(2) $f(z)=\dfrac{a\bar{z}+b}{c\bar{z}+d}$ $\quad a,b,c,d \in \mathbb{R}$

$$f(i)=\frac{-ai+b}{-ci+d}=$$

$$=\frac{(b-ai)(d+ci)}{c^2+d^2}=$$

$$=\frac{ac+bd}{c^2+d^2}-\frac{ad-bc}{c^2+d^2}i$$

>0

$\Rightarrow ad-bc<0$

$G_{\mathbb{H}} = \{ f \in \text{Möb} : f(\mathbb{H}) = \mathbb{H} \}$

queremos $G_{\mathbb{H}} = \text{Iso}(\mathbb{H})$

$d_{\mathbb{H}}(z, w) = ?$

$f \in G_{\mathbb{H}}$

• $|z - w|$?

$a, b, c, d \in \mathbb{R}$

$|f(z) - f(w)| =$

$|\cdot| \Rightarrow$ = para o conj.

$= \left| \dfrac{az+b}{cz+d} - \dfrac{aw+b}{cw+d} \right|$

$f \in \text{Möb}$

$$\left|\frac{az+b}{cz+d} - \frac{aw+b}{cw+d}\right| =$$

$$= \left|\frac{(az+b)(cw+d) - (aw+b)(cz+d)}{(cz+d)(cw+d)}\right|$$

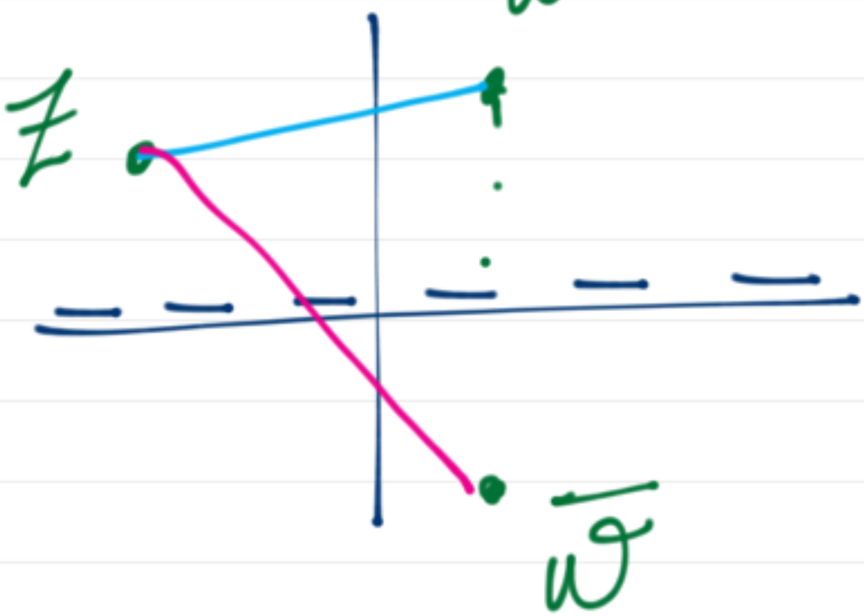
$$= \cdots = \frac{|ad-bc|}{|cz+d||cw+d|} \cdot |z-w|$$

$$\Rightarrow \quad \delta_{\mathbb{H}}(z,w) = \frac{|z-w|}{|z-\overline{w}|}$$

NOTA:

$$\left|\frac{f(z)-f(w)}{f(z)-f(\overline{w})}\right| = \left|\frac{z-w}{z-\overline{w}}\right|$$

$$\delta_{\mathbb{H}}(z,w) = \frac{|z-w|}{|z-\overline{w}|}$$



$$\delta_{\mathbb{H}}(z,w) < 1$$

$z, w \in \mathbb{H}$

Distancia

$$\delta_{\mathbb{H}}(z,w) = \frac{|z-w|}{|z-\overline{w}|}$$

- $\delta_{\mathbb{H}}(z,w) \geq 0$
- $\delta_{\mathbb{H}}(z,w) = 0 \Longrightarrow z = w$
- $\delta_{\mathbb{H}}(z,w) = \delta_{\mathbb{H}}(w,z)$
- Desigualdade triangular

→ Veremos + tarde que é verificada. ✓

No entanto temos sempre desigualdade estrita

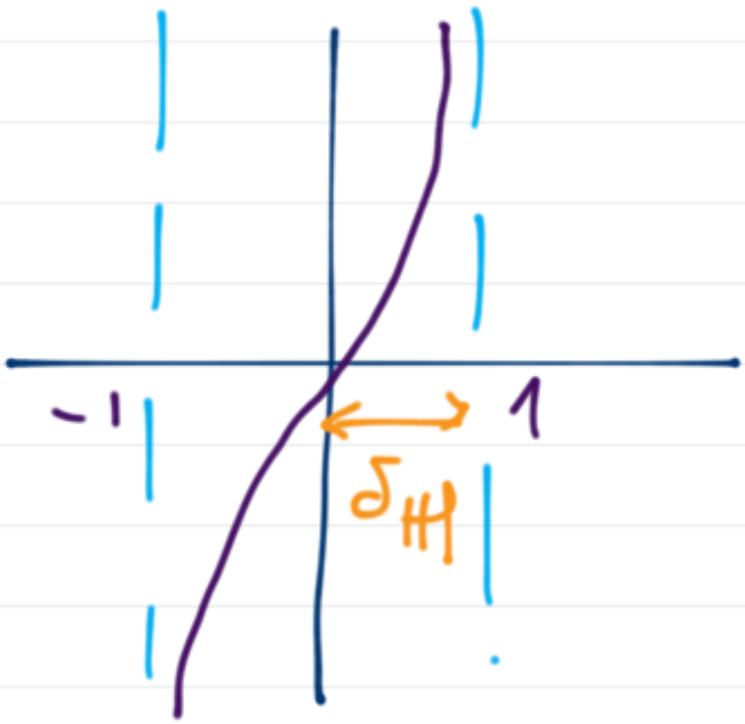
$$\delta_{\mathbb{H}}(z_1, z_3) < \delta_{\mathbb{H}}(z_1, z_2) + \delta_{\mathbb{H}}(z_2, z_3)$$

Não podemos definir segmentos de reta com as propriedades usuais

Para ultrapassar o problema

$$d_{\mathbb{H}}(z, w) = 2 \operatorname{arctanh}\left(\delta_{\mathbb{H}}(z, w)\right)$$
$$= \ln\left(\frac{1 + \delta_{\mathbb{H}}(z, w)}{1 - \delta_{\mathbb{H}}(z, w)}\right) \in [0, +\infty[$$

$$d_{\mathbb{H}}(z,w) = 2\,\operatorname{arctanh}\, \delta_{\mathbb{H}}(z,w)$$



$\delta_{\mathbb{H}}(z,w) \in [0,1[$

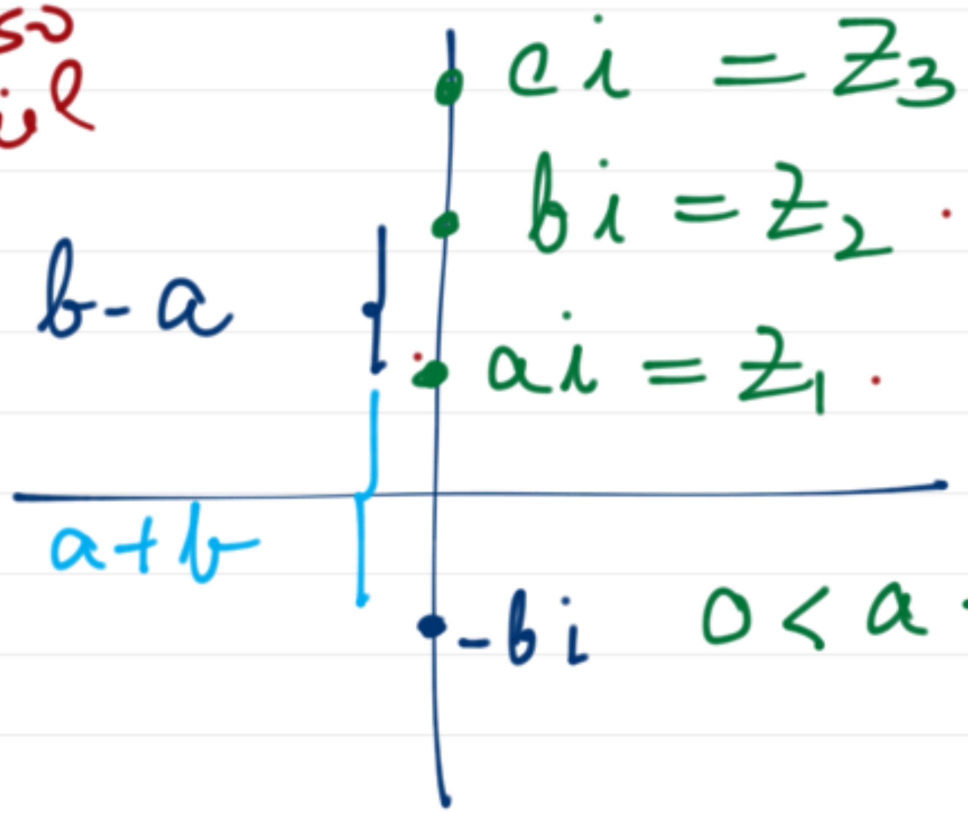
$d_{\mathbb{H}}(z,w) \in [0,+\infty[$

Propriedades

- $d_{\mathbb{H}}(z,w) \geq 0$
- $d_{\mathbb{H}}(z,w) = 0 \iff z = w$
- $d_{\mathbb{H}}(z,w) = d_{\mathbb{H}}(w,z)$

Antes de ver a desigualdade triangular vamos ver onde temos igualdade para então definirmos "segmentos de recta" hiperbólicos.

Caso fácil



$\mathbb{H}$

$0<a<b<c$

$$d_{\mathbb{H}}(z_1,z_2)=2\operatorname{arctgh}\left(\frac{b-a}{a+b}\right)$$

$$d_{\mathbb{H}}(z_2,z_3)=2\operatorname{arctgh}\left(\frac{c-b}{b+c}\right)$$

$$d_{\mathbb{H}}(z_1,z_3)=2\operatorname{arctgh}\left(\frac{c-a}{a+c}\right)$$

$$\boxed{\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}}$$

$$\tanh\left(\frac{d_{\mathbb{H}}(z_1,z_2)}{2} + \frac{d_{\mathbb{H}}(z_2,z_3)}{2}\right)$$

$$= \tanh\left(\operatorname{arctgh}\left(\frac{b-a}{a+b}\right) + \operatorname{arctgh}\left(\frac{c-b}{b+c}\right)\right)$$

$$= \frac{\dfrac{b-a}{a+b} + \dfrac{c-b}{b+c}}{1 + \dfrac{b-a}{a+b}\cdot\dfrac{c-b}{b+c}} = \dots$$

$$= \frac{\frac{b-a}{a+b} + \frac{c-b}{b+c}}{1 + \frac{b-a}{a+b} \cdot \frac{c-b}{b+c}} =$$

$$= \frac{(b-a)(b+c) + (c-b)(a+b)}{(a+b)(b+c) + (b-a)(c-b)}$$

$$= \cdots = \frac{c-a}{a+c} = \operatorname{tgh}\left(\frac{d_{H}(z_1, z_3)}{2}\right)$$

$$= d_H(z_1, z_2) + d_H(z_2, z_3) = d_H(z_1, z_3)$$

- No eixo imaginário temos "igualdade triangular"

$$\ell \cup \{\infty\} \perp \text{eixo real}$$

eixo imaginário

$$\Rightarrow f(\ell) \text{ com } f \in G_{\mathbb{H}}$$

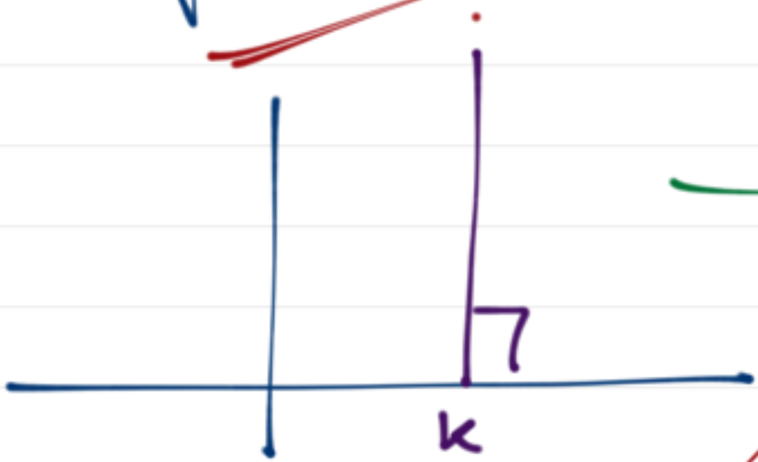
São circ. de $\overline{\mathbb{C}}$ $\perp$ ao eixo real

$$\left(f(\overline{\mathbb{R}}) = \overline{\mathbb{R}} \right)$$

$\Rightarrow f(\ell) \cap \mathbb{H}$ $\quad m \cup \infty$

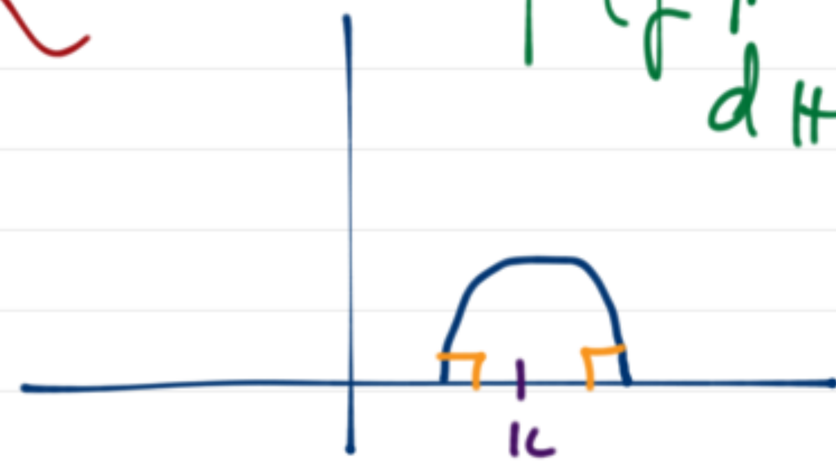
$\perp$

Nestes pts a "igualdade Δ" é verificada (f preserva $d_{\mathbb{H}}$)



$\operatorname{Re} z = k$

$k \in \mathbb{R}$



$|z - k| = r$

$k \in \mathbb{R}$

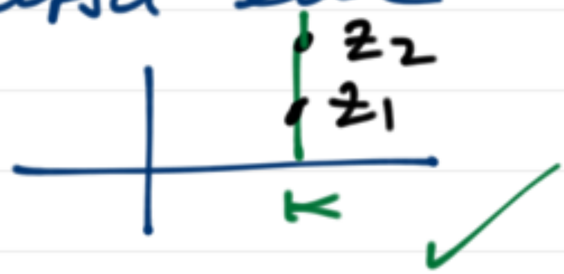
Definição: As rectas hiperbólicas de $\mathbb{H}$ são as curvas de $\mathbb{H}$ de eq.

- $\operatorname{Re} z = k$
- $|z - k| = r$

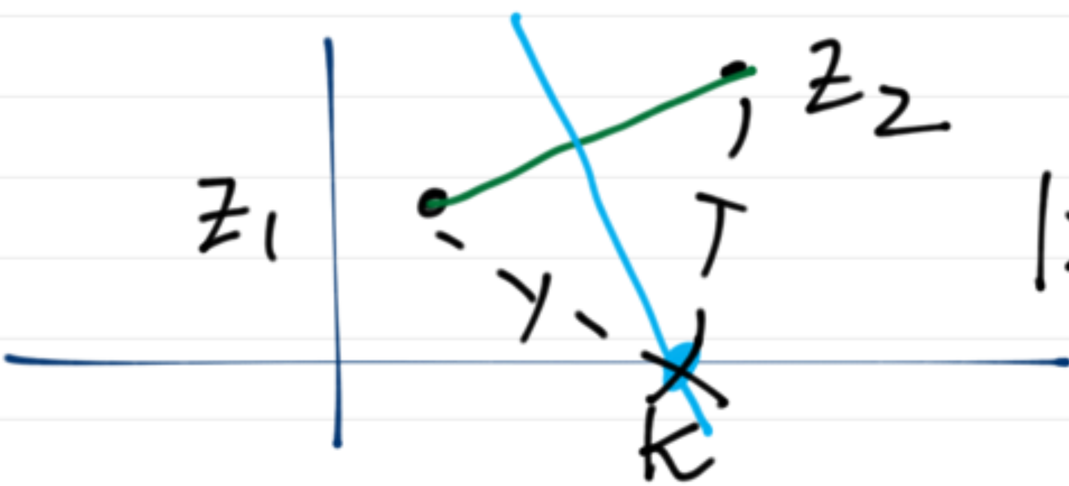
com $k \in \mathbb{R}$ e $r \in \mathbb{R}^+$

NOTA:

Dados 2 pts $z_1, z_2 \in \mathbb{H}$ existe uma única recta hiperbólica que passa em z_1 e z_2



- Se $\operatorname{Re} z_1 = \operatorname{Re} z_2 = k$
- Se $\operatorname{Re} z_1 \neq \operatorname{Re} z_2$



$|z-k| = |z_1-k|$

Falta ver que se os pts não pertencem a uma reta hiperbólica a igualdade ñ se verifica e temos desigualdade estrita

Para isso precisamos de saber o que são circunferências hiperbólicas ...

As circunferências hiperbólicas de $\mathbb{H}$ são circunferências euclidianas

$$C_r(z_0) = \{ z \in \mathbb{H} : d_{\mathbb{H}}(z, z_0) = r \}$$

é a circunf. euclidiana de centro euclidiano

$$\frac{z_0 - \bar{z}_0 m^2}{1 - m^2}$$ e raio (euclidiano) $(\sinh r) \operatorname{Im} z_0$

$$m = \operatorname{tgh}\left(\frac{r}{2}\right)$$

$$\frac{z_0 - \overline{z_0}\, m^2}{1-m^2}$$ — centro euclidiano

z_0 — centro hiperbólico

→ têm a mesma parte real

$$\operatorname{Re}\left(\frac{z_0 - \overline{z_0}\, m^2}{1-m^2}\right) =$$

$$= \frac{1}{1-m^2}\left(\operatorname{Re} z_0 - \operatorname{Re} z_0\, m^2\right)$$

$$= \operatorname{Re} z_0\, \frac{(1-m^2)}{1-m^2} = \operatorname{Re} z_0$$

$$\operatorname{Im}\left(\frac{z_0 - \bar{z}_0\, m^2}{1-m^2}\right) =$$

$$= \frac{1}{1-m^2}\left(\operatorname{Im} z_0 + \operatorname{Im} z_0\, m^2\right)$$

$$= \operatorname{Im} z_0\, \frac{(1+m^2)}{1-m^2} > \operatorname{Im} z_0$$

$$m = \operatorname{tgh}\left(\frac{r}{2}\right)$$