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Gauge Invariance versus Masslessness in de Sitter Spaces*

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The connection between gauge invariance, masslessness and null cone propagation is a flat space property which does not persist even in constant curvature geometries. In particular, we show that both the gauge invariant spin 3/2 and 2 fields in anti-de Sitter space have support inside the cone, whereas there are conformally invariant, but gauge variant, models which do propagate on the light cone. The Maxwell field in constant curvature spaces of dimension other than four also does not have null cone propagation; again there is a conformally invariant model which does.

1. INTRODUCTION

"Gauge invariance makes the photon massless" is a correct statement in four-dimensional Minkowski space because gauge invariance prohibits an explicit mass term in the vector field's action, and because the field equations imply propagation of gauge invariant quantities (such as the transverse vector potential or the field strengths) on null cones. In flat space in other dimensions, the statement is incorrect: in $D = 3$ there exists a gauge invariant term which gives the photon explicit mass and off-cone propagation support [1]. As we shall see, it is also incorrect in conformally flat spaces with $D \neq 4$, where the Maxwell field propagates off-cone. Indeed, in general curved spaces, the connection between gauge invariance and masslessness becomes obscured; scattering off the background geometry makes propagation complicated even locally, nor is there any immediate analog of mass as a Casimir operator there. However, spaces of constant curvature are sufficiently "like" Minkowski space and, because they are conformal to it, have equivalent null cones. In addition to serving as useful laboratories for studying the effect of curvature on propagation, spaces of constant curvature are of interest in their own right, both for cosmological reasons [2] and because they appear naturally in certain Kaluza–Klein and supergravity theories (see, e.g., [3]). This is the case especially for anti-de Sitter space (AdS), which we will be studying primarily here, although many of our results carry over to de Sitter spaces as well.

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We shall see that whereas (for $D=4$) fields of spin $s \leqslant 1$ do propagate on the AdS light cone because their actions are conformal- (Weyl-) invariant, this is no longer the case for $s > 1$. In particular, the dynamical fields of linearized supergravity ($s=\frac{3}{2}, 2$), while gauge invariant, do not propagate only on the null cone. On the other hand, we find two classes of gauge variant models which do have null cone propagation. The first class consists of Weyl invariant theories, which in flat space reduce to known [4, 5] global $O(4,2)$ (conformal) invariant models. The AdS models in the second class can also be mapped to the corresponding flat space conformal invariant theories; however, these mappings become singular as the radius of curvature of the space tends to infinity. Similar results hold for spin 1 in AdS for $D \neq 4$. We shall also link both the gauge invariant and gauge variant $s=\frac{3}{2}$, 2 models with the familiar higher-derivative[1] Weyl invariant theories.

The paper is organized as follows: In Section 2 we review the notions of massive and massless field propagation in four-dimensional flat space, and elaborate on the problems encountered when going over to a space of constant curvature. In Section 3 we point out that restricted Weyl invariance is the essential factor needed for null cone field propagation in a constant curvature space, and we demonstrate how this applies to fields of spin $s \leqslant 1$. We also study the propagation properties of $s=1$ in AdS in dimensions different from four. Our principal results are contained in Section 4, where we treat the $\frac{3}{2}$, 2 cases. We close in Section 5 with a brief summary and discussion of these results. There are three Appendices: the first lists our conventions and explains the properties of AdS used in the text; the second establishes the connection between restricted Weyl invariance and null cone support of Green functions in constant curvature spaces; and the last uses the "projection technique" to analyze the propagation properties of certain models discussed in text.

2. Review of Massive/Massless Propagation

Consider first the case of fields propagating in four-dimensional Minkowski space. A classical free scalar field $\phi(x)$ obeying the Klein–Gordon equation $(\square_0 - m^2)\phi = 0$, where $\square_0 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$, is propagated by the symmetric Green function

$$G_0(x,x') \equiv \tfrac{1}{2}[G_{\text{adv}}(x,x') + G_{\text{ret}}(x,x')],$$

which for $\sigma_0 \equiv (x-x')^2 \sim 0$ is [6]

$$G_0(x,x') = \frac{1}{4\pi}\delta(\sigma_0) - \frac{m^2}{8\pi}\theta(-\sigma_0)\left[\frac{1}{2} + \frac{m^2\sigma_0}{2^2\cdot 4} + \frac{m^4\sigma_0^2}{2^2\cdot 4^2\cdot 6} + \cdots\right]. \tag{2.1}$$

Note that G_0 has a "wavefront" δ-function term (whose presence is guaranteed by the

[1] We exclude throughout the Weyl invariance achieved by introducing an explicit compensating scalar field.

theory of characteristics of this hyperbolic problem) as well as a "tail" part with support inside the cone. This tail characterizes massive propagation both here and for higher-spin fields. Indeed, for $m=0$ the tail is absent, and the symmetric Green function manifestly has only cone support.[2] We shall see that this degeneracy between masslessness and null cone propagation will be lifted when we later come to curved geometries.

The massive vs massless question can also be studied from a group-theoretical point of view. Roughly speaking, free (quantum) fields in $D=4$ Minkowski space carry a finite-dimensional representation of the Poincaré group; and to these fields there correspond Fourier modes (particle states) constituting infinite-dimensional irreducible unitary representations. We denote these representations by $\mathscr{P}(E_0, s)$, where E_0 and s ($=0, \frac{1}{2}, 1 \cdots$) are the smallest eigenvalues of the Poincaré generators P_0 and J_3, respectively. The quadratic Casimir operator $P_\mu P^\mu$ and E_0 are related:

$$P^2 = -E_0^2 \leqslant 0. \tag{2.2}$$

It is not hard to see for $s=0$, say, that states with $P^2=-m^2$ are the Fourier modes of a field ϕ obeying $(\Box_0 - m^2)\phi = 0$, thereby justifying the identification of the parameter m appearing in the wave equation as a "mass." (One can also check that P_0 coincides with the Hamiltonian for the field ϕ.) States with $P^2=0$ (with $s>0$) can have only two degrees of freedom $\pm s$, thereby implying [7] that for $s \geqslant 1$ these states must emerge from a gauge field—the gauge invariance is needed to reduce to two the number of physical degrees of freedom described by the covariant field. Hence, group theory alone demands that gauge fields in $D=4$ Minkowski space be massless, and thus that their excitations propagate on the null cone.

In a general curved space background, the situation is rather different. A scalar field propagating according to $\nabla^\mu \nabla_\mu \Phi \equiv \Box \Phi = 0$ (no mass term) scatters from the background, thereby propagating both on and inside the local null cones [6]. In this sense, the field appears to be "massive"; however, one must remember that in general space (with no global or asymptotic [8] timelike Killing vector) the concept of energy (let alone mass) is not well defined. We restrict our attention to negative constant curvature spaces, i.e., AdS. As we discuss below, in these spaces it may be possible to compensate for the background scattering with suitable "mass" terms, thereby achieving null cone propagation. However, it is not obvious from inspection of a given field equation—in particular, for a gauge field— whether or not it provides null cone propagation. Moreover, this question is not readily settled by appealing to group theory. The group of motions of AdS is the de Sitter group $SO(3,2)$, whose algebra is generated by $J_{AB} = -J_{BA}$ $(A, B = 0, 1, 2, 3, 5)$. The infinite-dimensional irreducible representations of the de Sitter group are designated $\mathscr{D}(E_0, s)$, where E_0 and s are the smallest eigenvalues of J_{05} and J_{12}, respectively. In contrast to the Poincaré case (2.2), the de Sitter Casimir operator satisfies [9]

$$\tfrac{1}{2} J_{AB} J^{AB} = E_0(E_0 - 3) + s(s+1). \tag{2.3}$$

[2] This is not true for other Green functions, such as the Feynman propagator.

The unitary representations are given by $E_0 \geqslant s + \frac{1}{2}$ for $s = 0, \frac{1}{2}$, and by $E_0 \geqslant s + 1$ for $s = 1, \frac{3}{2}, \ldots$; in particular, those representations with $E_0 = s + 1$ $(s \geqslant 1)$ have two degrees of freedom and, like their Poincaré counterparts, correspond to gauge fields. It is important to note that E_0 is *not* a mass, as it is dimensionless. However, one can define the quantities $P_\mu \equiv mJ_{\mu 5}$ in terms of which the de Sitter algebra contracts to Poincaré for $m \to \infty$ ($1/m$ is the radius of curvature of AdS). In particular, this flat space limit defines a mass m_0

$$m_0 \equiv \lim_{m \to \infty} mE_0.$$

Thus, any representation with E_0 finite corresponds to a massless theory in the flat space limit [9]. Clearly, this criterion does not suffice to determine which of the representations $\mathscr{D}(E_0, s)$ is "massless" in AdS.

3. Lower Spin Propagation

Given a field equation in an AdS background, does it provide null propagation? Because the null cone $ds^2 = 0$ is preserved by Weyl transformations and AdS is conformally flat, it is clear that theories which are Weyl invariant (and which have null cone propagation in flat space) will describe null propagation in AdS. For completeness, a proof is provided in Appendix B, where invariance under the special Weyl transformation that maps flat space to AdS is shown to imply this. In this section, we simply remind the reader how $s \leqslant 1$ systems are Weyl invariant. As we shall see, the non-gauge $(s \leqslant \frac{1}{2})$ field theories are quite different from the vector case.

A. *Spin* 0

The "improved" equation for a scalar field $\Phi(x)$ in D dimensions

$$(\Box + \zeta R)\,\Phi = 0, \qquad \zeta \equiv \frac{1}{4}\frac{(D-2)}{(D-1)} \tag{3.1}$$

is covariant under the following simultaneous transformation of the metric and the field

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x)\, g_{\mu\nu}(x), \qquad \tilde{\Phi}(x) = \Omega^w(x)\,\Phi(x), \qquad w = (2-D)/2 \tag{3.2}$$

with the coordinate unchanged. [The value of the Weyl weight w can be deduced by inspection of the kinetic term in the action for Φ.] In AdS with $D = 4$ (for example), $R = 12m^2$, and (3.1) reduces to

$$(\Box + 2m^2)\,\Phi = 0. \tag{3.3}$$

That this equation describes null propagation in AdS is well known (see, e.g., [10]).

Had we not known the correct form (3.1), we could have arrived at (3.3) as follows: Let

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}, \qquad \Omega \equiv \left(1 - \frac{m^2 x^2}{4}\right)^{-1} \tag{3.4}$$

be the AdS metric in a suitable frame. (See Appendix A.) It then follows that[3]

$$\begin{aligned} \Box\Phi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi &= \Omega^{-2}\eta^{\mu\nu}\{\partial_\mu\partial_\nu(\Omega^{-1}\phi) - \Gamma^\alpha_{\nu\mu}\partial_\alpha(\Omega^{-1}\phi)\} \\ &= \Omega^{-3}\Box_0\phi - 2m^2(\Omega^{-1}\phi). \end{aligned} \tag{3.5}$$

We therefore see that $(\Box + 2m^2)\,\Phi = \Omega^{-3}\Box_0\phi$. This elementary remark will be useful in considering higher-spin fields. Finally, we note that invariance of the action $-\frac{1}{2}\int d^Dx\,\sqrt{-g}\;\Phi(\Box + \zeta R)\,\Phi$ under (3.2) is also straightforward to establish, since $\Phi\sqrt{-g}\;\Omega^{-2+w} = \phi$.

B. *Spin* $\frac{1}{2}$

For spin $\frac{1}{2}$, the massless Dirac equation $\nabla\!\!\!/\,\Psi = 0$ is well known to be Weyl invariant with Ψ of Weyl weight $(1-D)/2$, so this equation provides null propagation in AdS [11]. This is also verified by considering the second-order equation (we take $D = 4$ for simplicity)

$$0 = \nabla\!\!\!/^{\,2}\Psi = \gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu\Psi = \{\Box + \sigma^{\mu\nu}[\nabla_\mu, \nabla_\nu]\}\,\Psi = (\Box + R/4)\,\Psi = (\Box + 3m^2)\,\Psi. \tag{3.6}$$

Proceeding as in (3.5) above, we find

$$(\Box + 3m^2)\,\Psi = \Omega^{-7/2}\Box_0\psi = 0. \tag{3.7}$$

On the other hand, the system $\Box\Psi = 0$ (which is the second-order form of $(\nabla\!\!\!/ + \sqrt{3}\,m)\,\Psi = 0$) does not give null propagation, but corresponds to the flat space wave equation $(\Box_0 - 3m^2\Omega^2)\,\psi = 0$ with an x-dependent mass.

Weyl invariance of the action $(-i/2)\int d^Dx\, e\bar{\Psi}\nabla\!\!\!/\,\Psi$ can also be established directly. Indeed, for a Majorana spinor, each term in $\nabla\!\!\!/$ is separately invariant: the action reads

$$\begin{aligned} I_{1/2}(e_\mu{}^a, \Psi) &= (-i/2)\int d^4x\, e\bar{\Psi}e^{\mu c}\gamma_c(\partial_\mu - \tfrac{1}{2}\omega_{\mu ab}\sigma^{ab})\,\Psi \\ &= (-i/2)\int d^4x\, e\bar{\Psi}e^{\mu c}\gamma_c(\partial_\mu - \tfrac{1}{2}\gamma_5{}^*\omega_\mu)\,\Psi, \qquad {}^*\omega_\mu \equiv \tfrac{1}{2}\varepsilon_{\mu a}{}^{bc}\omega^a{}_{bc}. \end{aligned} \tag{3.8}$$

[3] We adopt throughout the convention that upper-case letters (Φ, Ψ, A_μ, $H_{\alpha\beta}\cdots$) denote curved space (AdS) fields, whereas lower-case letters (ϕ, ψ, a_μ, $h_{\alpha\beta}\cdots$) represent the corresponding rescaled, "flat space," fields.

But with Ω given by (3.4), $\omega_{abc} \sim (x_b \eta_{ac} - x_c \eta_{ab})$, so $^*\omega_\mu$ vanishes identically; also the rescaling of Ψ is "transparent" to ∂_μ because $\bar{\psi}\gamma_\mu \psi \equiv 0$. Consequently, $I_{1/2}(e_\mu{}^a, \Psi) = I_{1/2}(\delta_\mu{}^a, \psi)$.

C. *Spin* 1

Whereas spins 0, $\frac{1}{2}$ are Weyl invariant in any dimension, Maxwell theory has this property only for $D = 4$, as is clear from the fact that

$$T_\rho{}^\rho = g_{\mu\nu} \frac{\delta I_{\mathrm{Max}}}{\delta g_{\mu\nu}} = g_{\mu\nu} \left(F^{\mu\rho} F^\nu{}_\rho - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) = (4 - D) \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma}$$

is an expression which cannot be "improved" gauge invariantly. In $D = 4$, there are a number of ways of checking this invariance, the most elementary of which are manifestly gauge invariant. We present them here for later comparison with the $D \neq 4$ situation. At the level of the action, noting that A_μ has Weyl weight zero,

$$\int d^4x \sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} = \int d^4x\, \eta^{\mu\rho} \eta^{\nu\sigma} f_{\mu\nu} f_{\rho\sigma} \tag{3.9}$$

while the field equation reads

$$0 = \Box A_\mu - \nabla^\rho \nabla_\mu A_\rho = (\Box + 3m^2) A_\mu - \nabla_\mu (\nabla^\rho A_\rho) = \Omega^{-2} [\Box_0 a_\mu - \partial_\mu (\partial^\rho a_\rho)] = 0 \tag{3.10}$$

and so clearly represents the usual flat space propagation. Finally, the cyclic identity $\nabla_{[\rho} F_{\mu\nu]} \equiv 0$ may be used to write

$$\Box F_{\mu\nu} \equiv \nabla^\rho \nabla_\rho F_{\mu\nu} = \nabla^\rho (\nabla_\mu F_{\rho\nu} - \nabla_\nu F_{\rho\mu}).$$

Upon commuting covariant derivatives and using the field equation $\nabla^\mu F_{\mu\nu} = 0$ we then obtain

$$\begin{aligned} 0 = -(\Delta F)_{\mu\nu} &\equiv \Box F_{\mu\nu} + R_\mu{}^\rho F_{\rho\nu} + R_\nu{}^\rho F_{\mu\rho} - 2 R_\mu{}^\rho{}_\nu{}^\sigma F_{\rho\sigma} \\ &= (\Box + 4m^2) F_{\mu\nu} = \Omega^{-2} \Box_0 F_{\mu\nu}. \end{aligned} \tag{3.11}$$

Again, a "mass" term appears in the AdS wave equation in order to maintain null cone propagation. The operator Δ here is just the Laplacian (or "de Rham") operator [12] which is defined on an antisymmetric tensor $T_{\alpha_1 \cdots \alpha_p}$ according to

$$\begin{aligned} -(\Delta T)_{\alpha_1 \cdots \alpha_p} \equiv \Box T_{\alpha_1 \cdots \alpha_p} &+ \sum_k R_{\alpha_k}{}^\rho T_{\alpha_1 \cdots \rho \cdots \alpha_p} \\ &- \sum_{k \neq l} R_{\alpha_k}{}^\rho{}_{\alpha_l}{}^\sigma T_{\alpha_1 \cdots \rho \cdots \sigma \cdots \alpha_p} \end{aligned} \tag{3.12}$$

where the indices ρ, σ in the last term appear in the k, l positions, respectively. This operator is self-adjoint and commutes with contractions; for constant Ricci tensor it also commutes with covariant differentiations on vectors, and with divergences on the 2-tensors.

If one wishes to fix gauges *ab initio*, care must be taken because in general gauge-fixing is not covariant under Weyl rescaling; e.g., under (3.4)

$$\nabla^\mu A_\mu = \Omega^{-2}[\partial^\mu a_\mu + m^2 \Omega (x^\mu a_\mu)]. \tag{3.13}$$

There also exist Weyl-covariant gauge choices such as $x^\mu A_\mu = 0$ [13].[4] In either case, the complete set of field equations plus gauge condition transforms in a well-defined way: i.e., according to (3.10) and, e.g., $x \cdot A = 0$ or (3.13). To show that a given set describes null cone propagation becomes a pure flat space question, since one could have insisted on such "bizarre" gauge conditions as (3.13) even there. For example, in $x \cdot a = 0$ gauge, the equations read

$$\begin{aligned} &\Box a_i^T = 0, \qquad && (x \cdot \partial)\,\lambda = -x^i a_i^T, \qquad && a_0 = \dot{\lambda}, \\ &a_i \equiv a_i^T + \partial_i \lambda, && \partial^i a_i{}^T \equiv 0. && \end{aligned} \tag{3.14}$$

The gauge invariant transverse fields propagate normally, but the gauge set (λ, a_0) is here specified in terms of the dynamical variables. In Lorentz gauge (3.13), the reduction would be obscured.

As noted above, for $D \neq 4$, Maxwell theory is not in general Weyl invariant. However, in order to study AdS propagation, one need consider only the special Weyl transformation (3.4). Thus, assigning to A_μ the Weyl weight $(4-D)/2$, the AdS Maxwell equation

$$[\Box + (D-1)\, m^2]\, A_\mu - \nabla_\mu (\nabla \cdot A) = 0 \tag{3.15}$$

in $x \cdot A = 0$ gauge becomes

$$\Box_0 a_\mu - \partial_\mu (\partial \cdot a) + \tfrac{1}{4}(D-4)\, m^2 \Omega x_\mu (\partial \cdot a) - \tfrac{1}{4}(D-2)(D-4)\, m^2 \Omega^2 a_\mu = 0. \tag{3.16}$$

The requirement that $a_\mu(x)$ be less singular than $1/x$ implies that on shell $\partial \cdot a = 0$ (see footnote 4), and (3.16) reduces to

$$\Box_0 a_\mu - \tfrac{1}{4}(D-2)(D-4)\, m^2 \Omega^2 a_\mu = 0. \tag{3.17}$$

The vector field has an x-dependent "mass" (except for $D = 2, 4$),[5] indicating that it

[4] A field configuration $a_\mu(x)$ can be brought to this gauge by the transformation $a_\mu(x) \to a'_\mu(x) \equiv a_\mu(x) + \partial_\mu w(x)$, with $w(x) = -\int_P a_\mu(y)\, dy^\mu$, where P is the straight-line path from the origin to x. This transformation is well defined only for potentials $a_\mu(y)$ which are less singular that $1/y$ along P; hence, only for such potentials can this gauge be attained. [The excluded configurations have infinite actions for $D \leqslant 4$.] Finally, we remark that in this gauge $0 = \Box(x \cdot a) = 2(\partial \cdot a) + x^\mu \Box a_\mu$, so that on-shell $(x \cdot \partial)(\partial \cdot a) = -2(\partial \cdot a)$. That is, $\partial \cdot a$ is homogeneous of degree -2 in x; however, the requirement that this gauge be well defined then implies $\partial \cdot a = 0$.

[5] As in flat space, the $D = 2$ Maxwell equations $\partial_\mu(\sqrt{-g}\, F^{\mu\nu}) = 0$ have no dynamics, as they imply $\sqrt{-g}\, F^{01}$ is constant in both space and time.

does not propagate on the null cone; we have not found any further field redefinition to improve this situation. A similar conclusion can be reached by examining the gauge invariant equation of motion for $F_{\mu\nu}$:

$$\begin{aligned} 0 = -(\Delta F)_{\mu\nu} &= [\Box + 2(D-2)\, m^2]\, F_{\mu\nu} \\ &= \Omega^{-2}\{\Box_0 F_{\mu\nu} + (D-4)\, m^2 \Omega F_{\mu\nu} + \tfrac{1}{2}(D-4)\, m^2 \Omega (x \cdot \partial)\, F_{\mu\nu} \\ &\quad + \tfrac{1}{4}(D-4)\, m^4 \Omega^2 [x_\mu x^\alpha F_{\alpha\nu} - x_\nu x^\alpha F_{\alpha\mu}]\}. \end{aligned} \tag{3.18}$$

In contrast to (3.11), here it is no longer true in general that $\Box_0 F_{\mu\nu} = 0$. Of course, the flat space field strength $f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu$ is not invariant under the original AdS gauge transformation; it, too, satisfies a complicated equation of motion which we shall not record here.

Even though Maxwell theory does not provide null cone propagation in AdS for $D \neq 4$, there is a conformally invariant, but gauge variant, model whose modes do propagate on the null cone. Indeed, consider the action (in an arbitrary space-time background)

$$\begin{aligned} I = \frac{1}{2}\int d^D x \sqrt{-g}\, \Big\{ &\nabla^\rho A^\mu \nabla_\rho A_\mu - \frac{4}{D} \nabla^\rho A^\mu \nabla_\mu A_\rho \\ &+ \frac{2(D-4)}{D(D-2)} \left[R^{\mu\rho} A_\mu A_\rho - \frac{D^2}{8(D-1)} R A^\mu A_\mu \right] \Big\} \end{aligned} \tag{3.19}$$

with the corresponding field equation

$$\Box A_\mu - \frac{4}{D} \nabla^\rho \nabla_\mu A_\rho - \frac{2(D-4)}{D(D-2)} \left[R_\mu{}^\rho A_\rho - \frac{D^2}{8(D-1)} R A_\mu \right] = 0. \tag{3.20}$$

For $D = 4$ this reduces to the Maxwell theory, and for $D = 2$ (where $R^\rho_\mu \equiv \frac{1}{2}\delta^\rho_\mu R$) the "mass" term vanishes identically. Taking the divergence of (3.20) yields (for $D \neq 4$) a constraint, which on Einstein shell ($R_{\mu\nu} = \lambda g_{\mu\nu}$) reads $(\Box + \zeta R)(\nabla \cdot A) = 0$; i.e., $\nabla \cdot A$ obeys the conformal scalar equation (3.1). This consistency condition is milder than an "algebraic" constraint, such as $R^{\alpha\beta} \nabla_\alpha A_\beta = 0$.

One can check that this model is invariant under general Weyl transformations; it follows [14] that both the AdS equation

$$\Box A_\mu + \tfrac{1}{4}(D^2 - 2D + 4)\, m^2 A_\mu - (4/D)\, \nabla_\mu (\nabla \cdot A) = 0 \tag{3.21}$$

as well as the flat space equation

$$\Box_0 a_\mu - (4/D)\, \partial_\mu (\partial \cdot a) = 0 \tag{3.22}$$

are covariant under global $O(D, 2)$ (conformal) transformations.[6] Let us consider the

[6] AdS has $D(D+1)/2$ Killing vectors and $D(D+1)/2$ conformal Killing vectors; of these, $(D+2)(D+1)/2$ are linearly independent, and satisfy the algebra $O(D, 2)$.

flat space equation (3.22): for $D \neq 4$, the divergence yields the condition $\Box_0(\partial \cdot a) = 0$. Moreover, we find $\Box_0 a_i^T = 0$, implying that the $(D-2)$ transverse degrees of freedom a_i^T propagate along the null cone; also $\Box_0{}^2 a_0 = 0 = \Box_0{}^2 a_L$, and $\Box_0(\partial_i \partial^i a_L - \dot{a}_0) = 0 = \Box_0(\dot{a}_L - a_0)$, where $a_i \equiv a_i^T + a_i^L$, $a_i^L \equiv \partial_i a_L$. We conclude that the corresponding AdS theory (3.21) also describes null cone propagation. (Further arguments are presented in Appendix C.)

Actually, (3.21) is not the only AdS model which has modes propagating on the cone. Consider the related, but inequivalent, Proca system

$$\nabla^\rho F_{\rho\mu} + \tfrac{1}{4}(D-2)(D-4)\, m^2 A_\mu = [\Box + \tfrac{1}{4}(D^2 - 2D + 4)\, m^2]\, A_\mu - \nabla_\mu(\nabla \cdot A) = 0. \tag{3.23}$$

Taking the divergence yields (for $D \neq 2, 4$) the condition $\nabla \cdot A = 0$ which, when Weyl-rescaled to flat space, reads

$$\partial \cdot a + m^2(D/4)\, \Omega(x \cdot a) = 0. \tag{3.24}$$

Performing the Weyl rescaling on the field equation (3.23) gives

$$\Box_0 a_\mu - m^2 \Omega x_\mu(\partial \cdot a) + m^2 \Omega \partial_\mu(x \cdot a) + \tfrac{1}{4}(2-D)\, m^4 \Omega^2 x_\mu(x \cdot a) = 0. \tag{3.25}$$

Finally, using (3.24) to eliminate the $m^2(x \cdot a)$ terms leads once again to the conformally covariant equation

$$\Box_0 a_\mu - (4/D)\, \partial_\mu(\partial \cdot a) = 0 \tag{3.26}$$

indicating that the AdS model (3.23) has "massless" modes. However, since we have used the "on-shell" condition (3.24), we cannot directly argue (as we have done in Appendix B for Weyl invariant theories) that the Green function for this model has only null cone support. We also note that the transformation mapping (3.23) to (3.26) is singular for $m^2 = 0$, since in this case the condition (3.24) becomes simply $\partial \cdot a = 0$. That more than one AdS model describes null cone propagation is a feature we shall encounter again when we treat higher-spin fields.

4. Higher-Spin Propagation

As is well known, neither the spin $\frac{3}{2}$ nor linearized spin 2 theories can be made Weyl invariant for any choice of field Weyl weight. However, as noted above, in order to investigate AdS propagation, it suffices to consider the special Weyl transformation (3.4) mapping AdS to flat space.

A. *Spin $\frac{3}{2}$*

For spin $\frac{3}{2}$, the locally gauge invariant free action in an AdS background is given by [15]

$$I_{3/2} = \frac{-i}{2} \int d^4x\, \varepsilon^{\mu\nu\alpha\beta} \bar{\Psi}_\mu \gamma_5 \gamma_\nu \mathscr{D}_\alpha \Psi_\beta, \qquad \mathscr{D}_\alpha \equiv \nabla_\alpha + \tfrac{1}{2} m \gamma_\alpha. \tag{4.1}$$

The corresponding field equations read

$$\mathscr{R}_\mu(\Psi) \equiv \not{\mathscr{D}} \Psi_\mu - \mathscr{D}_\mu(\gamma \cdot \Psi) + \gamma_\mu[\not{\mathscr{D}}(\gamma \cdot \Psi) - (\mathscr{D} \cdot \Psi)] = 0. \tag{4.2}$$

In terms of the field strength $\Psi_{\mu\nu} \equiv \mathscr{D}_\mu \Psi_\nu - \mathscr{D}_\nu \Psi_\mu$, these have the equivalent forms

$$0 = \gamma^\mu {}^*\Psi_{\mu\nu} = \gamma^\mu \Psi_{\mu\nu}, \qquad {}^*\Psi_{\mu\nu} \equiv \tfrac{1}{2} \varepsilon_{\mu\nu}{}^{\alpha\beta} \Psi_{\alpha\beta}. \tag{4.3}$$

The action (as well as $\Psi_{\mu\nu}$) is invariant under $\delta\Psi_\mu(x) = \mathscr{D}_\mu \alpha(x)$ because $[\mathscr{D}_\mu, \mathscr{D}_\nu]\, \alpha(x) \equiv 0$ in the "matched" AdS background. Being gauge invariant, it has just two degrees of freedom. If we look at the field equation in the standard gauge $\gamma \cdot \Psi = 0$ (which is Weyl covariant), we then have $\nabla \cdot \Psi = 0$, and also

$$(\not{\nabla} + m)\, \Psi_\mu = 0 \tag{4.4a}$$

implying

$$(\Box + 3m^2)\, \Psi_\mu = 0. \tag{4.4b}$$

When Ψ_μ is assigned the Weyl weight $-\frac{1}{2}$, then $\gamma^\mu \Psi_{\mu\nu} = 0$ transforms under (3.4) to

$$\not{\partial} \psi_\beta - \partial_\beta(\gamma \cdot \psi) + m\Omega\psi_\beta + \frac{m}{2}\Omega \left[1 + \frac{m}{2}(x \cdot \gamma) \right] \gamma_\beta(\gamma \cdot \psi) + \frac{m^2}{2} \Omega \gamma_\beta (x \cdot \psi) = 0 \tag{4.5}$$

which has an x-dependent "mass" term. (Clearly, setting a gauge, such as $\gamma \cdot \psi = 0$ or $x \cdot \psi = 0$, cannot make this term vanish.) Moreover, unlike the action of the massless flat space theory, (4.1) is not invariant under the chiral transformation $\delta\Psi_\mu = \gamma_5 \Psi_\mu$ [16]. That the gauge invariant spin $\frac{3}{2}$ theory does not have null cone propagation can also be seen by considering the equation obeyed by the gauge invariant field strength $\Psi_{\mu\nu}$. By virtue of (4.3), as well as $\mathscr{D}_\mu \Psi^{\mu\nu} = 0 = \mathscr{D}_\mu {}^*\Psi^{\mu\nu}$, we find that $\not{\nabla} \Psi_{\mu\nu} = 0$ and hence

$$(\Box + 5m^2)\, \Psi_{\mu\nu} = 0. \tag{4.6}$$

On performing the Weyl rescaling to flat space, this becomes

$$\square_0 \Psi_{\mu\nu} + m^2\Omega^2 \left[1 + \frac{m}{2}(x \cdot \gamma) - \frac{5}{16} m^2 x^2\right] \Psi_{\mu\nu} + \frac{1}{2} m^2 \Omega (x \cdot \partial) \Psi_{\mu\nu}$$
$$+ \frac{m^3}{4} \Omega^2 [-2\gamma_\mu + m(x \cdot \gamma)\gamma_\mu](x^\alpha \Psi_{\alpha\nu}) - \frac{m^3}{4} \Omega^2 [-2\gamma_\nu + m(x \cdot \gamma)\gamma_\nu](x^\alpha \Psi_{\alpha\mu}) = 0 \tag{4.7}$$

which contains an x-dependent mass term, as well as $x^\alpha \Psi_{\alpha\beta}$ and $(x \cdot \partial) \Psi_{\alpha\beta}$ pieces. Again, rescaling Ψ_μ in order to obtain an equation for the (gauge variant) field strength $\psi_{\mu\nu} \equiv \partial_\mu \psi_\nu - \partial_\nu \psi_\mu$ is of no help: the latter also obeys a complicated "massive" equation.

Having found that the gauge invariant theory does not propagate as expected, we consider alternative spin $\frac{3}{2}$ models. First, it can be shown that any AdS equation with arbitrary nonvanishing mass term will, like (4.2), necessarily yield off-cone propagation. Clearly, there are a number of candidate systems without an explicit mass term; being gauge variant, these also involve additional lower-spin excitations, presumably of a ghost nature. This question aside, we find two models which have especially interesting properties. The first model is invariant under conformal transformations in AdS; the second is an AdS model which also has null cone propagation, and which is consistent in that the divergence of the field equation does not lead to constraints.[7] We now describe these in turn.

Consider the action (in an arbitrary space-time background)

$$I_{3/2} = \frac{-i}{2} \int d^4x\, e \bar{\tilde{\Psi}}^\mu \nabla\!\!\!/ \tilde{\Psi}_\mu, \qquad \tilde{\Psi}_\mu \equiv \Psi_\mu - \frac{1}{4} \gamma_\mu (\gamma \cdot \Psi) \tag{4.8}$$

which is the only available expression involving the γ-traceless $\tilde{\Psi}_\mu$ alone. The field equations read

$$0 = \tilde{R}_\mu(\tilde{\Psi}) = \nabla\!\!\!/ \tilde{\Psi}_\mu - \tfrac{1}{2}\gamma_\mu(\nabla \cdot \tilde{\Psi}) = \nabla\!\!\!/ \Psi_\mu - \tfrac{1}{2}\gamma_\mu(\nabla \cdot \Psi) - \tfrac{1}{2}\nabla_\mu(\gamma \cdot \Psi) + \tfrac{3}{8}\gamma_\mu \nabla\!\!\!/ (\gamma \cdot \Psi),$$
$$\tilde{R}_\mu \equiv R_\mu - \tfrac{1}{4}\gamma_\mu(\gamma \cdot R) \tag{4.9}$$

where $R^\mu(\Psi) \equiv e^{-1}\varepsilon^{\mu\nu\alpha\beta}\gamma_5\gamma_\nu\nabla_\alpha\Psi_\beta$ is the usual Rarita–Schwinger operator without mass term. One can check that this system is Weyl invariant and therefore [14] is also conformally invariant in AdS, as well as in flat space.[8] Indeed, that the action depends only on $\tilde{\Psi}_\mu$ is tantamount to invariance under $\delta\Psi_\mu(x) = \gamma_\mu \alpha(x)$, which also corresponds to a Weyl transformation on Ψ_μ. The field equation is also manifestly γ-traceless, and hence yields no information on $\nabla \cdot \tilde{\Psi}$ itself; there is, however, a consistency constraint: the divergence of (4.9) requires that $\nabla\!\!\!/ (\nabla \cdot \tilde{\Psi}) = 0$. That is, $\nabla \cdot \tilde{\Psi}$ must obey the Dirac equation.

[7] We have not addressed the Velo–Zwanziger consistency problems; however, it is likely [15] that in fact these problems do not arise in the models discussed here.

[8] We note a discrepancy with [5] where flat space conformal covariance of higher-spin equations is discussed: the coefficient of the $\gamma_\mu \partial\!\!\!/ (\gamma \cdot \psi)$ term there is 1 instead of $\frac{3}{8}$.

For comparison with the gauge invariant theory (4.4), we observe that in AdS the field equations (4.9) imply

$$(\Box + 4m^2)\,\tilde{\Psi}_\mu - \nabla_\mu(\nabla \cdot \tilde{\Psi}) = 0. \tag{4.10}$$

Moreover, performing the Weyl transformation to flat space gives

$$\Box_0 \tilde{\psi}_\mu - \partial_\mu(\partial \cdot \tilde{\psi}) = 0 \tag{4.11}$$

from which it is clear that the transverse components $\tilde{\psi}_i^T$ propagate on the null cone. A further indication that the gauge variant theory has modes with null cone support is that the action and field equations are invariant under chiral ($\delta\Psi_\mu = \gamma_5 \Psi_\mu$) transformations as in the massless flat space theory. Yet another argument is presented in Appendix C.

Let us now consider a second gauge variant model, having the field equation

$$0 = R_\mu(\tilde{\Psi}) = \nabla\!\!\!/\, \tilde{\Psi}_\mu - \gamma_\mu(\nabla \cdot \tilde{\Psi}) = \nabla\!\!\!/\, \Psi_\mu - \gamma_\mu(\nabla \cdot \Psi) - \tfrac{1}{2}\nabla_\mu(\gamma \cdot \Psi) + \tfrac{1}{2}\gamma_\mu \nabla\!\!\!/\,(\gamma \cdot \Psi). \tag{4.12}$$

In AdS, this model is consistent,[9] i.e., $\nabla_\mu R^\mu(\tilde{\Psi}) \equiv 0$. Taking the γ-trace yields

$$\nabla \cdot \tilde{\Psi} = 0 \tag{4.13}$$

so the field equation is simply

$$\nabla\!\!\!/\, \tilde{\Psi}_\mu = 0 \tag{4.14a}$$

which includes (4.13) and implies

$$(\Box + 4m^2)\,\tilde{\Psi}_\mu = 0. \tag{4.14b}$$

We have been unable to find a general coordinate invariant action for this model, even using auxiliary spinor fields, although it is closely related to the system $R_\mu(\Psi) = 0$ (which does have an action): In AdS, the latter is not consistent, as $\nabla_\mu R^\mu(\Psi) = 0$ implies the constraint $\gamma \cdot \Psi = 0$; however, it is clear that solutions to $R_\mu(\Psi) = 0$ satisfying this constraint in fact also satisfy $R_\mu(\tilde{\Psi}) = 0$.

In order to investigate the AdS propagation properties of (4.12) we perform the Weyl transformation to flat space, and find

$$\partial\!\!\!/\, \tilde{\psi}_\mu - \gamma_\mu(\partial \cdot \tilde{\psi}) - (m^2/2)\Omega\gamma_\mu(x \cdot \tilde{\psi}) = 0 \tag{4.15}$$

[9] It may appear paradoxical that gauge invariance and consistency are not equivalent here, but this can be understood from the fact that the Rarita–Schwinger operator ($\varepsilon^{\mu\nu\alpha\beta}\gamma_5\gamma_\nu\nabla_\alpha$) does not commute with the traceless projection operator ($\delta_\mu^\lambda - \frac{1}{4}\gamma^\lambda\gamma_\mu$). Thus, $\tilde{R}_\mu(\tilde{\Psi})$, $R_\mu(\tilde{\Psi})$, and $\tilde{R}_\mu(\Psi)$ provide distinct wave equations. The first and second correspond to the models we discuss in text; the third, although gauge invariant (but subject to a $\nabla \cdot \tilde{R}(\Psi) = 0$ constraint), will not be considered further here, as it gives off-cone propagation.

while the condition (4.13) gives

$$\partial \cdot \tilde{\psi} + \Omega m^2 (x \cdot \tilde{\psi}) = 0. \tag{4.16}$$

Using this to eliminate the $m^2(x \cdot \tilde{\psi})$ term in the previous equation, we obtain the flat space conformally covariant wave equation $\not\partial \tilde{\psi}_\mu - \frac{1}{2}\gamma_\mu(\partial \cdot \tilde{\psi}) = 0$. Evidently, the AdS model indeed has "massless" modes. This is in complete analogy with the discussion in Section 3 on the second $D \neq 4$ vector model, and the same remarks apply here. In particular, we stress that this transformation to the flat space conformally invariant theory is discontinuous at $m^2 = 0$, as it relies on (4.16).

Finally, let us briefly compare the above results with the higher-derivative spin $\frac{3}{2}$ theory [17, 18], which is invariant under arbitrary Weyl transformations with Ψ_μ assigned the Weyl weight $+\frac{1}{2}$. In an AdS background in $\gamma \cdot \Psi = 0 = \nabla \cdot \Psi$ gauge, the (third-order) field equation implies that

$$0 = (\Box + 3m^2)(\Box + 4m^2)\, \Psi_\mu = \Omega^{-3/2} \Box_0^2\, \psi_\mu. \tag{4.17}$$

Note how the differential operator here is just the product of those in the gauge invariant (4.4b) and gauge variant ((4.10), (4.14b)) models treated earlier.

B. *Spin* 2

Very similar results hold for the gauge invariant linearized excitations of cosmological gravity, $I_2 = (1/\kappa^2) \int d^4x \sqrt{-\bar{g}}\, (-\frac{1}{2}\bar{R} + 3m^2)$, which is the supersymmetric companion to $I_{3/2}$ of (4.1). (We write the metric as $\bar{g}_{\mu\nu}$, and denote functions of the metric with a bar.) Here we linearize the field equations $\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} + 3m^2 \bar{g}_{\mu\nu} = 0$ about the AdS background $g_{\mu\nu}$ by setting $\bar{g}_{\mu\nu} = g_{\mu\nu} + H_{\mu\nu}$, thereby obtaining

$$\mathscr{G}^L_{\mu\nu} \equiv \tfrac{1}{2}\{\Box H_{\mu\nu} + 2m^2 H_{\mu\nu} + m^2 g_{\mu\nu} H - \nabla_\mu V_\nu - \nabla_\nu V_\mu + g_{\mu\nu}(\nabla_\rho V^\rho - \tfrac{1}{2}\Box H)\} = 0,$$
$$V_\mu \equiv \nabla^\rho H_{\rho\mu} - \tfrac{1}{2}\nabla_\mu H, \qquad H \equiv H_\rho{}^\rho \tag{4.18}$$

after some reordering of covariant derivatives. This system is invariant under $\delta H_{\mu\nu}(x) = \nabla_\mu \zeta_\nu(x) + \nabla_\nu \zeta_\mu(x)$ and hence describes two degrees of freedom. In harmonic gauge $V_\mu = 0$, the field equation reads

$$\Box H_{\mu\nu} + 2m^2(H_{\mu\nu} - g_{\mu\nu} H) = 0. \tag{4.19}$$

Assigning to $H_{\mu\nu}$ a Weyl weight $+1$, we find that this transforms to the flat space equation

$$\begin{aligned} &\Box_0 h_{\mu\nu} - 2m^2\Omega^2 h_{\mu\nu} - m^4\Omega^2[x_\mu(x^\rho h_{\rho\nu}) + x_\nu(x^\rho h_{\rho\mu})] \\ &\quad + \frac{m^2\Omega^2}{2}[m^2 x_\mu x_\nu h + \eta_{\mu\nu}(m^2 x^\rho x^\sigma h_{\rho\sigma} - 4h)] \\ &\quad + m^2\Omega[-x_\mu(\partial^\rho h_{\rho\nu}) - x_\nu(\partial^\rho h_{\rho\mu}) + \partial_\mu(x^\rho h_{\rho\nu}) + \partial_\nu(x^\rho h_{\rho\mu})] = 0 \end{aligned} \tag{4.20}$$

with an x-dependent mass (which clearly cannot be eliminated using the $V_\mu = 0$ gauge condition), indicating that the theory does not have AdS null cone propagation. Presumably, this could also be checked by transforming to flat space the AdS equation of motion

$$(\Box + 6m^2)\, C^L_{\mu\nu\alpha\beta} = 0 \tag{4.21}$$

obeyed by the linearized Weyl tensor ("field strength") $C^L_{\mu\nu\alpha\beta}$.

Nonetheless, as in the $D \neq 4$ spin 1 and the spin $\frac{3}{2}$ cases, there are gauge variant AdS spin 2 systems with modes that do propagate on the null cone. Again, we present two such models.

First consider the field equation (in an arbitrary space-time background)

$$\begin{aligned}
&\tilde{S}_{\mu\nu} \equiv S_{\mu\nu} - \tfrac{1}{4}\, g_{\mu\nu} S_\alpha{}^\alpha = 0, \qquad \tilde{H}_{\mu\nu} \equiv H_{\mu\nu} - \tfrac{1}{4}\, g_{\mu\nu} H,\\
&S_{\mu\nu} \equiv \Box \tilde{H}_{\mu\nu} - \tfrac{2}{3}(\nabla^\rho \nabla_\mu \tilde{H}_{\rho\nu} + \nabla^\rho \nabla_\nu \tilde{H}_{\rho\mu}) + \tfrac{4}{3} R_\mu{}^\rho{}_\nu{}^\sigma \tilde{H}_{\rho\sigma}\\
&\qquad + \tfrac{1}{3}(R_\mu{}^\rho \tilde{H}_{\rho\nu} + R_\nu{}^\rho \tilde{H}_{\rho\mu}) - \tfrac{1}{6} R \tilde{H}_{\mu\nu}
\end{aligned} \tag{4.22}$$

which is derivable from the action $\int \tilde{H} S(\tilde{H})$. Since $\tilde{S}_{\mu\nu}$ depends only on $\tilde{H}_{\mu\nu}$, it is invariant under $\delta H_{\mu\nu}(x) = g_{\mu\nu}\omega(x)$; also, the field equation is manifestly traceless. However, the divergence $\nabla^\mu \tilde{S}_{\mu\nu} = 0$ provides an additional constraint, which—for example, in AdS—reads

$$\Box(\nabla^\rho \tilde{H}_{\rho\nu}) - \nabla_\nu(\nabla^\rho \nabla^\sigma \tilde{H}_{\rho\sigma}) + 3m^2(\nabla^\rho \tilde{H}_{\rho\nu}) = 0; \tag{4.23}$$

i.e., $\nabla^\rho \tilde{H}_{\rho\nu}$ obeys the AdS Maxwell equation (3.10). One can verify that this model is Weyl invariant; consequently, both the wave equations in an AdS background

$$\Box \tilde{H}_{\mu\nu} - \tfrac{2}{3}[\nabla_\mu(\nabla^\rho \tilde{H}_{\rho\nu}) + \nabla_\nu(\nabla^\rho \tilde{H}_{\rho\mu})] + \tfrac{1}{3}\, g_{\mu\nu} \nabla^\rho \nabla^\sigma \tilde{H}_{\rho\sigma} + 4m^2 \tilde{H}_{\mu\nu} = 0 \tag{4.24}$$

and also in flat space

$$\Box_0 \tilde{h}_{\mu\nu} - \tfrac{2}{3}[\partial_\mu(\partial^\rho \tilde{h}_{\rho\nu}) + \partial_\nu(\partial^\rho \tilde{h}_{\rho\mu})] + \tfrac{1}{3}\, \eta_{\mu\nu} \partial^\rho \partial^\sigma \tilde{h}_{\rho\sigma} = 0 \tag{4.25}$$

are covariant under conformal transformations.[10] The flat space equation was proposed in [4, 5]. Since it has modes that propagate on the null cone, so does the corresponding AdS equation (4.24). (See also Appendices B and C.) In passing, we note that (4.24) has the residual local invariance $\delta \tilde{H}_{\mu\nu}(x) = (\nabla_\mu \nabla_\nu - \frac{1}{4}\, g_{\mu\nu} \Box)\, \lambda(x)$, while (4.25) has the corresponding flat space invariance $\delta \tilde{h}_{\mu\nu}(x) = (\partial_\mu \partial_\nu - \frac{1}{4}\eta_{\mu\nu} \Box_0)\, \lambda(x)$. The latter can be used to eliminate the longitudinal part of $\tilde{h}_{0i}$, since $\delta \tilde{h}_{0i} = \partial_i \lambda$.

Next, let us turn to another model, with an explicit Pauli–Fierz mass term[11]

[10] The standard gauge invariant spin 2 theory in flat space is of course not conformally invariant [19].

[11] This mass term is *not* the linearization of the (only possible) covariant $\sqrt{-g}$.

$\sim m^2 \int dx^4 \sqrt{-g}\ (H_{\mu\nu} H^{\mu\nu} - H^2)$ added to the linearized gauge invariant action for (4.18). Its (AdS) field equation is

$$\mathscr{G}^L_{\mu\nu} + m^2 (H_{\mu\nu} - H g_{\mu\nu}) = 0. \tag{4.26}$$

From the Bianchi identity $\nabla^\mu \mathscr{G}^L_{\mu\nu} \equiv 0$, it follows that

$$\nabla^\mu H_{\mu\nu} - \nabla_\nu H = 0. \tag{4.27}$$

This reduces the field equation to

$$(\Box H_{\mu\nu} - \nabla_\mu \nabla_\nu H) + 4m^2 (H_{\mu\nu} - \tfrac{1}{4} H g_{\mu\nu}) = 0 \tag{4.28}$$

which is manifestly traceless. Although this model is also gauge variant, it has the residual invariance $\delta H_{\mu\nu}(x) = (\nabla_\mu \nabla_\nu - m^2 g_{\mu\nu})\, \omega(x)$. This allows us to set $H = 0$, in which case the field equation becomes simply

$$(\Box + 4m^2)\, H_{\mu\nu} = 0 \tag{4.29}$$

with $\nabla^\mu H_{\mu\nu} = H = 0$.

In order to study the model's propagation properties, we Weyl-transform the AdS equation (4.28) to flat space, finding

$$\begin{aligned} &\Box_0 h_{\mu\nu} - \partial_\mu \partial_\nu h - m^4 \Omega^2 [x_\mu (x^\rho h_{\rho\nu}) + x_\nu (x^\rho h_{\rho\mu})] \\ &\quad + m^2 \Omega [-x_\mu (\partial^\rho h_{\rho\nu} - \partial_\nu h) - x_\nu (\partial^\rho h_{\rho\mu} - \partial_\mu h) + \partial_\mu (x^\rho h_{\rho\nu}) + \partial_\nu (x^\rho h_{\rho\mu})] \\ &\quad - (m^2/2)\, \Omega \eta_{\mu\nu} [h + (x \cdot \partial)\, h - m^2 \Omega x^\rho x^\sigma h_{\rho\sigma}] = 0. \end{aligned} \tag{4.30}$$

The condition (4.27) becomes

$$\partial^\rho h_{\rho\nu} - \partial_\nu h + \tfrac{3}{2} m^2 \Omega x^\rho h_{\rho\nu} = 0 \tag{4.31}$$

which we use to eliminate $m^2 x^\alpha h_{\alpha\beta}$ terms from the preceding equation. This leads again to the flat space conformally covariant wave equation (4.25), from which we conclude that the AdS model also describes null cone propagation. (See also Appendix C.) As remarked in the parallel discussions on lower-spin systems, this "equivalence" between the AdS and flat space models breaks down at $m^2 = 0$.

These results can be compared with the higher-derivative Weyl invariant spin 2 theory obtained by linearizing the field equations of the Weyl action $I_W = \int d^4x \sqrt{-g}\ C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}$. In an AdS background in the combined harmonic-Weyl gauge $V_\mu = 0 = H$, the linearized Weyl equations [20] read

$$0 = (\Box + 2m^2)(\Box + 4m^2)\, H_{\mu\nu} = \Omega^{-2} \Box_0^2\, h_{\mu\nu} \tag{4.32}$$

where we have used the fact that $H_{\mu\nu}$ has Weyl weight +2. Again, we see that the differential operator is the product of the operators of the gauge invariant (4.19) and gauge variant ((4.24), (4.29)) models.

5. DISCUSSION

We have found that the gauge invariant spin $s = \frac{3}{2}$, 2 theories, because they are not conformally invariant, are not "massless" even in a "matched" AdS background; propagation of the potentials or of the gauge invariant field strengths is not restricted to the null cone, but includes "massive" interior support. Likewise, the Maxwell theory for $D \neq 4$ has massive behavior in these backgrounds. Of course, the gauge invariance of these models restricts the number of degrees of freedom to two for $D = 4$ (more generally, $D - 2$ for Maxwell, etc.).

We have also found $s = 1$ theories for $D \neq 4$, and $s = \frac{3}{2}$, 2 theories in four dimensions that are Weyl invariant (and hence, conformally invariant in AdS) and which do have null cone propagation. These models are not gauge invariant, and include lower-spin (ghost) degrees of freedom as well. Evidently, the fields in these models carry an irreducible representation of the conformal group, which is reducible with respect to the de Sitter (Poincaré) group.

Still other, inequivalent, AdS wave equations have been found which can be mapped by Weyl transformations to the corresponding flat space conformally covariant equations. These mappings become singular as the radius of curvature of the de Sitter space tends to infinity. An $s = \frac{3}{2}$ model of this type is particularly notable, as it is consistent. Our search for such models was not exhaustive, so that those discussed in text may not constitute a complete set.

The inverse propagators of the higher-derivative $s = \frac{3}{2}$, 2 theories of conformal supergravity in AdS are each a product of two operators, one belonging to the gauge invariant, the other to the gauge variant models. From this one could infer the irreducible representation content carried by the fields of conformal supergravity. It would be of interest to further understand the relationships among these gauge and conformally invariant theories.

Note added in proof. The relations given in text among the gauge theories, the conformally invariant models, and the corresponding higher-derivative theories (e.g., in (4.17), (4.32)) have been further elucidated in [25] and [26]. In [26], the four-dimensional conformally invariant spin-two model (4.22) is also generalized to arbitrary dimensions.

APPENDIX A: CONVENTIONS AND PROPERTIES OF AdS

Here we list our conventions, as well as some useful identities:

$$\eta_{ab} = \mathrm{diag}(-+++), \qquad \{\gamma^a, \gamma^b\} = 2\eta^{ab}, \qquad \tfrac{1}{4}[\gamma^a, \gamma^b] \equiv \sigma^{ab}, \tag{A.1}$$

$$\varepsilon^{0123} = -\varepsilon_{0123} = 1, \qquad \gamma_5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3, \qquad \gamma_5^2 = -1, \tag{A.2}$$

$$e\varepsilon_{\mu\nu\lambda\sigma}\gamma_5\gamma^\sigma = 2\sigma_{\nu\lambda}\gamma_\mu + g_{\nu\mu}\gamma_\lambda - g_{\lambda\mu}\gamma_\nu = 2\gamma_\mu\sigma_{\nu\lambda} - g_{\nu\mu}\gamma_\lambda + g_{\lambda\mu}\gamma_\nu, \tag{A.3}$$

$$\omega_{\mu ab} \equiv \tfrac{1}{2}(C_{\mu,ab} - C_{a,b\mu} - C_{b,\mu a}), \quad C^a{}_{,\mu\nu} \equiv \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a, \quad e \equiv \det e_{\mu a}, \tag{A.4}$$

$$R_{\mu\nu ab} \equiv \partial_\mu \omega_{\nu ab} - \omega_{\mu ac}\omega_\nu{}^c{}_b - \mu \leftrightarrow \nu, \tag{A.5}$$

$$R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_\beta \Gamma^{\mu}_{\nu\alpha} - \cdots = g^{\mu\lambda} e_\alpha{}^a e_\beta{}^b R_{\lambda\nu ab}, \tag{A.6}$$

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu}, \qquad \sigma^{\alpha\beta} R_{\alpha\beta\mu\nu} \sigma^{\mu\nu} = -\tfrac{1}{2} R. \tag{A.7}$$

The covariant derivative operator on spinors λ is

$$\nabla_\mu \equiv \partial_\mu - \tfrac{1}{2} \omega_{\mu ab} \sigma^{ab} \tag{A.8}$$

obeying the Ricci identity

$$[\nabla_\mu, \nabla_\nu]\, \lambda = -\tfrac{1}{2} R_{\mu\nu ab} \sigma^{ab} \lambda. \tag{A.9}$$

The corresponding vector identity is

$$[\nabla_\mu, \nabla_\nu]\, A_\rho = R^{\beta}{}_{\rho\mu\nu} A_\beta. \tag{A.10}$$

A constant curvature space in D dimensions is defined by

$$R_{\mu\nu\alpha\beta} = m^2 (g_{\mu\alpha}\, g_{\nu\beta} - g_{\mu\beta}\, g_{\nu\alpha}) \tag{A.11}$$

implying

$$R_{\mu\nu} = (D-1)\, m^2 g_{\mu\nu}, \qquad R = D(D-1)\, m^2. \tag{A.12}$$

The constant curvature space with $m^2 > 0$ is anti-de Sitter (AdS). In local coordinates, the AdS metric (in any dimension) can be written in the conformal form

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}, \qquad e_{\mu a} = \Omega \eta_{\mu a}, \qquad \Omega^2 \equiv \left(1 - \frac{m^2 x^2}{4}\right)^{-2} \tag{A.13}$$

where $x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu$, and the vielbein is in symmetric (Lorentz) gauge. Introducing the $(D+1)$-vector ζ^A $(A = 0, 1,..., D-1, D+1)$

$$\zeta^\mu = \Omega x^\mu, \qquad \zeta^{D+1} = m^{-2}(1 - 2\Omega) \tag{A.14}$$

and the flat metric $\eta^{AB} \equiv \mathrm{diag}(- + \cdots + -)$, one can readily check that

$$\eta_{AB} \zeta^A \zeta^B = -m^{-2} \tag{A.15}$$

and that the x^μ are projective coordinates for this D-(pseudo)-sphere. The AdS metric (A.13) is in fact $\partial_\mu \zeta^A \partial_\nu \zeta^B \eta_{AB}$, the metric induced by the immersion (A.14) of the D-(pseudo)-sphere in the flat $(D+1)$-space.

From (A.13) it is clear that AdS is locally Weyl ("conformally") equivalent to Minkowski space. Although this is sufficient for our purposes, we remark that globally the correspondence is more intricate: the entire Minkowski space can only be conformally mapped to a portion of the Einstein static universe (ESU); and, in turn, half of ESU can be conformally mapped to the universal covering of the AdS [21]. We also note that the points $x^2 = 4/m^2$ at which the metric (A.13) is singular correspond to null and space-like infinity in AdS.

APPENDIX B: WEYL INVARIANCE AND NULL CONE PROPAGATION

The null cone $ds^2 = 0$ is preserved by a Weyl transformation. This suggests that a Weyl invariant field theory describes null cone propagation in a space of constant curvature.[12] Here we sketch a proof of this assertion (in $D = 4$, for concreteness), relying on the use of Weyl transformations; an alternative argument is presented in Appendix C.

First, consider the case of a scalar field in the constant curvature background $g_{\mu\nu}$ obeying the Weyl invariant equation (2.1). The symmetric Green function $G(x, x')$ satisfies

$$\left(\Box_x + \frac{1}{6} R\right) G(x, x') = \frac{-\delta^4(x, x')}{\sqrt[4]{g_x}\,\sqrt[4]{g_{x'}}}, \qquad R = 12m^2, \qquad g_x \equiv |\det g_{\mu\nu}(x)|. \tag{B.1}$$

Under the substitution

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}, \qquad \Omega^2 \equiv \left(1 - \frac{m^2 x^2}{4}\right)^{-2} \tag{B.2}$$

it follows that

$$\left(\Box_x + \frac{1}{6} R\right) G(x, x') = \Omega_x^{-3}(\Box_{0_x})(\Omega_x G(x, x')) = \frac{-\delta^4(x, x')}{\sqrt[4]{g_x}\,\sqrt[4]{g_{x'}}} = \frac{-\delta^4(x, x')}{\Omega^3{}_x \Omega_{x'}}$$

or

$$(\Box_{0_x})(\Omega_x G(x, x')\,\Omega_{x'}) = -\delta^4(x, x')$$

giving the standard result

$$G(x, x') = \Omega_x^{-1} G_0(x, x')\,\Omega_{x'}^{-1}. \tag{B.3}$$

In flat space we know

$$G_0(x, x') = (1/4\pi)\,\delta(\sigma_0) \tag{B.4}$$

where σ_0 is the square of the distance from x to x', $\sigma_0 \equiv \eta_{\mu\nu}(x - x')^\mu (x - x')^\nu$. With these results, we will now demonstrate that the Green function for the wave equation in the curved background also has null cone support: i.e., $G(x, x') \sim \delta(\sigma)$, where σ is the square of the geodesic distance [6] from x to x' in this background.

[12] Presumably, the result can be extended to general conformally flat spaces; however, we treat the only case of real interest, since those Einstein spaces which are conformally flat have constant curvature [22].

To this end, we observe (see, e.g., [23]) that since $\sigma_0 = 0$ iff $\sigma = 0$, then

$$\sigma = a_1 \sigma_0 + a_2 \sigma_0{}^2 + \cdots . \tag{B.5}$$

But

$$\eta^{\mu\nu} \partial_\mu \sigma_0 \partial_\nu \sigma_0 = 4\sigma_0 \tag{B.6a}$$

$$g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma = 4\sigma. \tag{B.6b}$$

Substituting (B.5) into (B.6b) yields the relation

$$\Omega^2 = \frac{d}{ds}(sa_1)$$

or

$$a_1 = \frac{1}{s} \int_0^{s(x')} dt\, \Omega^2 \tag{B.7}$$

where s is the affine parameter for the geodesic $x^\mu(s)$ from x to x'. Using (B.2) and remembering that null geodesics are preserved by the Weyl transformation, we find

$$a_1 = \int_0^1 \frac{dt}{\{1 - (m^2/4)[x + t(x' - x)]^2\}^2} = \Omega_{x'} \Omega_x \tag{B.8}$$

and

$$\delta(\sigma) = \delta(\sigma_0) \Big/ \left(\frac{d\sigma}{d\sigma_0}\right) \Bigg|_{\sigma_0 = 0} = \frac{1}{a_1} \delta(\sigma_0) = \Omega_{x'}^{-1} \delta(\sigma_0)\, \Omega_x^{-1} \sim G(x, x'). \tag{B.9}$$

Thus, we have shown that the Green function in a constant curvature background has support entirely on the null cone, as desired.

The spin $\frac{1}{2}$ Green function satisfies

$$\nabla\!\!\!/_x S(x, x') = \frac{-\delta^4(x, x')}{\sqrt[4]{g_x}\, \sqrt[4]{g_{x'}}}. \tag{B.10a}$$

A second-order form is found by setting $S = \nabla\!\!\!/ H$, where H obeys

$$\nabla\!\!\!/_x^2 H(x, x') = \left(\Box + \frac{R}{4}\right) H(x, x') = \frac{-\delta^4(x, x')}{\sqrt[4]{g_x}\, \sqrt[4]{g_{x'}}}. \tag{B.10b}$$

Proceeding as in the scalar case, one finds $S(x, x') = \Omega_x^{-3/2} S_0(x, x')\, \Omega_{x'}^{-3/2}$, with $S_0 = \partial\!\!\!/_x \delta(\sigma_0)$; one can then argue that the AdS Green function has only null cone support.

Next, consider the case of the Maxwell field. The symmetric Green function $G_{\mu\alpha'}(x, x')$ satisfies

$$\Delta_{x\mu}{}^{\nu} G_{\nu\alpha'}(x, x') = \frac{-\bar{g}_{\mu\alpha'}}{\sqrt[4]{g_x}\,\sqrt[4]{g_{x'}}}\,\delta^4(x, x'), \qquad \Delta_{\mu}{}^{\nu} \equiv \Box g_{\mu}{}^{\nu} + R_{\mu}{}^{\nu} - \nabla_{\mu}\nabla^{\nu}, \qquad R_{\mu\nu} \equiv 3m^2 g_{\mu\nu} \tag{B.11}$$

where $\bar{g}_{\mu\alpha'}(x, x')$ is the bi-vector of geodesic parallel displacement [6]. Of course, the Green function is not well defined unless a gauge is fixed. For the moment, let us overlook this point and proceed formally. Inserting (B.2), we have

$$\Delta_{\mu}{}^{\nu} = \Omega^{-2}\{\Box_0 \eta_{\mu}{}^{\nu} - \partial_{\mu}\partial^{\nu}\} \equiv \Omega^{-2}\,\Delta_{0\mu}{}^{\nu} \tag{B.12}$$

and also $\bar{g}_{\mu\alpha'}(x, x') = a_1(x, x')\,\eta_{\mu\alpha'} = \Omega_x \eta_{\mu\alpha'} \Omega_{x'}$. Thus,

$$\Omega^{-2}\,\Delta_{0\mu}{}^{\nu} G_{\nu\alpha'} = \frac{-\Omega_x \eta_{\mu\alpha'} \Omega_{x'}}{\Omega_x^2 \Omega_{x'}^2}\,\delta^4(x, x') = \frac{-1}{\Omega_x^2}\,\eta_{\mu\alpha'}\,\delta^4(x, x')$$

or (B.13)

$$G_{\nu\alpha'}(x, x') = G_{0\nu\alpha'}(x, x').$$

One would now like to quote the flat space result $G_{0\nu\alpha'}(x, x') = (1/4\pi)\,\eta_{\nu\alpha'}\,\delta(\sigma_0)$, and conclude from (B.9) and (B.13) that

$$G_{\nu\alpha'}(x, x') = (1/4\pi)\,\bar{g}_{\nu\alpha'}\,\delta(\sigma).$$

However, this would not be quite right. The above expression for G_0 is valid in Lorentz gauge $\partial^{\mu} a_{\mu} = 0$; but, this gauge is not Weyl covariant, so that the simple relation (B.12) is not satisfied by the gauge-fixed operators. Another possibility is to work in the gauge $x \cdot a = 0$, but here the expression for G_0 is more complicated. This (calculational) difficulty with gauge artifacts notwithstanding, it should be clear that the physical degrees of freedom propagate with null cone support. Indeed, one can easily repeat the steps (B.11)–(B.13) for $G_{\mu\nu\alpha'\beta'}$, the Green function for the field strength $F_{\mu\nu}$, and thereby avoid the gauge complications altogether.

Finally, consider the conformal (gauge variant) spin 2 theory (4.22). Let $G_{\rho\sigma\alpha'\beta'}(x, x')$ be the symmetric Green function obeying

$$\Delta_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} G_{\rho\sigma\alpha'\beta'} = \frac{-1}{\sqrt[4]{g_x}\,\sqrt[4]{g_{x'}}}\,(\bar{g}_{\mu\alpha'}\,\bar{g}_{\nu\beta'} + \bar{g}_{\mu\beta'}\,\bar{g}_{\nu\alpha'})\,\delta^4(x, x') \tag{B.14}$$

where

$$\begin{aligned}\Delta_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} \equiv \tfrac{1}{2}\{&\Box g_{\mu}{}^{\rho} g_{\nu}{}^{\sigma} - \tfrac{1}{3}\Box g_{\mu\nu}\,g^{\rho\sigma} - \tfrac{2}{3}\nabla^{\rho}(\nabla_{\mu}\,g_{\nu}{}^{\sigma} + \nabla_{\nu}\,g_{\mu}{}^{\sigma}) + \tfrac{1}{3}(\nabla_{\mu}\nabla_{\nu}\,g^{\rho\sigma} + \nabla^{\rho}\nabla^{\sigma} g_{\mu\nu})\\ &+ \tfrac{4}{3}\,R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} - \tfrac{1}{2}(R_{\mu\nu}\,g^{\rho\sigma} + R^{\rho\sigma} g_{\mu\nu}) + \tfrac{1}{3}(R_{\mu}{}^{\rho} g_{\nu}{}^{\sigma} + R_{\nu}{}^{\rho} g_{\mu}{}^{\sigma})\\ &+ \tfrac{1}{6}R(g_{\mu\nu}\,g^{\rho\sigma} - g_{\mu}{}^{\rho} g_{\nu}{}^{\sigma}) + \rho \leftrightarrow \sigma\}.\end{aligned}$$

Again, we refrain from fixing a "gauge" (although setting $H=0$ poses no problems, as this condition is Weyl covariant), and simply observe that

$$\Delta_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} = \Omega^{-1} \Delta_{0\mu}{}^{\rho}{}_{\nu}{}^{\sigma} \Omega^{-1} \tag{B.15}$$

which implies

$$G_{\rho\sigma\alpha'\beta'}(x, x') = \Omega_x G_{0\rho\sigma\alpha'\beta'}(x, x') \, \Omega_{x'}. \tag{B.16}$$

Since the flat space Green function has null cone support, evidently so does the AdS Green function $G_{\rho\sigma\alpha'\beta'}$. (However, we note that since Δ_0 is not simply $\Box_0$, then $G_{0\rho\sigma\alpha'\beta'}$ is not proportional to $[\eta_{\rho\alpha'}\eta_{\sigma\beta'} + \eta_{\rho\beta'}\eta_{\sigma\alpha'}]\,\delta(\sigma_0)$; and hence, we cannot conclude that $G_{\rho\sigma\alpha'\beta'} \sim [\bar{g}_{\rho\alpha'}\,\bar{g}_{\sigma\beta'} + \bar{g}_{\rho\beta'}\,\bar{g}_{\sigma\alpha'}]\,\delta(\sigma)$.)

APPENDIX C: PROJECTION TECHNIQUE

In this section, we use the "projection technique" of Gutzwiller [24] to analyze null cone propagation in a space of constant curvature, which we have taken to be AdS. This method makes use of the immersion of AdS in a flat $(D+1)$-space, M_{D+1}. We begin by recalling a few elementary facts about the geometry of subspaces [22]. As in Appendix A, M_{D+1} has the metric $ds^2 = \eta_{AB}\, d\zeta^A\, d\zeta^B$, and AdS is the subspace

$$\eta_{AB}\, \zeta^A \zeta^B = -m^{-2} \tag{C.1}$$

with metric $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$. Both metrics must agree on AdS,

$$\eta_{AB}\, d\zeta^A\, d\zeta^B = g_{\mu\nu}\, dx^\mu\, dx^\nu.$$

By (C.1), we have

$$d\zeta^A = \partial_\mu \zeta^A\, dx^\mu \tag{C.2}$$

so that

$$\eta_{AB} \partial_\mu \zeta^A \partial_\nu \zeta^B = g_{\mu\nu}. \tag{C.3}$$

Treating ζ^A as invariants under coordinate transformations in AdS implies

$$\partial_\mu \zeta^A = \nabla_\mu \zeta^A \tag{C.4}$$

so that (C.3) becomes

$$\eta_{AB} \nabla_\mu \zeta^A \nabla_\nu \zeta^B = g_{\mu\nu}. \tag{C.5}$$

To obtain another useful result, we first differentiate (C.1) twice

$$\eta_{AB}(\nabla_\nu \nabla_\mu \zeta^A)\, \zeta^B + \eta_{AB} \nabla_\mu \zeta^A \nabla_\nu \zeta^B = 0$$

and use (C.5)

$$\eta_{AB}(\nabla_\nu \nabla_\mu \zeta^A)\, \zeta^B = -g_{\mu\nu}.$$

Comparing it with (C.1), we obtain the Gauss–Codazzi relation

$$\nabla_\nu \nabla_\mu \zeta^A = m^2 \zeta^A g_{\mu\nu}. \tag{C.6}$$

We are now ready to describe the projection technique. The idea is to project AdS fields of arbitrary integer (half-integer) spin to fields on M_{D+1}, thereby reducing the problem to a set of AdS scalar (spin $\frac{1}{2}$) fields. To illustrate this, consider the case of a vector gauge field A^μ in AdS. Following (C.2), we define the projection

$$V^A \equiv (\partial_\mu \zeta^A)\, A^\mu. \tag{C.7}$$

For $D = 4$, using (C.1), (C.4), and (C.6) above, one can readily prove the identities

$$\eta_{AB} \zeta^A V^B = 0 \tag{C.8}$$

$$(\Box + 2m^2)\, V^A = (\partial_\mu \zeta^A)(\Box + 3m^2)\, A^\mu + 2m^2 \zeta^A (\nabla_\mu A^\mu) \tag{C.9}$$

where $\Box \equiv \nabla_\mu \nabla^\mu$, and V^A is treated as a set of AdS scalars. In AdS, the Maxwell equations in Lorentz gauge read (see Section 3)

$$(\Box + 3m^2)\, A_\mu = 0, \qquad \nabla \cdot A = 0. \tag{C.10}$$

Hence, the Maxwell equations imply

$$(\Box + 2m^2)\, V^A = 0 \tag{C.11}$$

which demonstrates that each of the AdS scalars V^A propagates on the null cone, as (C.11) is the "improved" scalar equation (2.3). This, in turn, indicates [24] that the vector A_μ also has null cone propagation.

This argument is readily extended to AdS spaces of arbitrary dimension D. Here one has the identity

$$(\Box + \zeta R)\, V^A = (\partial_\mu \zeta^A)[\Box + \tfrac{1}{4}(D^2 - 2D + 4)\, m^2]\, A^\mu + 2m^2 \zeta^A (\nabla_\mu A^\mu), \tag{C.12}$$

since $\zeta \equiv \frac{1}{4}((D-2)/(D-1))$ and $R = D(D-1)\, m^2$. Hence, the Proca equations (3.23)

$$\begin{aligned} [\Box + \tfrac{1}{4}(D^2 - 2D + 4)\, m^2]\, A_\mu &= 0, \\ \nabla \cdot A &= 0 \end{aligned} \tag{C.13}$$

imply $(\Box + \zeta R)\, V^A = 0$; that is, the scalars V^A propagate on the null cone, and thus presumably so does A^{μ}. A similar result is implied by the equations of the conformally invariant theory (3.21),

$$\left[\Box+\frac{1}{4}(D^2-2D+4)\,m^2\right]A_{\mu}-\frac{4}{D}\nabla_{\mu}(\nabla\cdot A)=0. \tag{C.14}$$

Indeed, observe that the divergence of (C.14) yields (for $D\neq 4$)

$$\Box(\nabla\cdot A)+\frac{D}{4}(D-2)\,m^2(\nabla\cdot A)=0 \tag{C.15}$$

and note the identity

$$\begin{aligned}(\Box+\zeta R)&\left[V^A-\frac{2}{D}\zeta^A(\nabla\cdot A)\right]\\ &=(\partial_{\mu}\zeta^A)\left[\Box A^{\mu}+\frac{1}{4}(D^2-2D+4)\,m^2A^{\mu}-\frac{4}{D}\nabla^{\mu}(\nabla\cdot A)\right]\\ &\quad-\frac{2}{D}\zeta^A\left[\Box(\nabla\cdot A)+\frac{D}{4}(D-2)\,m^2(\nabla\cdot A)\right].\end{aligned} \tag{C.16}$$

Hence, (C.14) and (C.15) imply that the scalars $V'^A\equiv[V^A-(2/D)\,\zeta^A(\nabla\cdot A)]$ propagate on the null cone.

A similar analysis can be performed on the symmetric Green function $G_{\mu\alpha'}(x,x')$. We define the projection

$$G^{AA'}(x,x')\equiv(\partial_{\mu}\zeta^A\partial_{\mu'}\zeta^{A'})\,G^{\mu\mu'}(x,x')=G^{A'A}(x',x) \tag{C.17}$$

and (for $D=4$, say) note the identity

$$(\Box_x+2m^2)\,G^{AA'}=(\partial_{\mu}\zeta^A\partial_{\mu'}\zeta^{A'})(\Box_x+3m^2)\,G^{\mu\mu'}+2m^2\zeta^A(\partial_{\mu'}\zeta^{A'})(\nabla_{\mu}G^{\mu\mu'}). \tag{C.18}$$

In unitary gauge,

$$\begin{aligned}(\Box_x+3m^2)\,G_{\mu\mu'}(x,x')&=-\frac{1}{\sqrt[4]{g_x}\,\sqrt[4]{g_{x'}}}\,\bar{g}_{\mu\mu'}\,\delta^4(x,x'),\\ \nabla^{\mu}G_{\mu\mu'}(x,x')&=0\end{aligned} \tag{C.19}$$

so that (C.12) becomes

$$(\Box_x+2m^2)\,G^{AA'}=\eta^{AA'}\frac{-1}{\sqrt[4]{g_x}\,\sqrt[4]{g_{x'}}}\,\delta^4(x,x'). \tag{C.20}$$

The $G^{AA'}$ of (C.20) are treated as a set of scalar AdS Green functions, which we know have support entirely on the null cone.

Next, consider the case of the spin 2 field $H_{\mu\nu}$. Let

$$V^{AB} = (\partial_\mu \zeta^A \partial_\nu \zeta^B)\, H^{\mu\nu} \tag{C.21}$$

and note the identity

$$(\Box + 2m^2)\, V^{AB} = (\partial_\mu \zeta^A \partial_\nu \zeta^B)(\Box + 4m^2)\, H^{\mu\nu} + 2m^4 \zeta^A \zeta^B H + 2m^2 \{(\zeta^A \nabla_\nu \zeta^B)(\nabla_\mu H^{\mu\nu}) + (\zeta^B \nabla_\mu \zeta^A)(\nabla_\nu H^{\mu\nu})\}. \tag{C.22}$$

Hence, the relations (4.29)

$$(\Box + 4m^2)\, H_{\mu\nu} = 0, \qquad \nabla^\mu H_{\mu\nu} = H = 0 \tag{C.23}$$

reduce (C.22) to $(\Box + 2m^2)\, V^{AB} = 0$, which, as already observed, corresponds to "massless" scalars in AdS. Thus, the field equations (C.23) evidently describe null propagation, in agreement with the result found in text. The conformally covariant spin 2 equations (4.24), (4.23) imply a similar result.

Finally, consider the spin $\frac{3}{2}$ field Ψ_μ and its projection

$$\psi_A \equiv (\nabla_\mu \zeta_A)\, \Psi^\mu. \tag{C.24}$$

We treat ψ_A as a set of five spin $\frac{1}{2}$ fields, and find the identity

$$\not{\nabla} \psi_A = (\nabla_\mu \zeta_A)(\not{\nabla} \Psi^\mu) + m^2 \zeta_A (\gamma \cdot \Psi). \tag{C.25}$$

Thus, the field equation (4.14a)

$$\not{\nabla} \Psi_\mu = 0, \qquad \gamma \cdot \Psi = 0 \tag{C.26}$$

implies $\not{\nabla} \psi_A = 0$, which clearly has null propagation. Moreover, the conformally covariant equation (4.10a)

$$\not{\nabla} \Psi_\mu - \tfrac{1}{2} \gamma_\mu \nabla \cdot \Psi = 0, \qquad \gamma \cdot \Psi = 0 \tag{C.27}$$

yields an analogous result. Indeed, note the identity

$$\not{\nabla} [\psi_A - \tfrac{1}{2} \zeta_A (\nabla \cdot \Psi)] = (\nabla_\mu \zeta_A)[\not{\nabla} \Psi^\mu - \tfrac{1}{2} \gamma^\mu (\nabla \cdot \Psi)] + m^2 \zeta_A (\gamma \cdot \Psi) - \tfrac{1}{2} \zeta_A \not{\nabla} (\nabla \cdot \Psi). \tag{C.28}$$

But, since (27) also requires $\not{\nabla}(\nabla \cdot \Psi) = 0$, we in fact have

$$\not{\nabla} [\psi_A - \tfrac{1}{2} \zeta_A (\nabla \cdot \Psi)] = 0. \tag{C.29}$$

That is, $\psi'_A \equiv [\psi_A - \frac{1}{2} \zeta_A (\nabla \cdot \Psi)]$ has null propagation.

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