*}$ and the orientation of the fiber induced by the complex orientation of $\mathbb{CP}^{N-1*}$ and the orientation of $\partial\Omega$, and we also use the 2-connectedness of $\partial\Omega$.

We deduce firstly that the fiber must be connected. Thus our space is a fiber bundle with fiber S^1. Because the $\mathrm{Diff}^+(S^1)$ retracts onto $SO(2)$, our fiber bundle admits a principal S^1-bundle structure. Thus it is classified by its Euler class, which is nothing but $e(1)$ where e is the first transgression map in (11). Therefore the Euler class is ± 1, and the principal bundle structure is diffeomorphic to the Hopf fibration (or the opposite one). Hence, $\partial\Omega$ is C^{r-1}-diffeomorphic to S^{2N-1}.

We still need to rule out the possibility of $\bar{V}$ being different from the disk. There are two different cases. The simplest one is when $N > 2$. By dimension count the inclusion

$$\partial\Omega\backslash\Delta \hookrightarrow \partial\Omega$$

induces an isomorphism in homotopy groups up to degree 2, and the restriction map

$$\phi_L \colon \partial\Omega\backslash\Delta \to D^2_\Lambda$$

is a locally trivial fibration. Then comparison of its homotopy exact sequence up to degree two with the one given by the complex Gauss map leads to a contradiction.

If $N=2$ the above argument cannot rule out the case $\Lambda=1$, so one has to discard the possibility of $\partial\Omega$ being diffeomorphic to $S^1\times S^2$ by different means. The advantage is that in dimension 2 complex lines and complex hyperplanes are the same. We pick $x\in\partial\Omega$, and consider the pencil of complex lines having it as its base point. This gives a map

$$\partial\Omega\backslash x\to\mathbb{C}.$$

More precisely, the fiber of each point is the intersection of the corresponding complex line with $\partial\Omega\backslash x$. A point $y\in\partial\Omega\backslash x$ fails to be regular if and only if the complex line corresponding to its value is tangent to $\partial\Omega$ at y. By remark 1 such points are isolated on its fiber. Thus the fibers are a collection of points and 1-manifolds. At x there is just one tangent complex line. The other $\mathbb{C}$-worth of lines are transverse. Thus their fibers in $\partial\Omega\backslash x$ contain one open interval which compactifies into a circle by adding x. That $\mathbb{C}$-worth of interval is an open subset of Ω which is diffeomorphic to $\mathbb{R}^3$ (a locally trivial fiber bundle over $\mathbb{R}^2$ with fiber $\mathbb{R}$), and its 1-point compactification is a connected component of $\partial\Omega$. Thus $\partial\Omega$ is diffeomorphic to the 3-sphere, and this finishes the proof of theorem 1.

It is known that the dual map of a strictly $\mathbb{C}$-convex hypersurface $\partial\Omega$ maps it homeomorphically to $\partial\Omega^*$, which is also strictly $\mathbb{C}$-convex.

Corollary 2. *Let $\partial\Omega\subset\mathbb{C}^N$ be a strictly $\mathbb{C}$-convex hypersurface. Then the dual map $\nu\colon\partial\Omega\to\partial\Omega^*$ is an embedding of class C^{h-1}, and $\partial\Omega^*$ is a C^{h-1}-strictly $\mathbb{C}$-convex hypersurface.*

Proof. By lemma 1 the dual map is an immersion. Assume that it is not bijective and let $x^*\in\partial\Omega^*$ be a point contained in several branches. Take L to be a Lefschetz pencil containing x^* and a hyperplane not intersecting $\partial\Omega$; for example a suitable small perturbation of the hyperplane at infinity. Using the notation of the proof of theorem 1 we have

$$L\cap\partial\Omega^*=K,$$

where $x^*\in K$. Because L is Lefschetz $K\subset\mathbb{C}$ must have as many 1-dimensional branches at x^* as branches $\partial\Omega^*$ has at x^*. Because L is a pencil in the hypothesis of corollary 1, by item (v) in the proof theorem 1 K is a circle, and this contradicts the existence of several branches.

The dual map for $\partial\Omega^*$ is

$$\nu^{-1}\colon\partial\Omega^*\to\partial\Omega,$$

in particular it is a local embedding and thus $\partial\Omega^*\subset\mathbb{C}^{N^*}$ has immersed dual map. Thus its Euclidean shape operator along the CR distribution D^* is non-degenerate. One checks that at the point $x^*\in\partial\Omega^*\subset\mathbb{C}^{N^*}$ in which the norm attains a (local) maximum, the Euclidean shape operator is positive definite, and this proves the corollary. □

Remark 4. *It would be interesting to explore the implications of existence of CR Morse functions for hypersurfaces with immersed dual subset.*

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