- $d_{\mathbb{H}}(z, z_0) = r \iff$

$$2\,\text{arctgh}\,\delta_{\mathbb{H}}(z, z_0) = r$$

$$\Rightarrow \delta_{\mathbb{H}}(z, z_0) = \text{tgh}\left(\frac{r}{2}\right) =: \underline{\underline{m}}$$

- $\delta_{\mathbb{H}}(z, z_0) = m \iff \left|\dfrac{z - z_0}{\bar{z} - z_0}\right|^2 = m^2$

$$\iff \frac{(z - z_0)(\bar{z} - \bar{z}_0)}{(\bar{z} - z_0)(z - \bar{z}_0)} = m^2$$

$$|w|^2 = w\bar{w}$$

$$\iff (z - z_0)(\bar{z} - \bar{z}_0) = m^2(\bar{z} - z_0)(z - \bar{z}_0)$$

$$\Leftrightarrow |z|^2 - \left(\frac{\overline{z_0} - m^2 z_0}{1-m^2}\right) z - \boxed{\left(\frac{z_0 - m^2 \overline{z_0}}{1-m^2}\right)} \bar{z} + |z_0|^2 = 0$$

centro

circunf.

$$\text{raio} = \sqrt{-|z_0|^2 + \left|\frac{z_0 - m^2 \overline{z_0}}{1-m^2}\right|^2}$$

$$= \dots = \frac{2m}{1-m^2} \operatorname{Im} z_0 = (\sinh r) \operatorname{Im} z_0$$

Proposição:

- $z_1, z_2, z_3 \in \mathbb{H}$ 3 ptos distintos que não estão numa mesma reta hiperbólica

$\Rightarrow$

$$d_{\mathbb{H}}(z_1, z_3) < d_{\mathbb{H}}(z_1, z_2) + d_{\mathbb{H}}(z_2, z_3)$$

- Se $d_{\mathbb{H}}(z_1, z_2) \geq d_{\mathbb{H}}(z_1, z_3)$

ou $d_{\mathbb{H}}(z_2, z_3) \geq d_{\mathbb{H}}(z_1, z_3)$

Temos então que apenas considerar o caso

$$d_{\mathbb{H}}(z_1, z_2) < d_{\mathbb{H}}(z_1, z_3)$$

e

$$d_{\mathbb{H}}(z_2, z_3) < d_{\mathbb{H}}(z_1, z_3)$$

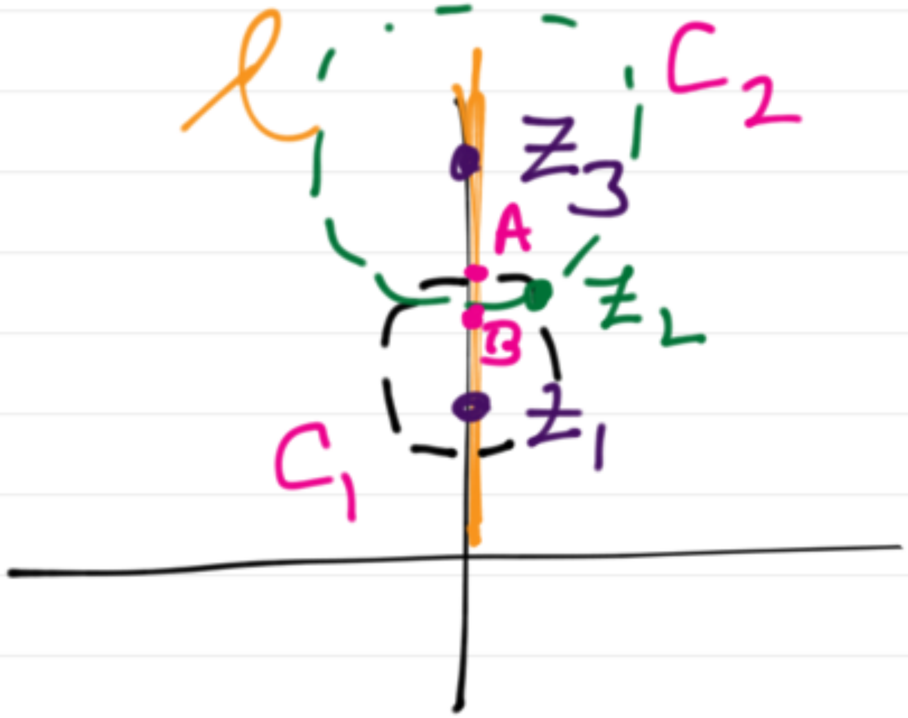
l - recta hiperbólica que passa em z_1 e z_3

As duas circ. ñ são tgs a.c. $z_2 \in$ eixo imaginario e os pts estariam na mesma recta

Usando uma transf. de Möbius apropriada posso supor ℓ tem eq. $\operatorname{Re} z = 0$

tem centro no eixo imag.

- $C_{d_{\mathbb{H}}(z_1, z_2)}(z_1)$ $\ni z_2$
- $C_{d_{\mathbb{H}}(z_2, z_3)}(z_3)$ $\ni z_2$



$A \in C_1 \cap \ell \qquad B \in C_2 \cap \ell$

$$d_{\mathbb{H}}(z_1, z_2) + d_{\mathbb{H}}(z_2, z_3) =$$

$$= d_{\mathbb{H}}(z_1, A) + d_{\mathbb{H}}(B, z_3)$$

$$= d_{\mathbb{H}}(z_1, A) + d_{\mathbb{H}}(B, A) + d_{\mathbb{H}}(A, z_3)$$

$$= d_{\mathbb{H}}(z_1, z_3) + d_{\mathbb{H}}(A, B) > d_{\mathbb{H}}(z_1, z_3)$$

Teorema:

$d_{\mathbb{H}}(\cdot,\cdot)$ verifica a desigualdade triangular

$$d_{\mathbb{H}}(z_1,z_3) \leq d_{\mathbb{H}}(z_1,z_2) + d_{\mathbb{H}}(z_2,z_3)$$

e, além disso,

$$d_{\mathbb{H}}(z_1,z_3) = d_{\mathbb{H}}(z_1,z_2) + d_{\mathbb{H}}(z_2,z_3)$$

sse z_2 está no segmento de recta hiperbólico que une z_1 a z_3

Finalmente vamos ver que $\delta_{\mathbb{H}}$ verifica sempre a desigualdade estrita:

$$\delta_{\mathbb{H}}(z_1, z_3) = \operatorname{tgh}\left(\frac{d_{\mathbb{H}}(z_1 z_3)}{2}\right)$$

$$\leq \operatorname{tgh}\left(\frac{d_{\mathbb{H}}(z_1, z_2)}{2} + \frac{d_{\mathbb{H}}(z_2, z_3)}{2}\right)$$

$$= \frac{\operatorname{tgh}\left(\frac{d_{\mathbb{H}}(z_1, z_2)}{2}\right) + \operatorname{tgh}\left(\quad\right)}{1 + \operatorname{tgh}\left(\quad\right)\operatorname{tgh}\left(\quad\right)}$$

$$<$$

$$\frac{\operatorname{tgh}\left(\frac{d_{\mathbb{H}}(z_1,z_2)}{2}\right)+\operatorname{tgh}(\qquad)}{1+\operatorname{tgh}(\qquad)\operatorname{tgh}(\qquad)}$$

$$< \operatorname{tgh}(\qquad)+\operatorname{tg}(\qquad)$$

$$= \delta_{\mathbb{H}}(z_1,z_2)+\delta_{\mathbb{H}}(z_2,z_3)$$