# Aula anterior

## transformações de Möbius

$f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ $\quad \overset{\mathbb{C}}{\in} a, b, c, d$

$\to f(z) = \dfrac{az+b}{cz+d} \quad ad-bc \neq 0$

$\to \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad$ Möb$^+$ $\quad ad-bc \neq 0$

$A \sim \lambda A \quad PGL(2,\mathbb{C})$

Propriedades: $f \in \text{Möb}^+$

- $f \in \text{Möb}^+$ e $f \neq id$

$\Rightarrow$ $f$ tem 1 ou 2 pts fixos em $\overline{\mathbb{C}}$

- $f \in \text{Möb}^+$ fixa 3 pts distintos

$\Rightarrow$ $f = id$

- $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ distintos

  $w_1, w_2, w_3 \in \overline{\mathbb{C}}$ distintos

$\hookrightarrow$ $\exists!\ f \in \text{Möb}^+$ t.q. $f(z_i) = w_i$

- $f \in Möb^+ \Rightarrow f$ transforma circunferencias de $\overline{\mathbb{C}}$ em circunferencias de $\overline{\mathbb{C}}$

- $\left.\begin{array}{l} C_1 \\ C_2 \end{array}\right\}$ circunf. de $\overline{\mathbb{C}}$

  $\Rightarrow \exists\, f \in Möb^+$ t.q.

  $$f(C_1) = C_2$$

- $f \in Möb^+ \Rightarrow f$ preserva angulos

$\text{Möb} \equiv$ grupo de Möbius generalizado

$$= \underbrace{\text{Möb}^+}_{\text{transf. de Möbius}} \cup \left\{ \frac{a\bar{z}+b}{c\bar{z}+d},\ a,b,c,d \in \mathbb{C},\ ad-bc \neq 0 \right\}$$

$z \mapsto \bar{z}$

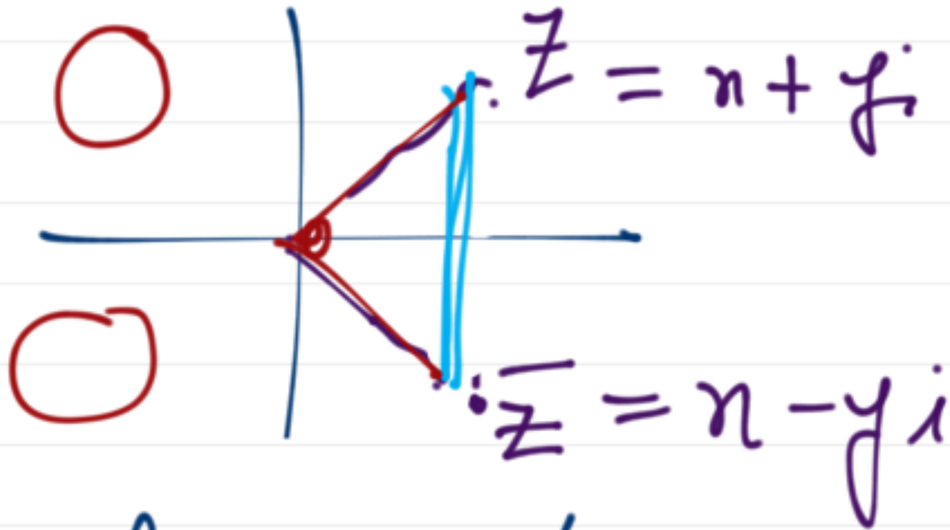
Ex:

$I(z) = 1/\bar{z}$

$I_c$

$\rightarrow z \longmapsto \bar{z}$ reflexão no eixo real

$\infty \qquad \infty$




$\rightarrow$ transforma retas e circunf. de $\mathbb{C}$ em retas e circunf. de $\mathbb{C}$

$\rightarrow \infty \longmapsto \infty$

$\Longrightarrow$ transf. circunf. de $\bar{\mathbb{C}}$ em circunf. de $\bar{\mathbb{C}}$

· t.b. preserva $\measuredangle$s

Teorema:

$f \in \text{Möb} \Rightarrow f$ transforma circ. de $\overline{\mathbb{C}}$ em circ. de $\overline{\mathbb{C}}$ e preserva $\measuredangle$s.

Pts fixos?

**Teorema:** Se $f \in \text{Möb} \setminus \text{Möb}^+$ fixa 3 pts distintos então $f$ fixa a circ. de $\overline{\mathbb{C}}$ que passa nos 3 pts

$\cdot\, z_2$

$z_1 \cdot$

$\cdot\, z_3$

**Porquê?:**

- 3 pts não colineares $z_1, z_2, z_3$

$C$ – circ. de $\overline{\mathbb{C}}$ que passa em $z_1, z_2, z_3$

$I_C$ – inversão em $C$
$\in \text{Möb} \setminus \text{Möb}^+$

$$(I_C \circ f)(z_i) = z_i \qquad i = 1,2,3$$

$$I_C \circ f \in \text{Möb}^+$$

$$\Rightarrow I_C \circ f = id \Rightarrow f = I_C$$

$$(I_C \circ I_C = id)$$

$\Rightarrow f$ fixa todos os pts de $C$ //

- 3 pts colineares (ou um deles $\infty$)

$C = r \cup \{\infty\}$ passa por $z_1, z_2, z_3$ $r$ $\infty$

$$R_r = I_C \in \text{Möb} \setminus \text{Möb}^+ \quad \ldots$$

↪ circ. de $\overline{\mathbb{C}}$ q passa por $\infty$

**Teorema:**

$z_1, z_2, z_3$ - 3 pts distintos do $\overline{\mathbb{C}}$

$w_1, w_2, w_3$ - " " " " $\overline{\mathbb{C}}$

$\Rightarrow \exists!\, f \in \text{Möb} \setminus \text{Möb}^+$ t.q.

$f(z_i) = w_i$

**Unicidade:** $f, g \in \text{Möb} \setminus \text{Möb}^+$

$f(z_i) = g(z_i) = w_i$

$(g^{-1} \circ f)(z_i) = z_i$ e $g^{-1} \circ f \in \text{Möb}^+$

Existência:

$$\begin{tikzcd}
 & \mathbb{C} \\
 & \bar{z}_1 \arrow[u, phantom, "\cup" description] & \\
z_1 \arrow[r] & \bar{z}_1 \arrow[r, "h \in \text{Möb}^+"] & w_1 \\
z_2 \arrow[r] & \bar{z}_2 \arrow[r] & w_2 \\
z_3 \arrow[r, "g"'] \arrow[rr, bend right, "h \circ g \in \text{Möb} \setminus \text{Möb}^+"'] & \bar{z}_3 \arrow[r] & w_3
\end{tikzcd}$$

$g(z) = \bar{z}$

$g \in \text{Möb} \setminus \text{Möb}^+$

Corolário

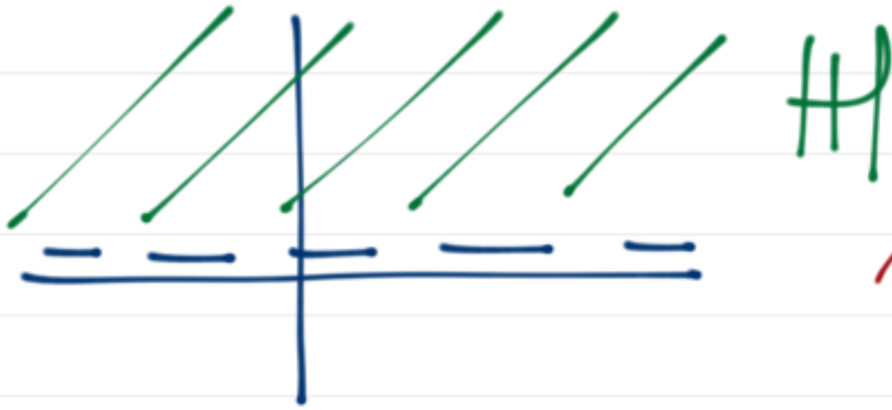
Dadas 2 circunf. $C_1, C_2$ de $\overline{\mathbb{C}}$

existe $\underline{\underline{f}} \in \text{Möb} \setminus \text{Möb}^+$ t.q.

$f(C_1) = C_2$

# Geometria Hiperbólica

$$\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$$




$$G_{\mathbb{H}} = \{ f \in \text{Möb} : f(\mathbb{H}) = \mathbb{H} \}$$

?

Teorema: $\left[ f \in \text{Möb} \text{ t.q. } f(\mathbb{H}) = \mathbb{H} \right.$

$\Longrightarrow$

- $f$ é uma transf. de Möbius

$$f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$$

da forma

$$f(z) = \frac{az+b}{cz+d}$$

$a, b, c, d \in \mathbb{R}$, $ad - bc > 0$

- $f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$:

$f(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc < 0$

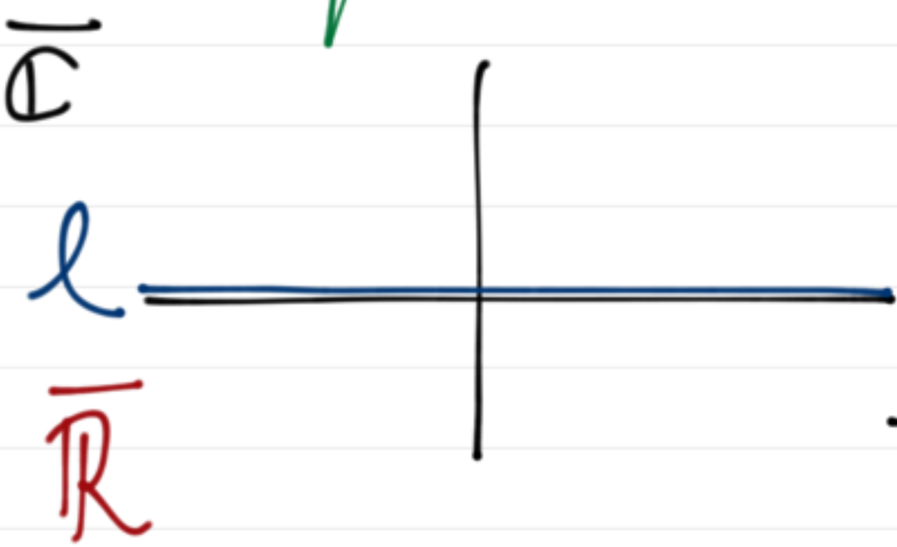
Porquê?

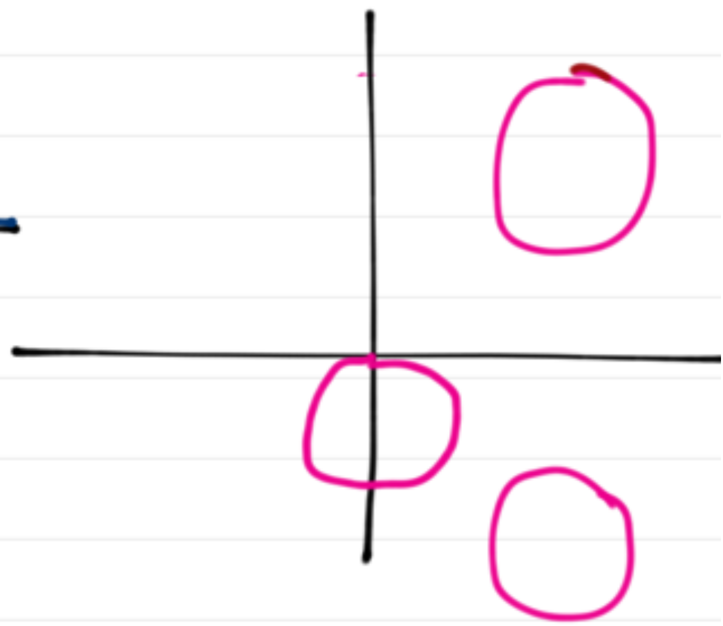
$$f(\mathbb{H})=\mathbb{H} \Rightarrow f(\overline{\mathbb{R}})=\overline{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$$

$$\cap$$

$$\overline{\mathbb{C}}$$

(continuidade ou usando o fato que $f$ preserva circunferências de $\overline{\mathbb{C}}$ e r.s)



- $f(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$

- $f \in \text{Möb}$

$f(z) = \dfrac{az+b}{cz+d}$ ou $f(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$

$ad-bc \neq 0$

i) $c=0$

$ad \neq 0$

$\boxed{f(\infty) = \infty}$ $(\Rightarrow f(0) \neq \infty)$

- $\boxed{f(z) = \dfrac{a}{d} z + \dfrac{b}{d}}$ ou $\boxed{f(z) = \dfrac{a}{d} \bar{z} + \dfrac{b}{d}}$

$f(0) = \dfrac{b}{d} \in \mathbb{R} \Rightarrow b = md$

$m \in \mathbb{R}$

$f(1) = \dfrac{a+b}{d} \in \mathbb{R} \Rightarrow a + md = kd$

$a = (k-m)d$

$$b = md \quad , \quad m \in \mathbb{R}$$

$$a = \tilde{k} d \, , \quad \tilde{k} \in \mathbb{R}$$

$$f(z) = \frac{a}{d} z + \frac{b}{d} = \tilde{k} z + m \qquad \Big| \qquad f(z) = \frac{a}{d} \bar{z} + \frac{b}{d} = \tilde{k} \bar{z} + m$$

$$\tilde{k}, m \in \mathbb{R}$$

ii) $c \neq 0 \quad \wedge \quad \boxed{d \neq 0}$

$$f(0) = \frac{b}{d} \in \mathbb{R} \quad (d \neq 0)$$

$$\Rightarrow \quad b = md \quad m \in \mathbb{R}$$

$$f(x) = \frac{a}{c} \in \mathbb{R} \quad (c \neq 0)$$

$$a = kc, \quad k \in \mathbb{R}$$

$$\left.\begin{array}{l} f(1) = \dfrac{a+b}{c+d} \in \overline{\mathbb{R}} \\[2em] f(-1) = \dfrac{-a+b}{-c+d} \in \overline{\mathbb{R}} \end{array}\right\} \text{um destes está em } \mathbb{R}$$

$\text{fa}\ e\alpha,\ \text{fe}\quad f(1)\in\mathbb{R}$

$\underbrace{a}_{kc}+\underbrace{b}_{md} = t(c+d) \quad t\in\mathbb{R}$

$kc + md = tc + td$

$(t-k)c = (m-t)d$

$$\left.\begin{array}{l} \cdot\ d\neq 0 \\ \cdot\ ad-bc\neq 0 \end{array}\right\} \Rightarrow t\neq k$$

o.e. $m = t = k$

$$\Rightarrow ad-bc = \underbrace{kc}_{a}d - \underbrace{kd}_{b}c = 0$$

$$\Rightarrow c = \frac{m-t}{t-k}\,d = \lambda d \quad \lambda\in\mathbb{R}$$

$$\Rightarrow a = \frac{k(m-t)}{t-k}\,d\ ,\ b = md$$

- Similar para $f(-1)$

iii) $\underline{c \neq 0}$ , $\boxed{d=0}$

$$f(z) = \frac{az+b}{cz+0} = \frac{a}{c} + \frac{b}{cz}$$

ou

$$f(z) = \frac{a}{c} + \frac{b}{cz}$$

- $f(0) = \infty$

$f(\infty) = \frac{a}{c} \in \mathbb{R} \Rightarrow \underline{\underline{a}} = mc$
$m \in \mathbb{R}$

$f(1) = \frac{a}{c} + \frac{b}{c} \in \mathbb{R} \Rightarrow \frac{b}{c} \in \mathbb{R}$

$\underline{\underline{b}} = kc \quad k \in \mathbb{R}$

... ✓

temos então

(1) $$f(z) = \frac{az+b}{cz+d}$$ ou (2) $$f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$$

com $a, b, c, d \in \mathbb{R}$

$$ad - bc \neq 0$$

$$f(\mathbb{H}) = \mathbb{H} \Longrightarrow f(i) \in \mathbb{H}$$

por ex:

(1) $$f(i) = \frac{b+ai}{d+ci} = \frac{(b+ai)(d-ci)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{ad-bc}{c^2+d^2}\, i$$

$> 0$

$\Rightarrow ad - bc > 0$