• <u>Exercício 4.2</u>: Considere o par aleatório contínuo (X,Y) com função densidade de probabilidade conjunta dada por
$$f_{X,Y}(x,y) = \begin{cases} 2, & 0<x<y<1 \\ 0, & \text{c.c.} \end{cases}$$

a) $\text{corr}(X,Y) = ?$

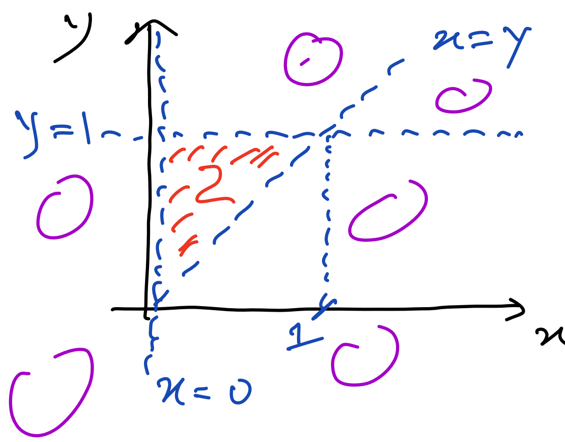
b) $V(X \mid Y=y) = ?$

c) Verifique que $E(X) = E[E(X|Y)]$.

<u>Resolução</u>

a) $$\text{corr}(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{V(X)\,V(Y)}} = \frac{E(XY) - E(X)E(Y)}{\sqrt{(E(X^2)-E^2(X))(E(Y^2)-E^2(Y))}}$$

• $$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X,Y}(x,y)\, dy\, dx$$



$$= \int_0^1 \left(\int_x^1 (xy \cdot 2)\, dy \right) dx$$

$$= \int_0^1 x \left[\left[y^2 \right]_x^1 \right] dx$$

$$= \int_0^1 x(1-x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = 1/4$$

• F.d.p. marginais:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy = \begin{cases} \int_x^1 2\, dy = 2(1-x), & 0<x<1 \\ 0, & \text{c.c.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^y 2\,dx = 2y, & 0<y<1 \\ 0, & \text{c.c.} \end{cases}$$

- $$E(X) = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_0^1 x \cdot 2(1-x)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 =$$
$$= 2\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = 1 - \tfrac{2}{3} = \tfrac{1}{3}$$

- $$E(X^2) = \int_0^1 x^2\, 2(1-x)\,dx = 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\left(\tfrac{1}{3} - \tfrac{1}{4}\right) =$$
$$= 2\left(\tfrac{1}{12}\right) = \tfrac{1}{6}$$

- $$V(X) = E(X^2) - E^2(X) = \tfrac{1}{6} - \tfrac{1}{9} = \tfrac{1}{18}$$

- $$E(Y) = \int_{-\infty}^{+\infty} y\, f_Y(y)\,dy = \int_0^1 y\, 2y\,dy = 2\left[\frac{y^3}{3}\right]_0^1 = \tfrac{2}{3}$$

- $$E(Y^2) = \int_0^1 y^2 \cdot 2y\,dy = 2\left[\frac{y^4}{4}\right]_0^1 = \tfrac{1}{2}$$

- $$V(Y) = E(Y^2) - E^2(Y) = \tfrac{1}{2} - \tfrac{4}{9} = \frac{1}{18}$$

- $$\text{cov}(X,Y) = E(XY) - E(X)E(Y) = \tfrac{1}{4} - \tfrac{1}{3}\cdot\tfrac{2}{3} = \tfrac{1}{4} - \tfrac{2}{9} = \tfrac{1}{36}$$

- $$\text{Corr}(X,Y) = \frac{1/36}{\sqrt{1/18 \times 1/18}} = \frac{18}{36} = \tfrac{1}{2} \neq 0$$

$\Rightarrow$ X e Y não são independentes e estão moderadam/ correlacionadas de forma positiva.

b) $V(X \mid Y=y) = ?$

- $$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y}, \quad 0<x<y<1$$

• $E(X|Y=y) = \int_0^y x \cdot \frac{1}{y}\, dx = \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y = \frac{1}{y}\cdot\frac{y^2}{2} = y/2, \quad 0<y<1$

• $E(X^2|Y=y) = \int_0^y x^2 \cdot \frac{1}{y}\, dx = \frac{1}{y}\left[\frac{x^3}{3}\right]_0^y = \frac{1}{y}\cdot\frac{y^3}{3} = y^2/3, \quad 0<y<1$

• $V(X|Y=y) = \frac{y^2}{3} - \left(\frac{y}{2}\right)^2 = y^2\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{y^2}{12}$ //

c) Verifique que $E(X) = E[E(X|Y)]$.

$E(X|Y)$ = v.a. com valor $E(X|Y=y) = \frac{Y}{2}$ e f.d.p

$$f_Y(y) = 2y, \quad 0<y<1$$

$$\Rightarrow E(E(X|Y)) = \int_0^1 \frac{y}{2}\cdot 2y\, dy = \left[\frac{y^3}{3}\right]_0^1 = 1/3 = E(X)$$

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(2) X = nº de fusíveis defeituosos em 14 selecionados ao acaso e com reposição de um lote. P(fusível ser defeituoso) = 0.2.

$\Rightarrow X \sim Bin(n=14, p=1/5)$, $f_X(x) = \binom{14}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{14-x}$, $x=0,\dots,14$.

• Mediana $\mu_e = ?$ • $P(X \leq \mu_e | X \geq 1) = ?$

• $F_X(\mu_e^-) \leq 1/2 \leq F_X(\mu_e)$

Tabela $\Rightarrow F_X(2) = 0.4481$ e $F_X(3) = 0.6982$

$$\Rightarrow \boxed{\mu_e = 3}$$

• $P(X\leq 3 | X \geq 1) = \frac{P(1\leq X\leq 3)}{P(X\geq 1)} = \frac{F_X(3)-F_X(0)}{1-F_X(0)}$

$\underset{\text{(tabela)}}{=} \frac{0.6982-0.0440}{1-0.0440} = \frac{0.6542}{0.9560} = 0,6843$ //

• Modas das distribuições binomial, geométrica e Poisson

$\boxed{\text{Binomial}}$

$$\frac{P(X=x)}{P(X=x-1)} = \frac{\binom{n}{x} p^{x} (1-p)^{n-x}}{\binom{n}{x-1} p^{x-1} (1-p)^{n-x+1}} = \frac{(x-1)!\,(n-x+1)!}{x!\,(n-x)!} \cdot \frac{p}{1-p} =$$

$$= \frac{n-x+1}{x} \cdot \frac{p}{1-p} \geq 1 \Rightarrow (n-x+1)p \geq x(1-p) \Rightarrow \boxed{x \leq p(n+1)}$$

$$\frac{P(X=x)}{P(X=x+1)} = \frac{\binom{n}{x} p^{x} (1-p)^{n-x}}{\binom{n}{x+1} p^{x+1} (1-p)^{n-x-1}} = \frac{(x+1)!\,(n-x-1)!}{x!\,(n-x)!} \frac{(1-p)}{p} =$$

$$= \frac{(x+1)(1-p)}{(n-x)\,p} \geq 1 \Rightarrow (x+1)(1-p) \geq (n-x)\,p \Rightarrow \boxed{x \geq p(n+1) - 1}$$

$$\Downarrow$$

$$\boxed{p(n+1) - 1 \leq M_o \leq p(n+1)}$$

$\boxed{\text{Geométrica}}$ $\quad P(X=x) = (1-p)^{x-1} \cdot p, \quad x = 1, 2, 3, \ldots$

$$\Rightarrow \boxed{M_o = 1}$$

$\boxed{\text{Poisson}}$ $\quad P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$

$$\frac{P(X=x)}{P(X=x-1)} = \frac{\dfrac{\cancel{e^{-\lambda}}\,\lambda^{x}}{x!}}{\dfrac{\cancel{e^{-\lambda}}\,\lambda^{x-1}}{(x-1)!}} = \frac{\lambda}{x} \geq 1 \Rightarrow \boxed{x \leq \lambda}$$

$$\frac{P(X=x)}{P(X=x+1)} = \frac{\lambda^{x}/x!}{\lambda^{x+1}/(x+1)!} = \frac{x+1}{\lambda} \geq 1 \Rightarrow \boxed{x \geq \lambda - 1}$$

$$\Rightarrow \boxed{\lambda - 1 \leq M_o \leq \lambda}$$

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(3) X = vida útil de televisor em anos

$X \sim N(\mu, \sigma^2 = 3^2)$ $\left(\Rightarrow \frac{X-\mu}{\sigma} \sim N(0,1) \right)$

$P(X \leq 5) = 0.025 \Leftrightarrow F_X(5) = 0.025$

• $\mu = ?$ $E(0.5X^2 + 1.5X) = ?$

• $0.025 = P(X \leq 5) = P\left(\frac{X-\mu}{3} \leq \frac{5-\mu}{3}\right) = \Phi\left(\frac{5-\mu}{3}\right)$

com Φ = função distribuição de prob. de $N(0,1)$

$\Leftrightarrow \phi\left(\frac{\mu-5}{3}\right) = 1 - 0.025 = 0.975$

$\underset{\text{tabela}}{\Leftrightarrow} \frac{\mu-5}{3} = 1.96 \Leftrightarrow \mu = 1.96 \times 3 + 5 = \underline{\underline{10,88}}$ anos

• $E(0.5X^2 + 1.5X) = 0.5\,E(X^2) + 1.5\,E(X)$

$= 0.5\,(V(X) + E^2(X)) + 1.5\,E(X)$

$= 0.5\,(9 + 10.88^2) + 1.5 \times 10.88$

$= 80.0072$ //

(4) (não chegou a ser resolvido na aula por falta de tempo)

X \ Y	0	1	2	P(X=x)
0	0.3	0	0	0.3
1	0.1	0.3	0	0.4
2	0	0.1	0.2	0.3
P(Y=y)	0.4	0.4	0.2	1

$Corr(X,Y) = ?$

Comente o valor obtido.

$Corr(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)\,V(Y)}}$

$= \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)\,V(Y)}}$

- $E(XY) = 1\times 1\times 0.3 + 2\times 1\times 0.1 + 2\times 2\times 0.2 =$
$= 0.3 + 0.2 + 0.8 = 1.3$,

- $E(X) = 1\times 0.4 + 2\times 0.3 = 1$
$E(Y) = 1\times 0.4 + 2\times 0.2 = 0.8$

- $E(X^2) = 1\times 0.4 + 4\times 0.3 = 1.6$
$E(Y^2) = 1\times 0.4 + 4\times 0.2 = 1.2$

- $V(X) = E(X^2) - E^2(X) = 1.6 - 1 = 0.6$
$V(Y) = E(Y^2) - E^2(Y) = 1.2 - 0.64 = 0.56$

- $$\text{Corr}(X,Y) = \frac{1.3 - 1\times 0.8}{\sqrt{0.6\times 0.56}} = \frac{0.5}{\sqrt{0.336}}$$
$$\simeq 0.862582$$

- Comentários:

(i) $\text{corr}(X,Y) \neq 0 \Rightarrow X$ e Y não são independentes

(ii) $\text{corr}(X,Y) > 0 \Rightarrow X$ e Y tendem a variar no mesmo sentido em relação aos seus valores médios.

(iii) $\text{Corr}(X,Y) \simeq 0.86$ perto de $1 \Rightarrow X$ e Y estão fortemente linearmente correlacionadas, com declive positivo.