4
Manifolds, Vector Fields, and Differential Forms

In preparation for later chapters, it will be necessary for the reader to learn a little bit about manifold theory. We recall a few basic facts here, beginning with the finite-dimensional case. (See Abraham, Marsden, and Ratiu [1988] for a full account.) The reader need not master all of this material now, but it suffices to read through it for general sense and come back to it repeatedly as our development of mechanics proceeds.

4.1 Manifolds

Our first goal is to define the notion of a manifold. Manifolds are, roughly speaking, abstract surfaces that locally look like linear spaces. We shall assume at first that the linear spaces are $\mathbb{R}^n$ for a fixed integer n, which will be the dimension of the manifold.

Coordinate Charts. Given a set M, a ***chart*** on M is a subset U of M together with a bijective map $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$. Usually, we denote $\varphi(m)$ by $(x^1, \ldots, x^n)$ and call the x^i the ***coordinates*** of the point $m \in U \subset M$.

Two charts (U, φ) and (U', φ') such that $U \cap U' \neq \varnothing$ are called ***compatible*** if $\varphi(U \cap U')$ and $\varphi'(U' \cap U)$ are open subsets of $\mathbb{R}^n$ and the maps

$$\varphi' \circ \varphi^{-1} | \varphi(U \cap U') : \varphi(U \cap U') \longrightarrow \varphi'(U \cap U')$$

and

$$\varphi \circ (\varphi')^{-1} | \varphi'(U \cap U') : \varphi'(U \cap U') \longrightarrow \varphi(U \cap U')$$

are C^∞. Here, $\varphi' \circ \varphi^{-1} | \varphi(U \cap U')$ denotes the restriction of the map $\varphi' \circ \varphi^{-1}$ to the set $\varphi(U \cap U')$. See Figure 4.1.1.

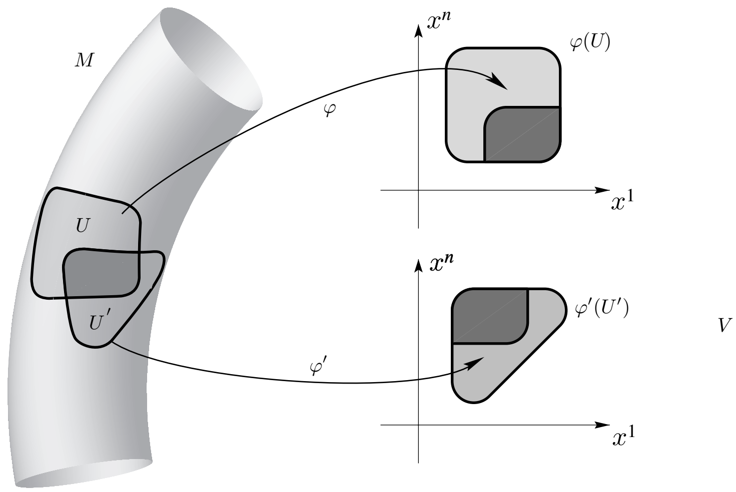

FIGURE 4.1.1. Overlapping charts on a manifold.

We call M a ***differentiable*** n***-manifold*** if the following hold:

M1. *The set M is covered by a collection of charts, that is, every point is represented in at least one chart.*

M2. *M has an* ***atlas****; that is, M can be written as a union of compatible charts.*

If a chart is compatible with a given atlas, then it can be included into the atlas itself to produce a new, larger, atlas. One wants to allow such charts, thereby enlarging a given atlas, and so one really wants to define a ***differentiable structure*** as a ***maximal atlas***. We will assume that this is done and resist the temptation to make this process overly formal.

A simple example will make what we have in mind clear. Suppose one considers Euclidean three-space $\mathbb{R}^3$ as a manifold with simply one (identity) chart. Certainly, we want to allow other charts such as those defined by spherical coordinates. Allowing all possible charts whose changes of coordinates with the standard Euclidean coordinates are smooth then gives us a maximal atlas.

A ***neighborhood*** of a point m in a manifold M is defined to be the inverse image of a Euclidean space neighborhood of the point $\varphi(m)$ under a chart map $\varphi : U \to \mathbb{R}^n$. Neighborhoods define open sets, and one checks that the open sets in M define a topology. *Usually, we assume without explicit mention that the topology is Hausdorff*: Two different points m, m' in M have nonintersecting neighborhoods.

Tangent Vectors. Two curves $t \mapsto c_1(t)$ and $t \mapsto c_2(t)$ in an n-manifold M are called ***equivalent*** at the point m if

$$c_1(0) = c_2(0) = m \quad \text{and} \quad (\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$$

in some chart φ. Here the prime denotes the differentiation of curves in Euclidean space. It is easy to check that this definition is chart independent and that it defines an equivalence relation. A ***tangent vector*** v to a manifold M at a point $m \in M$ is an equivalence class of curves at m.

It is a theorem that the set of tangent vectors to M at m forms a vector space. It is denoted by T_mM and is called the ***tangent space*** to M at $m \in M$.

Given a curve $c(t)$, we denote by $c'(s)$ the tangent vector at $c(s)$ defined by the equivalence class of $t \mapsto c(s+t)$ at $t = 0$. We have set things up so that tangent vectors to manifolds are thought of intuitively as tangent vectors to curves in M.

Let $\varphi : U \subset M \to \mathbb{R}^n$ be a chart for the manifold M, so that we get associated coordinates $(x^1, \ldots, x^n)$ for points in U. Let v be a tangent vector to M at m; i.e., $v \in T_mM$, and let c be a curve that is a representative of the equivalence class v. The ***components*** of v are the numbers $v^1, \ldots, v^n$ defined by taking the derivatives of the components, in Euclidean space, of the curve $\varphi \circ c$:

$$v^i = \frac{d}{dt}(\varphi \circ c)^i \bigg|_{t=0},$$

where $i = 1, \ldots, n$. From the definition, the components are independent of the representative curve chosen, but they do, of course, depend on the chart chosen.

Tangent Bundles. The ***tangent bundle*** of M, denoted by TM, is the set that is the disjoint union of the tangent spaces to M at the points $m \in M$, that is,

$$TM = \bigcup_{m \in M} T_mM.$$

Thus, a point of TM is a vector v that is tangent to M at some point $m \in M$.

If M is an n-manifold, then TM is a $2n$-manifold. To define the differentiable structure on TM, we need to specify how to construct local coordinates on TM. To do this, let $x^1, \ldots, x^n$ be local coordinates on M and let $v^1, \ldots, v^n$ be components of a tangent vector in this coordinate system. Then the $2n$ numbers $x^1, \ldots, x^n, v^1, \ldots, v^n$ give a local coordinate system on TM. This is the basic idea one uses to prove that indeed TM is a $2n$-manifold.

The ***natural projection*** is the map $\tau_M : TM \to M$ that takes a tangent vector v to the point $m \in M$ at which the vector v is attached (that is,

$v \in T_m M$). The inverse image $\tau_M^{-1}(m)$ of a point $m \in M$ under the natural projection τ_M is the tangent space $T_m M$. This space is called the ***fiber*** of the tangent bundle over the point $m \in M$.

Differentiable Maps and the Chain Rule. Let $f : M \to N$ be a map of a manifold M to a manifold N. We call f ***differentiable*** (resp. C^k) if in local coordinates on M and N, the map f is represented by differentiable (resp. C^k) functions. Here, by "represented" we simply mean that coordinate charts are chosen on both M and N so that in these coordinates f, suitably restricted, becomes a map between Euclidean spaces. One of course has to check that this notion of smoothness is independent of the charts chosen—this follows from the chain rule.

The ***derivative*** of a differentiable map $f : M \to N$ at a point $m \in M$ is defined to be the linear map

$$T_m f : T_m M \to T_{f(m)} N$$

constructed in the following way. For $v \in T_m M$, choose a curve $c :]-\epsilon, \epsilon[\to M$ with $c(0) = m$, and associated velocity vector $dc/dt\,|_{t=0} = v$. Then $T_m f \cdot v$ is the velocity vector at $t = 0$ of the curve $f \circ c : \mathbb{R} \to N$, that is,

$$T_m f \cdot v = \left. \frac{d}{dt} f(c(t)) \right|_{t=0} .$$

The vector $T_m f \cdot v$ does not depend on the curve c but only on the vector v, as is seen using the chain rule. If $f : M \to N$ is of class C^k, then $Tf : TM \to TN$ is a mapping of class C^{k-1}. Note that

$$\left. \frac{dc}{dt} \right|_{t=0} = T_0 c \cdot 1.$$

If $f : M \to N$ and $g : N \to P$ are differentiable maps (or maps of class C^k), then $g \circ f : M \to P$ is differentiable (or of class C^k), and the ***chain rule*** holds:

$$T(g \circ f) = Tg \circ Tf.$$

Diffeomorphisms. A differentiable (or of class C^k) map $f : M \to N$ is called a ***diffeomorphism*** if it is bijective and its inverse is also differentiable (or of class C^k).

If $T_m f : T_m M \to T_{f(m)} N$ is an isomorphism, the ***inverse function theorem*** states that f is a ***local diffeomorphism*** around $m \in M$, that is, there are open neighborhoods U of m in M and V of $f(m)$ in N such that $f|U : U \to V$ is a diffeomorphism. The set of all diffeomorphisms $f : M \to M$ forms a group under composition, and the chain rule shows that $T(f^{-1}) = (Tf)^{-1}$.

Submanifolds and Submersions. A ***submanifold*** of M is a subset $S \subset M$ with the property that for each $s \in S$ there is a chart (U, φ) in M with the ***submanifold property***, namely,

SM. $\varphi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^k \times \{\mathbf{0}\})$.

The number k is called the ***dimension*** of the submanifold S.

This latter notion is in agreement with the definition of dimension for a general manifold, since S is a manifold in its own right all of whose charts are of the form $(U \cap S, \varphi|(U \cap S))$ for all charts (U, φ) of M having the submanifold property. Note that any open subset of M is a submanifold and that a submanifold is necessarily ***locally closed***, that is, every point $s \in S$ admits an open neighborhood U of s in M such that $U \cap S$ is closed in U.

There are convenient ways to construct submanifolds using smooth mappings. If $f : M \to N$ is a smooth map, a point $m \in M$ is a ***regular point*** if $T_m f$ is surjective; otherwise, m is a ***critical point*** of f. If $C \subset M$ is the set of critical points of f, then $f(C) \subset N$ is the set of ***critical values*** of f and $N \backslash f(C)$ is the set of ***regular values*** of f.[1]

The ***submersion theorem*** states that if $f : M \to N$ is a smooth map and n is a regular value of f, then $f^{-1}(n)$ is a smooth submanifold of M of dimension $\dim M - \dim N$ and

$$T_m \left(f^{-1}(n)\right) = \ker T_m f.$$

The ***local onto theorem*** states that $T_m f : T_m M \to T_{f(m)} N$ is surjective if and only if there are charts $\varphi : U \subset M \to U'$ at m in M and $\psi : V \subset N \to V'$ at $f(m)$ in N such that φ maps into the product space $\mathbb{R}^{\dim M - \dim N} \times \mathbb{R}^{\dim N}$; the image of U' correspondingly has the form of a product $U' = U'' \times V'$; the point m gets mapped to the origin $\varphi(m) = (\mathbf{0}, \mathbf{0})$, as does $f(m)$, namely, $\psi(f(m)) = \mathbf{0}$; and the local representative of f is a projection:

$$(\psi \circ f \circ \varphi^{-1})(x, y) = x.$$

In particular, $f|U : U \to V$ is onto. If $T_m f$ is onto for every $m \in M$, then f is called a ***submersion***. It follows that submersions are open mappings (the images of open sets are open).

Immersions and Embeddings. A C^k map $f : M \to N$ is called an ***immersion*** if $T_m f$ is injective for every $m \in M$. The ***local 1-to-1 theorem*** states that $T_m f$ is injective if and only if there are charts $\varphi : U \subset M \to U'$ at m in M and $\psi : V \subset N \to V'$ at $f(m)$ in N such that V' is a product

[1] ***Sard's theorem*** states that if $f : M \to N$ is a C^k-map, $k \geq 1$, and if M has the property that every open covering has a countable subcovering, then if $k > \max(0, \dim M - \dim N)$, the set of regular values of f is residual and hence dense in N.

$V' = U' \times V'' \subset \mathbb{R}^{\dim M} \times \mathbb{R}^{\dim N - \dim M}$; both m and $f(m)$ get sent to zero, i.e., $\varphi(m) = \mathbf{0}$ and $\psi(f(m)) = (\mathbf{0}, \mathbf{0})$; and the local representative of f is the inclusion

$$(\psi \circ f \circ \varphi^{-1})(x) = (x, \mathbf{0}).$$

In particular, $f|U : U \to V$ is injective. The ***immersion theorem*** states that $T_m f$ is injective if and only if there is a neighborhood U of m in M such that $f(U)$ is a submanifold of N and $f|U : U \to f(U)$ is a diffeomorphism.

It should be noted that this theorem does *not* say that $f(M)$ is a submanifold of N. For example, f may not be injective and $f(M)$ may thus have self-intersections. Even if f is an injective immersion, the image $f(M)$ may *not* be a submanifold of N. An example is indicated in Figure 4.1.2.

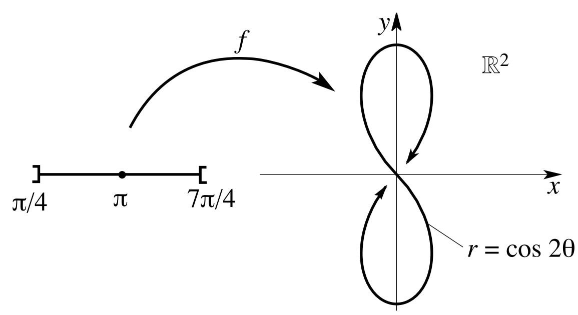

FIGURE 4.1.2. An injective immersion.

The map indicated in the figure (explicitly given by $f :]\pi/4, 7\pi/4[\to \mathbb{R}^2;\ \theta \mapsto (\sin\theta \cos 2\theta, \cos\theta \cos 2\theta))$ is an injective immersion, but the topology induced from $\mathbb{R}^2$ onto its image does not coincide with the usual topology of the open interval: Any neighborhood of the origin in the relative topology consists, in the domain interval, of the union of an open interval about π with two open segments $]\pi/4, \pi/4 + \epsilon[$, $]7\pi/4 - \epsilon, 7\pi/4[$. Thus, the image of f is not a submanifold of $\mathbb{R}^2$, but an ***injectively immersed submanifold***.

An immersion $f : M \to N$ that is a homeomorphism onto $f(M)$ with the relative topology induced from N is called an ***embedding***. In this case $f(M)$ is a submanifold of N and $f : M \to f(M)$ is a diffeomorphism. For example, if $f : M \to N$ is an injective immersion and if M is compact, then f is an embedding. Thus, the example given in the preceding figure is an example of an injective immersion that is not an embedding (and of course, M is not compact).

Another example of an injective immersion that is not an embedding is the linear flow on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with irrational slope: $f(t) = (t, \alpha t) \pmod{\mathbb{Z}^2}$. However, there is a difference between this injective immersion and the "figure eight" example above: In some sense, the second example is better behaved; it has some "uniformity" about its lack of being an embedding.

An injective immersion $f : M \to N$ is called ***regular*** if the following property holds: If $g : L \to M$ is any map of the manifold L into M, then g is C^k if and only if $f \circ g : L \to N$ is C^k for any $k \geq 1$. It is easy to see that all embeddings satisfy this property but that the previous example also satisfies it, without being an embedding, and that the "figure eight" example (see Figure 4.1.2) does not satisfy it. Varadarajan [1974] calls such maps ***quasi-regular embeddings***; they appear below in the Frobenius theorem and in the study of Lie subgroups.

Vector Fields and Flows. A ***vector field*** X on a manifold M is a map $X : M \to TM$ that assigns a vector $X(m)$ at the point $m \in M$; that is, $\tau_M \circ X =$ identity. The real vector space of vector fields on M is denoted by $\mathfrak{X}(M)$. An ***integral curve*** of X with initial condition m_0 at $t = 0$ is a (differentiable) map $c :]a, b[\to M$ such that $]a, b[$ is an open interval containing 0, $c(0) = m_0$, and

$$c'(t) = X(c(t))$$

for all $t \in]a, b[$. In formal presentations we usually suppress the domain of definition, even though this is technically important.

The ***flow*** of X is the collection of maps $\varphi_t : M \to M$ such that $t \mapsto \varphi_t(m)$ is the integral curve of X with initial condition m. Existence and uniqueness theorems from ordinary differential equations guarantee that φ is smooth in m and t (where defined) if X is. From uniqueness, we get the ***flow property***

$$\varphi_{t+s} = \varphi_t \circ \varphi_s$$

along with the initial condition $\varphi_0 =$ identity. The flow property generalizes the situation where $M = V$ is a *linear* space, $X(m) = Am$ for a (bounded) *linear* operator A, and where

$$\varphi_t(m) = e^{tA}m$$

to the *nonlinear* case.

A ***time-dependent vector field*** is a map $X : M \times \mathbb{R} \to TM$ such that $X(m, t) \in T_mM$ for each $m \in M$ and $t \in \mathbb{R}$. An ***integral curve*** of X is a curve $c(t)$ in M such that $c'(t) = X(c(t), t)$. In this case, the flow is the collection of maps

$$\varphi_{t,s} : M \to M$$

such that $t \mapsto \varphi_{t,s}(m)$ is the integral curve $c(t)$ with initial condition $c(s) = m$ at $t = s$. Again, the existence and uniqueness theorem from ODE theory applies, and in particular, uniqueness gives the ***time-dependent flow property***

$$\varphi_{t,s} \circ \varphi_{s,r} = \varphi_{t,r}.$$

If X happens to be time independent, the two notions of flows are related by $\varphi_{t,s} = \varphi_{t-s}$.

Differentials and Covectors. If $f : M \to \mathbb{R}$ is a smooth function, we can differentiate it at any point $m \in M$ to obtain a map $T_m f : T_m M \to T_{f(m)}\mathbb{R}$. Identifying the tangent space of $\mathbb{R}$ at any point with itself (a process we usually do in any vector space), we get a linear map $\mathbf{d}f(m) : T_m M \to \mathbb{R}$. That is, $\mathbf{d}f(m) \in T_m^* M$, the dual of the vector space $T_m M$. We call $\mathbf{d}f$ the ***differential*** of f. For $v \in T_m M$, we call $\mathbf{d}f(m) \cdot v$ the ***directional derivative*** of f in the direction v. In a coordinate chart or in linear spaces, this notion coincides with the usual notion of a directional derivative learned in vector calculus.

Explicitly, in coordinates, the directional derivative is given by

$$\mathbf{d}f(m) \cdot v = \sum_{i=1}^{n} \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} v^i,$$

where φ is a chart at m. We will employ the ***summation convention*** and drop the summation sign when there are repeated indices.

One can show that specifying the directional derivatives completely determines a vector, and so we can identify a basis of $T_m M$ using the operators $\partial / \partial x^i$. We write

$$\{e_1, \dots, e_n\} = \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$$

for this basis, so that $v = v^i \partial / \partial x^i$.

If we replace each vector space $T_m M$ with its dual $T_m^* M$, we obtain a new $2n$-manifold called the ***cotangent bundle*** and denoted by $T^* M$. The dual basis to $\partial / \partial x^i$ is denoted by dx^i. Thus, relative to a choice of local coordinates we get the basic formula

$$\mathbf{d}f(x) = \frac{\partial f}{\partial x^i} dx^i$$

for any smooth function $f : M \to \mathbb{R}$.

Exercises

◇ **4.1-1.** Show that the two-sphere $S^2 \subset \mathbb{R}^3$ is a 2-manifold.

◇ **4.1-2.** If $\varphi_t : S^2 \to S^2$ rotates points on S^2 about a fixed axis through an angle t, show that φ_t is the flow of a certain vector field on S^2.

◇ **4.1-3.** Let $f : S^2 \to \mathbb{R}$ be defined by $f(x, y, z) = z$. Compute $\mathbf{d}f$ relative to spherical coordinates (θ, φ).

4.2 Differential Forms

We next review some of the basic definitions, properties, and operations on differential forms, without proofs (see Abraham, Marsden, and Ratiu [1988] and references therein).

> *The main idea of differential forms is to provide a generalization of the basic operations of vector calculus,* div, grad, *and* curl, *and the integral theorems of Green, Gauss, and Stokes to manifolds of arbitrary dimension.*

Basic Definitions. We have already met one-forms, a term that is used in two ways—they are either members of a particular cotangent space T_m^*M or else, analogous to a vector field, an assignment of a covector in T_m^*M to each $m \in M$. A basic example of a one-form is the differential of a real-valued function.

A 2-***form*** Ω on a manifold M is a function $\Omega(m) : T_mM \times T_mM \to \mathbb{R}$ that assigns to each point $m \in M$ a skew-symmetric bilinear form on the tangent space T_mM to M at m. More generally, a k-***form*** α (sometimes called a ***differential form of degree*** k) on a manifold M is a function $\alpha(m) : T_mM \times \cdots \times T_mM$ (there are k factors) $\to \mathbb{R}$ that assigns to each point $m \in M$ a skew-symmetric k-multilinear map on the tangent space T_mM to M at m. Without the skew-symmetry assumption, α would be called a $(0, k)$-***tensor***. A map $\alpha : V \times \cdots \times V$ (there are k factors) $\to \mathbb{R}$ is ***multilinear*** when it is linear in each of its factors, that is,

$$\begin{aligned}\alpha(v_1, \ldots , av_j + bv_j', \ldots , v_k)& \\ &= a\alpha(v_1, \ldots , v_j, \ldots , v_k) + b\alpha(v_1, \ldots , v_j', \ldots , v_k)\end{aligned}$$

for all j with $1 \leq j \leq k$. A k-multilinear map $\alpha : V \times \ldots \times V \to \mathbb{R}$ is ***skew*** (or ***alternating***) when it changes sign whenever two of its arguments are interchanged, that is, for all $v_1, \ldots , v_k \in V$,

$$\alpha(v_1, \ldots , v_i, \ldots , v_j, \ldots , v_k) = -\alpha(v_1, \ldots , v_j, \ldots , v_i, \ldots , v_k).$$

Let $x^1, \ldots , x^n$ denote coordinates on M, let

$$\{e_1, \ldots , e_n\} = \{\partial/\partial x^1, \ldots , \partial/\partial x^n\}$$

be the corresponding basis for T_mM, and let

$$\{e^1, \ldots , e^n\} = \{dx^1, \ldots , dx^n\}$$

be the dual basis for T_m^*M. Then at each $m \in M$, we can write a 2-form as

$$\Omega_m(v, w) = \Omega_{ij}(m)v^iw^j, \quad \text{where} \quad \Omega_{ij}(m) = \Omega_m\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

and more generally, a k-form can be written

$$\alpha_m(v_1, \ldots, v_k) = \alpha_{i_1 \ldots i_k}(m) v_1^{i_1} \cdots v_k^{i_k},$$

where there is a sum on $i_1, \ldots, i_k$,

$$\alpha_{i_1 \ldots i_k}(m) = \alpha_m \left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}} \right),$$

and $v_i = v_i^j \partial / \partial x^j$, with a sum on j understood.

Tensor and Wedge Products. If α is a $(0, k)$-tensor on a manifold M and β is a $(0, l)$-tensor, their ***tensor product*** $\alpha \otimes \beta$ is the $(0, k+l)$-tensor on M defined by

$$(\alpha \otimes \beta)_m(v_1, \ldots, v_{k+l}) = \alpha_m(v_1, \ldots, v_k)\beta_m(v_{k+1}, \ldots, v_{k+l}) \tag{4.2.1}$$

at each point $m \in M$.

If t is a $(0, p)$-tensor, define the ***alternation operator*** $\mathbf{A}$ acting on t by

$$\mathbf{A}(t)(v_1, \ldots, v_p) = \frac{1}{p!} \sum_{\pi \in S_p} \operatorname{sgn}(\pi) t(v_{\pi(1)}, \ldots, v_{\pi(p)}), \tag{4.2.2}$$

where $\operatorname{sgn}(\pi)$ is the ***sign*** of the permutation π,

$$\operatorname{sgn}(\pi) = \begin{cases} +1 \text{ if } \pi \text{ is even }, \\ -1 \text{ if } \pi \text{ is odd }, \end{cases} \tag{4.2.3}$$

and S_p is the group of all permutations of the set $\{1, 2, \ldots, p\}$. The operator $\mathbf{A}$ therefore skew-symmetrizes p-multilinear maps.

If α is a k-form and β is an l-form on M, their ***wedge product*** $\alpha \wedge \beta$ is the $(k+l)$-form on M defined by[2]

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\, l!} \mathbf{A}(\alpha \otimes \beta). \tag{4.2.4}$$

For example, if α and β are one-forms, then

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1),$$

while if α is a 2-form and β is a 1-form,

$$(\alpha \wedge \beta)(v_1, v_2, v_3) = \alpha(v_1, v_2)\beta(v_3) + \alpha(v_3, v_1)\beta(v_2) + \alpha(v_2, v_3)\beta(v_1).$$

We state the following without proof:

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[2]The numerical factor in (4.2.4) agrees with the convention of Abraham and Marsden [1978], Abraham, Marsden, and Ratiu [1988], and Spivak [1976], but *not* that of Arnold [1989], Guillemin and Pollack [1974], or Kobayashi and Nomizu [1963]; it is the Bourbaki [1971] convention.

Proposition 4.2.1. *The wedge product has the following properties:*

(i) $\alpha \wedge \beta$ *is* ***associative***: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

(ii) $\alpha \wedge \beta$ *is* ***bilinear*** *in* α, β :

$$(a\alpha_1 + b\alpha_2) \wedge \beta = a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta),$$
$$\alpha \wedge (c\beta_1 + d\beta_2) = c(\alpha \wedge \beta_1) + d(\alpha \wedge \beta_2).$$

(iii) $\alpha \wedge \beta$ *is* ***anticommutative***: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$, *where* α *is a* k*-form and* β *is an* l*-form.*

In terms of the dual basis dx^i, any k-form can be written locally as

$$\alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the sum is over all i_j satisfying $i_1 < \cdots < i_k$.

Pull-Back and Push-Forward. Let $\varphi : M \to N$ be a C^∞ map from the manifold M to the manifold N and α be a k-form on N. Define the ***pull-back*** $\varphi^* \alpha$ of α by φ to be the k-form on M given by

$$(\varphi^* \alpha)_m (v_1, \ldots, v_k) = \alpha_{\varphi(m)} (T_m \varphi \cdot v_1, \ldots, T_m \varphi \cdot v_k). \tag{4.2.5}$$

If φ is a diffeomorphism, the ***push-forward*** φ_* is defined by $\varphi_* = (\varphi^{-1})^*$.

Here is another basic property.

Proposition 4.2.2. *The pull-back of a wedge product is the wedge product of the pull-backs:*

$$\varphi^* (\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta. \tag{4.2.6}$$

Interior Products and Exterior Derivatives. Let α be a k-form on a manifold M and X a vector field. The ***interior product*** $\mathbf{i}_X \alpha$ (sometimes called the ***contraction*** of X and α and written, using the "hook" notation, as $X \lrcorner \alpha$) is defined by

$$(\mathbf{i}_X \alpha)_m (v_2, \ldots, v_k) = \alpha_m (X(m), v_2, \ldots, v_k). \tag{4.2.7}$$

Proposition 4.2.3. *Let* α *be a* k*-form and* β *a* 1*-form on a manifold* M*. Then*

$$\mathbf{i}_X (\alpha \wedge \beta) = (\mathbf{i}_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X \beta). \tag{4.2.8}$$

In the "hook" notation, this proposition reads

$$X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner \beta).$$

The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α on a manifold M is the $(k+1)$-form on M determined by the following proposition:

Proposition 4.2.4. *There is a unique mapping* $\mathbf{d}$ *from* k*-forms on* M *to* $(k+1)$*-forms on* M *such that:*

(i) *If* α *is a* 0*-form* ($k = 0$)*, that is,* $\alpha = f \in \mathcal{F}(M)$*, then* $\mathbf{d}f$ *is the one-form that is the differential of* f*.*

(ii) $\mathbf{d}\alpha$ *is* ***linear*** *in* α*, that is, for all real numbers* c_1 *and* c_2*,*

$$\mathbf{d}(c_1\alpha_1 + c_2\alpha_2) = c_1\mathbf{d}\alpha_1 + c_2\mathbf{d}\alpha_2.$$

(iii) $\mathbf{d}\alpha$ *satisfies the* ***product rule****, that is,*

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta,$$

where α *is a* k*-form and* β *is an* l*-form.*

(iv) $\mathbf{d}^2 = 0$*, that is,* $\mathbf{d}(\mathbf{d}\alpha) = 0$ *for any* k*-form* α*.*

(v) $\mathbf{d}$ *is a* ***local operator****, that is,* $\mathbf{d}\alpha(m)$ *depends only on* α *restricted to any open neighborhood of* m*; in fact, if* U *is open in* M*, then*

$$\mathbf{d}(\alpha|U) = (\mathbf{d}\alpha)|U.$$

If α is a k-form given in coordinates by

$$\alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad (\text{sum on } i_1 < \cdots < i_k),$$

then the coordinate expression for the exterior derivative is

$$\mathbf{d}\alpha = \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad (\text{sum on all } j \text{ and } i_1 < \cdots < i_k). \tag{4.2.9}$$

Formula (4.2.9) can be taken as the definition of the exterior derivative, provided that one shows that (4.2.9) has the above-described properties and, correspondingly, is independent of the choice of coordinates.

Next is a useful proposition that in essence rests on the chain rule:

Proposition 4.2.5. *Exterior differentiation commutes with pull-back, that is,*

$$\mathbf{d}(\varphi^*\alpha) = \varphi^*(\mathbf{d}\alpha), \tag{4.2.10}$$

where α *is a* k*-form on a manifold* N *and* $\varphi : M \to N$ *is a smooth map between manifolds.*

A k-form α is called ***closed*** if $\mathbf{d}\alpha = 0$ and ***exact*** if there is a $(k-1)$-form β such that $\alpha = \mathbf{d}\beta$. By Proposition 4.2.4(iv) every exact form is closed. Exercise 4.4-2 gives an example of a closed nonexact one-form.

Proposition 4.2.6 (Poincaré Lemma)**.** *A closed form is locally exact; that is, if* $\mathbf{d}\alpha = 0$*, there is a neighborhood about each point on which* $\alpha = \mathbf{d}\beta$*.*

See Exercise 4.2-5 for the proof.

The definition and properties of vector-valued forms are direct extensions of those for usual forms on vector spaces and manifolds. One can think of a vector-valued form as an array of usual forms (see Abraham, Marsden, and Ratiu [1988]).

Vector Calculus. The table below entitled "Vector Calculus and Differential Forms" summarizes how forms are related to the usual operations of vector calculus. We now elaborate on a few items in this table. In item 4, note that

$$\mathbf{d}f = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (\operatorname{grad} f)^\flat = (\nabla f)^\flat,$$

which is equivalent to $\nabla f = (\mathbf{d}f)^\sharp$.

The ***Hodge star operator*** on $\mathbb{R}^3$ maps k-forms to $(3-k)$-forms and is uniquely determined by linearity and the properties in item 2. (This operator can be defined on general Riemannian manifolds; see Abraham, Marsden, and Ratiu [1988].)

In item 5, if we let $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$, so $F^\flat = F_1\,dx + F_2\,dy + F_3\,dz$, then

$$\begin{aligned}
\mathbf{d}(F^\flat) &= \mathbf{d}F_1 \wedge dx + F_1\mathbf{d}(dx) + \mathbf{d}F_2 \wedge dy + F_2\mathbf{d}(dy) \\
&\quad + \mathbf{d}F_3 \wedge dz + F_3\mathbf{d}(dz) \\
&= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dx \\
&\quad + \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dy \\
&\quad + \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dz \\
&= -\frac{\partial F_1}{\partial y}dx \wedge dy + \frac{\partial F_1}{\partial z}dz \wedge dx + \frac{\partial F_2}{\partial x}dx \wedge dy - \frac{\partial F_2}{\partial z}dy \wedge dz \\
&\quad - \frac{\partial F_3}{\partial x}dz \wedge dx + \frac{\partial F_3}{\partial y}dy \wedge dz \\
&= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx \\
&\quad + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz.
\end{aligned}$$

Hence, using item 2,

$$
\begin{aligned}
*(\mathbf{d}(F^\flat)) &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dy + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dx, \\
(*(\mathbf{d}(F^\flat)))^\sharp &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \mathbf{e}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathbf{e}_3 \\
&= \operatorname{curl} F = \nabla \times F.
\end{aligned}
$$

With reference to item 6, let $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$, so

$$F^\flat = F_1\, dx + F_2\, dy + F_3\, dz.$$

Thus $*(F^\flat) = F_1\, dy \wedge dz + F_2(-dx \wedge dz) + F_3\, dx \wedge dy$, and so

$$
\begin{aligned}
\mathbf{d}(*(F^\flat)) &= \mathbf{d}F_1 \wedge dy \wedge dz - \mathbf{d}F_2 \wedge dx \wedge dz + \mathbf{d}F_3 \wedge dx \wedge dy \\
&= \left(\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz\right) \wedge dy \wedge dz \\
&\quad - \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz\right) \wedge dx \wedge dz \\
&\quad + \left(\frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz\right) \wedge dx \wedge dy \\
&= \frac{\partial F_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y} dx \wedge dy \wedge dz + \frac{\partial F_3}{\partial z} dx \wedge dy \wedge dz \\
&= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz = (\operatorname{div}\ F)\, dx \wedge dy \wedge dz.
\end{aligned}
$$

Therefore, $*(\mathbf{d}(*(F^\flat))) = \operatorname{div}\ F = \nabla \cdot F$.

Vector Calculus and Differential Forms

1. **Sharp and Flat** (Using standard coordinates in $\mathbb{R}^3$)

   (a) $v^\flat = v^1\, dx + v^2\, dy + v^3\, dz$, the one-form corresponding to the vector $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$.

   (b) $\alpha^\sharp = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$, the vector corresponding to the one-form $\alpha = \alpha_1\, dx + \alpha_2\, dy + \alpha_3\, dz$.

2. **Hodge Star Operator**

   (a) $*1 = dx \wedge dy \wedge dz$.

   (b) $*dx = dy \wedge dz$, $*dy = -dx \wedge dz$, $*dz = dx \wedge dy$,
   $*(dy \wedge dz) = dx$, $*(dx \wedge dz) = -dy$, $*(dx \wedge dy) = dz$.

(c) $*(dx \wedge dy \wedge dz) = 1$.

3. Cross Product and Dot Product

(a) $v \times w = [*(v^\flat \wedge w^\flat)]^\sharp$.

(b) $(v \cdot w) dx \wedge dy \wedge dz = v^\flat \wedge *(w^\flat)$.

4. Gradient $\nabla f = \operatorname{grad} f = (\mathbf{d}f)^\sharp$.

5. Curl $\nabla \times F = \operatorname{curl} F = [*(\mathbf{d}F^\flat)]^\sharp$.

6. Divergence $\nabla \cdot F = \operatorname{div}\ F = *\mathbf{d}(*F^\flat)$.

Exercises

◇ **4.2-1.** Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $\varphi(x, y, z) = (x + z, xy)$. For

$$\alpha = e^v \, du + u \, dv \in \Omega^1(\mathbb{R}^2) \quad \text{and} \quad \beta = u \, du \wedge dv,$$

compute $\alpha \wedge \beta$, $\varphi^*\alpha$, $\varphi^*\beta$, and $\varphi^*\alpha \wedge \varphi^*\beta$.

◇ **4.2-2.** Given

$$\alpha = y^2 \, dx \wedge dz + \sin(xy) \, dx \wedge dy + e^x \, dy \wedge dz \in \Omega^2(\mathbb{R}^3)$$

and

$$X = 3\partial/\partial x + \cos z \partial/\partial y - x^2 \partial/\partial z \in \mathfrak{X}(\mathbb{R}^3),$$

compute $\mathbf{d}\alpha$ and $\mathbf{i}_X \alpha$.

◇ **4.2-3.**

(a) Denote by $\bigwedge^k(\mathbb{R}^n)$ the vector space of all skew-symmetric k-linear maps on $\mathbb{R}^n$. Prove that this space has dimension $n!/(k!\,(n-k)!)$ by showing that a basis is given by $\{\, e^{i_1} \wedge \cdots \wedge e^{i_k} \mid i_1 < \ldots < i_k \,\}$, where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ and $\{e^1, \ldots, e^n\}$ is its dual basis, that is, $e^i(e_j) = \delta^i_j$.

(b) If $\mu \in \bigwedge^n(\mathbb{R}^n)$ is nonzero, prove that the map $v \in \mathbb{R}^n \mapsto \mathbf{i}_v \mu \in \bigwedge^{n-1}(\mathbb{R}^n)$ is an isomorphism.

(c) If M is a smooth n-manifold and $\mu \in \Omega^n(M)$ is nowhere-vanishing (in which case it is called a volume form), show that the map $X \in \mathfrak{X}(M) \mapsto \mathbf{i}_X \mu \in \Omega^{n-1}(M)$ is an isomorphism.

◇ **4.2-4.** Let $\alpha = \alpha_i \, dx^i$ be a closed one-form in a ball around the origin in $\mathbb{R}^n$. Show that $\alpha = \mathbf{d}f$ for

$$f(x^1, \ldots, x^n) = \int_0^1 \alpha_j(tx^1, \ldots, tx^n) x^j \, dt.$$

◇ **4.2-5.**

(a) Let U be an open ball around the origin in $\mathbb{R}^n$ and $\alpha \in \Omega^k(U)$ a closed form. Verify that $\alpha = \mathbf{d}\beta$, where

$$\beta(x^1, \ldots, x^n) \\ = \left(\int_0^1 t^{k-1} \alpha_{j i_1 \ldots i_{k-1}}(tx^1, \ldots, tx^n) x^j \, dt \right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}},$$

and where the sum is over $i_1 < \cdots < i_{k-1}$. Here,

$$\alpha = \alpha_{j_1 \ldots j_k} \, dx^{j_1} \wedge \cdots \wedge dx^{j_k},$$

where $j_1 < \cdots < j_k$ and where α is extended to be skew-symmetric in its lower indices.

(b) Deduce the Poincaré lemma from (a).

◇ **4.2-6** (Construction of a homotopy operator for a retraction)**.** Let M be a smooth manifold and $N \subset M$ a smooth submanifold. A family of smooth maps $r_t : M \to M$, $t \in [0,1]$, is called a ***retraction of*** M ***onto*** N if $r_t|N =$ identity on N for all $t \in [0,1]$, $r_1 =$ identity on M, r_t is a diffeomorphism of M with $r_t(M)$ for every $t \neq 0$, and $r_0(M) = N$. Let X_t be the time-dependent vector field generated by r_t, $t \neq 0$. Show that the operator $\mathbf{H} : \Omega^k(M) \to \Omega^{k-1}(M)$ defined by

$$\mathbf{H} = \int_0^1 (r_t^* \mathbf{i}_{X_t} \alpha) \, dt$$

satisfies

$$\alpha - (r_0^* \alpha) = \mathbf{dH}\alpha + \mathbf{Hd}\alpha.$$

(a) Deduce the ***relative Poincaré lemma*** from this formula: If $\alpha \in \Omega^k(M)$ is closed and $\alpha|N = 0$, then there is a neighborhood U of N such that $\alpha|U = \mathbf{d}\beta$ for some $\beta \in \Omega^{k-1}(U)$ and $\beta|N = 0$. (Hint: Use the existence of a tubular neighborhood of N in M.)

(b) Deduce the ***global Poincaré lemma*** for contractible manifolds: If M is contractible, that is, there is a retraction of M to a point, and if $\alpha \in \Omega^k(M)$ is closed, then α is exact.

4.3 The Lie Derivative

Lie Derivative Theorem. The *dynamic definition* of the Lie derivative is as follows. Let α be a k-form and let X be a vector field with flow φ_t. The ***Lie derivative*** of α along X is given by

$$\pounds_X \alpha = \lim_{t \to 0} \frac{1}{t}[(\varphi_t^* \alpha) - \alpha] = \left. \frac{d}{dt} \varphi_t^* \alpha \right|_{t=0}. \tag{4.3.1}$$

This definition together with properties of pull-backs yields the following.

Theorem 4.3.1 (Lie Derivative Theorem)**.**

$$\frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* \pounds_X \alpha. \tag{4.3.2}$$

This formula holds also for *time-dependent* vector fields in the sense that

$$\frac{d}{dt} \varphi_{t,s}^* \alpha = \varphi_{t,s}^* \pounds_X \alpha,$$

and in the expression $\pounds_X \alpha$ the vector field X is evaluated at time t.

If f is a real-valued function on a manifold M and X is a vector field on M, the ***Lie derivative of f along X*** is the ***directional derivative***

$$\pounds_X f = X[f] := \mathbf{d} f \cdot X. \tag{4.3.3}$$

If M is finite-dimensional, then

$$\pounds_X f = X^i \frac{\partial f}{\partial x^i}. \tag{4.3.4}$$

For this reason one often writes

$$X = X^i \frac{\partial}{\partial x^i}.$$

If Y is a vector field on a manifold N and $\varphi : M \to N$ is a diffeomorphism, the ***pull-back*** $\varphi^* Y$ is a vector field on M defined by

$$(\varphi^* Y)(m) = \left(T_m \varphi^{-1} \circ Y \circ \varphi \right)(m). \tag{4.3.5}$$

Two vector fields X on M and Y on N are said to be ***φ-related*** if

$$T\varphi \circ X = Y \circ \varphi. \tag{4.3.6}$$

Clearly, if $\varphi : M \to N$ is a diffeomorphism and Y is a vector field on N, then $\varphi^* Y$ and Y are φ-related. For a diffeomorphism φ, the ***push-forward*** is defined, as for forms, by $\varphi_* = (\varphi^{-1})^*$.

Jacobi–Lie Brackets. If M is finite-dimensional and C^∞, then the set of vector fields on M coincides with the set of derivations on $\mathcal{F}(M)$. The same result is true for C^k manifolds and vector fields if $k \geq 2$. This property is false for infinite-dimensional manifolds; see Abraham, Marsden, and Ratiu [1988]. If M is C^∞ (that is, smooth), then the derivation $f \mapsto X[Y[f]] - Y[X[f]]$, where $X[f] = \mathbf{d}f \cdot X$, determines a unique vector field denoted by $[X, Y]$ and called the ***Jacobi–Lie bracket*** of X and Y. Defining $\pounds_X Y = [X, Y]$ gives the ***Lie derivative*** of Y along X. Then the Lie derivative formula (4.3.2) holds with α replaced by Y, and the pull-back operation given by (4.3.5).

If M is infinite-dimensional, then one defines the Lie derivative of Y along X by

$$\left.\frac{d}{dt}\right|_{t=0} \varphi_t^* Y = \pounds_X Y, \tag{4.3.7}$$

where φ_t is the flow of X. Then formula (4.3.2) with α replaced by Y holds, and the action of the vector field $\pounds_X Y$ on a function f is given by $X[Y[f]] - Y[X[f]]$, which is denoted, as in the finite-dimensional case, by $[X, Y][f]$. As before $[X, Y] = \pounds_X Y$ is also called the Jacobi–Lie bracket of vector fields.

If M is finite-dimensional, then

$$(\pounds_X Y)^j = X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} = (X \cdot \nabla) Y^j - (Y \cdot \nabla) X^j, \tag{4.3.8}$$

and in general, where we identify X, Y with their local representatives, we have

$$[X, Y] = \mathbf{D}Y \cdot X - \mathbf{D}X \cdot Y. \tag{4.3.9}$$

The formula for $[X, Y] = \pounds_X Y$ can be remembered by writing

$$\left[X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}\right] = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i}.$$

Algebraic Definition of the Lie Derivative. The *algebraic approach* to the Lie derivative on forms or tensors proceeds as follows. Extend the definition of the Lie derivative from functions and vector fields to differential forms, by requiring that the Lie derivative be a derivation; for example, for one-forms α, write

$$\pounds_X \langle \alpha, Y \rangle = \langle \pounds_X \alpha, Y \rangle + \langle \alpha, \pounds_X Y \rangle, \tag{4.3.10}$$

where X, Y are vector fields and $\langle \alpha, Y \rangle = \alpha(Y)$. More generally,

$$\pounds_X(\alpha(Y_1, \ldots, Y_k)) = (\pounds_X \alpha)(Y_1, \ldots, Y_k) + \sum_{i=1}^{k} \alpha(Y_1, \ldots, \pounds_X Y_i, \ldots, Y_k), \tag{4.3.11}$$

where $X, Y_1, \ldots, Y_k$ are vector fields and α is a k-form.

Proposition 4.3.2. *The dynamic and algebraic definitions of the Lie derivative of a differential k-form are equivalent.*

Cartan's Magic Formula. A very important formula for the Lie derivative is given by the following.

Theorem 4.3.3. *For X a vector field and α a k-form on a manifold M, we have*

$$\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha, \tag{4.3.12}$$

or, in the "hook" notation,

$$\pounds_X \alpha = \mathbf{d}(X \lrcorner\, \alpha) + X \lrcorner\, \mathbf{d}\alpha.$$

This is proved by a lengthy but straightforward calculation.

Another property of the Lie derivative is the following: If $\varphi : M \to N$ is a diffeomorphism, then

$$\varphi^* \pounds_Y \beta = \pounds_{\varphi^* Y} \varphi^* \beta$$

for $Y \in \mathfrak{X}(N)$ and $\beta \in \Omega^k(M)$. More generally, if $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are ψ related, that is, $T\psi \circ X = Y \circ \psi$ for $\psi : M \to N$ a smooth map, then $\pounds_X \psi^* \beta = \psi^* \pounds_Y \beta$ for all $\beta \in \Omega^k(N)$.

There are a number of valuable identities relating the Lie derivative, the exterior derivative, and the interior product that we record at the end of this chapter. For example, if Θ is a one-form and X and Y are vector fields, identity 6 in the table at the end of §4.4 gives the useful identity

$$\mathbf{d}\Theta(X, Y) = X[\Theta(Y)] - Y[\Theta(X)] - \Theta([X, Y]). \tag{4.3.13}$$

Volume Forms and Divergence. An n-manifold M is said to be ***orientable*** if there is a nowhere-vanishing n-form μ on it; μ is called a ***volume form***, and it is a basis of $\Omega^n(M)$ over $\mathcal{F}(M)$. Two volume forms μ_1 and μ_2 on M are said to define the same ***orientation*** if there is an $f \in \mathcal{F}(M)$ with $f > 0$ and such that $\mu_2 = f\mu_1$. Connected orientable manifolds admit precisely two orientations. A basis $\{v_1, \ldots, v_n\}$ of $T_m M$ is said to be ***positively oriented*** relative to the volume form μ on M if $\mu(m)(v_1, \ldots, v_n) > 0$. Note that the volume forms defining the same orientation form a convex cone in $\Omega^n(M)$, that is, if $a > 0$ and μ is a volume form, then $a\mu$ is again a volume form, and if $t \in [0, 1]$ and μ_1, μ_2 are volume forms defining the same orientation, then $t\mu_1 + (1 - t)\mu_2$ is again a volume form defining the same orientation as μ_1 or μ_2. The first property is obvious. To prove the second, let $m \in M$ and let $\{v_1, \ldots, v_n\}$ be a positively oriented basis of $T_m M$ relative to the orientation defined by μ_1, or equivalently (by hypothesis) by μ_2. Then $\mu_1(m)(v_1, \ldots, v_n) > 0$, $\mu_2(m)(v_1, \ldots, v_n) > 0$, so that their convex combination is again strictly positive.

If $\mu \in \Omega^n(M)$ is a volume form, since $\pounds_X \mu \in \Omega^n(M)$, there is a function, called the ***divergence*** of X relative to μ and denoted by $\operatorname{div}_\mu(X)$ or simply $\operatorname{div}(X)$, such that

$$\pounds_X \mu = \operatorname{div}_\mu(X)\mu. \tag{4.3.14}$$

From the dynamic approach to Lie derivatives it follows that $\operatorname{div}_\mu(X) = 0$ if and only if $F_t^* \mu = \mu$, where F_t is the flow of X. This condition says that F_t is ***volume preserving***. If $\varphi : M \to M$, since $\varphi^* \mu \in \Omega^n(M)$ there is a function, called the ***Jacobian*** of φ and denoted by $J_\mu(\varphi)$ or simply $J(\varphi)$, such that

$$\varphi^* \mu = J_\mu(\varphi)\mu. \tag{4.3.15}$$

Thus, *φ is volume preserving if and only if $J_\mu(\varphi) = 1$. From the inverse function theorem, we see that φ is a local diffeomorphism if and only if $J_\mu(\varphi) \neq 0$ on M.*

Frobenius' Theorem. We also mention a basic result called ***Frobenius' theorem***. If $E \subset TM$ is a vector subbundle, it is said to be ***involutive*** if for any two vector fields X, Y on M with values in E, the Jacobi–Lie bracket $[X, Y]$ is also a vector field with values in E. The subbundle E is said to be ***integrable*** if for each point $m \in M$ there is a local submanifold of M containing m such that its tangent bundle equals E restricted to this submanifold. If E is integrable, the local integral manifolds can be extended to get, through each $m \in M$, a connected maximal integral manifold, which is unique and is a regularly immersed submanifold of M. The collection of all maximal integral manifolds through all points of M is said to form a ***foliation***.

The Frobenius theorem states that the *involutivity of E is equivalent to the integrability of E.*

Exercises

◇ **4.3-1.** Let M be an n-manifold, $\mu \in \Omega^n(M)$ a volume form, $X, Y \in \mathfrak{X}(M)$, and $f, g : M \to \mathbb{R}$ smooth functions such that $f(m) \neq 0$ for all m. Prove the following identities:

(a) $\operatorname{div}_{f\mu}(X) = \operatorname{div}_\mu(X) + X[f]/f$;

(b) $\operatorname{div}_\mu(gX) = g \operatorname{div}_\mu(X) + X[g]$; and

(c) $\operatorname{div}_\mu([X, Y]) = X[\operatorname{div}_\mu(Y)] - Y[\operatorname{div}_\mu(X)]$.

◇ **4.3-2.** Show that the partial differential equation

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{n} X^i(x^1, \ldots, x^n) \frac{\partial f}{\partial x^i}$$

with initial condition $f(x, 0) = g(x)$ has the solution $f(x, t) = g(F_t(x))$, where F_t is the flow of the vector field $(X^1, \ldots, X^n)$ in $\mathbb{R}^n$ whose flow is assumed to exist for all time. Show that the solution is *unique*. Generalize this exercise to the equation

$$\frac{\partial f}{\partial t} = X[f]$$

for X a vector field on a manifold M.

◇ **4.3-3.** Show that if M and N are orientable manifolds, so is $M \times N$.

4.4 Stokes' Theorem

The basic idea of the definition of the integral of an n-form μ on an oriented n-manifold M is to pick a covering by coordinate charts and to sum up the ordinary integrals of $f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n$, where

$$\mu = f(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n$$

is the local representative of μ, being careful not to count overlaps twice. The change of variables formula guarantees that the result, denoted by $\int_M \mu$, is well-defined.

If one has an oriented manifold with boundary, then the boundary, ∂M, inherits a compatible orientation. This proceeds in a way that generalizes the relation between the orientation of a surface and its boundary in the classical Stokes' theorem in $\mathbb{R}^3$.

Theorem 4.4.1 (Stokes' Theorem)**.** *Suppose that M is a compact, oriented k-dimensional manifold with boundary ∂M. Let α be a smooth $(k-1)$-form on M. Then*

$$\int_M \mathbf{d}\alpha = \int_{\partial M} \alpha. \tag{4.4.1}$$

Special cases of Stokes' theorem are as follows:

The Integral Theorems of Calculus. Stokes' theorem generalizes and synthesizes the classical theorems of calculus:

(a) Fundamental Theorem of Calculus.

$$\int_a^b f'(x)\, dx = f(b) - f(a). \tag{4.4.2}$$

(b) Green's Theorem. For a region $\Omega \subset \mathbb{R}^2$,

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_{\partial\Omega} P\, dx + Q\, dy. \tag{4.4.3}$$

(c) Divergence Theorem. For a region $\Omega \subset \mathbb{R}^3$,

$$\iiint_\Omega \operatorname{div} \mathbf{F}\, dV = \iint_{\partial\Omega} \mathbf{F} \cdot n\, dA. \tag{4.4.4}$$

(d) Classical Stokes' Theorem. For a surface $S \subset \mathbb{R}^3$,

$$\begin{aligned} \iint_S \bigg\{ &\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz \\ &+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \bigg\} \\ &= \iint_S \mathbf{n} \cdot \operatorname{curl} \mathbf{F}\, dA = \int_{\partial S} P\, dx + Q\, dy + R\, dz, \end{aligned} \tag{4.4.5}$$

where $\mathbf{F} = (P, Q, R)$.

Notice that the Poincaré lemma generalizes the vector calculus theorems in $\mathbb{R}^3$, saying that if $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F} = \nabla f$, and if $\operatorname{div} \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$. Recall that it states that *if α is closed, then locally α is exact; that is, if $\mathbf{d}\alpha = 0$, then locally $\alpha = \mathbf{d}\beta$ for some β.* On contractible manifolds these statements hold globally.

Cohomology. The failure of closed forms to be globally exact leads to the study of a very important topological invariant of M, the ***de Rham cohomology***. The kth de Rham cohomology group, denoted by $H^k(M)$, is defined by

$$H^k(M) := \frac{\ker(\mathbf{d} : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{range}(\mathbf{d} : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

The de Rham theorem states that these Abelian groups are isomorphic to the so-called singular cohomology groups of M defined in algebraic topology in terms of simplices and that depend only on the topological structure of M and not on its differentiable structure. The isomorphism is provided by integration; the fact that the integration map drops to the preceding quotient is guaranteed by Stokes' theorem. A useful particular case of this theorem is the following: If M is an orientable compact boundaryless n-manifold, then $\int_M \mu = 0$ if and only if the n-form μ is exact. This statement is equivalent to $H^n(M) = \mathbb{R}$ for M compact and orientable.

Change of Variables. Another basic result in integration theory is the global change of variables formula.

Theorem 4.4.2 (Change of Variables)**.** *Let M and N be oriented n-manifolds and let $\varphi : M \to N$ be an orientation-preserving diffeomorphism. If α is an n-form on N (with, say, compact support), then*

$$\int_M \varphi^* \alpha = \int_N \alpha.$$

Identities for Vector Fields and Forms

1. Vector fields on M with the bracket $[X, Y]$ form a ***Lie algebra***; that is, $[X, Y]$ is real bilinear, skew-symmetric, and ***Jacobi's identity*** holds:

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0.$$

Locally,

$$[X, Y] = \mathbf{D}Y \cdot X - \mathbf{D}X \cdot Y = (X \cdot \nabla)Y - (Y \cdot \nabla)X,$$

and on functions,

$$[X, Y][f] = X[Y[f]] - Y[X[f]].$$

2. For diffeomorphisms φ and ψ,

$$\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y] \quad \text{and} \quad (\varphi \circ \psi)_* X = \varphi_* \psi_* X.$$

3. The forms on a manifold comprise a real associative algebra with $\wedge$ as multiplication. Furthermore, $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for k- and l-forms α and β, respectively.

4. For maps φ and ψ,

$$\varphi^*(\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta \quad \text{and} \quad (\varphi \circ \psi)^* \alpha = \psi^* \varphi^* \alpha.$$

5. $\mathbf{d}$ is a real linear map on forms, $\mathbf{dd}\alpha = 0$, and

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta$$

for α a k-form.

6. For α a k-form and $X_0, \ldots, X_k$ vector fields,

$$\begin{aligned}(\mathbf{d}\alpha)(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i[\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)] \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k),\end{aligned}$$

where $\hat{X}_i$ means that X_i is omitted. Locally,

$$\mathbf{d}\alpha(x)(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i \mathbf{D}\alpha(x) \cdot v_i(v_0, \ldots, \hat{v}_i, \ldots, v_k).$$

7. For a map φ,

$$\varphi^* \mathbf{d}\alpha = \mathbf{d}\varphi^*\alpha.$$

8. Poincaré Lemma. If $\mathbf{d}\alpha = 0$, then the k-form α is locally exact; that is, there is a neighborhood U about each point on which $\alpha = \mathbf{d}\beta$. This statement is global on contractible manifolds or more generally if $H^k(M) = 0$.

9. $\mathbf{i}_X\alpha$ is real bilinear in X, α, and for $h : M \to \mathbb{R}$,

$$\mathbf{i}_{hX}\alpha = h\mathbf{i}_X\alpha = \mathbf{i}_X h\alpha.$$

Also, $\mathbf{i}_X\mathbf{i}_X\alpha = 0$ and

$$\mathbf{i}_X(\alpha \wedge \beta) = \mathbf{i}_X\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{i}_X\beta$$

for α a k-form.

10. For a diffeomorphism φ,

$$\varphi^*(\mathbf{i}_X\alpha) = \mathbf{i}_{\varphi^*X}(\varphi^*\alpha), \quad \text{i.e.,} \quad \varphi^*(X \mathbin{\lrcorner} \alpha) = (\varphi^*X) \mathbin{\lrcorner} (\varphi^*\alpha).$$

If $f : M \to N$ is a mapping and Y is f-related to X, that is,

$$Tf \circ X = Y \circ f,$$

then

$$\mathbf{i}_X f^*\alpha = f^*\mathbf{i}_Y\alpha; \quad \text{i.e.,} \quad X \mathbin{\lrcorner} (f^*\alpha) = f^*(Y \mathbin{\lrcorner} \alpha).$$

11. $\pounds_X\alpha$ is real bilinear in X, α and

$$\pounds_X(\alpha \wedge \beta) = \pounds_X\alpha \wedge \beta + \alpha \wedge \pounds_X\beta.$$

12. Cartan's Magic Formula:

$$\pounds_X\alpha = \mathbf{d}\mathbf{i}_X\alpha + \mathbf{i}_X\mathbf{d}\alpha = \mathbf{d}(X \mathbin{\lrcorner} \alpha) + X \mathbin{\lrcorner} \mathbf{d}\alpha.$$

13. For a diffeomorphism φ,

$$\varphi^*\pounds_X\alpha = \pounds_{\varphi^*X}\varphi^*\alpha.$$

If $f : M \to N$ is a mapping and Y is f-related to X, then

$$\pounds_Y f^*\alpha = f^*\pounds_X\alpha.$$

14. $$(\pounds_X\alpha)(X_1,\ldots,X_k) = X[\alpha(X_1,\ldots,X_k)] - \sum_{i=0}^{k}\alpha(X_1,\ldots,[X,X_i],\ldots,X_k).$$

Locally,

$$(\pounds_X\alpha)(x)\cdot(v_1,\ldots,v_k) = (\mathbf{D}\alpha_x\cdot X(x))(v_1,\ldots,v_k) + \sum_{i=0}^{k}\alpha_x(v_1,\ldots,\mathbf{D}X_x\cdot v_i,\ldots,v_k).$$

15. The following identities hold:

(a) $\pounds_{fX}\alpha = f\pounds_X\alpha + \mathbf{d}f\wedge\mathbf{i}_X\alpha$;

(b) $\pounds_{[X,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$;

(c) $\mathbf{i}_{[X,Y]}\alpha = \pounds_X\mathbf{i}_Y\alpha - \mathbf{i}_Y\pounds_X\alpha$;

(d) $\pounds_X\mathbf{d}\alpha = \mathbf{d}\pounds_X\alpha$;

(e) $\pounds_X\mathbf{i}_X\alpha = \mathbf{i}_X\pounds_X\alpha$;

(f) $\pounds_X(\alpha\wedge\beta) = \pounds_X\alpha\wedge\beta + \alpha\wedge\pounds_X\beta$.

16. If M is a finite-dimensional manifold, $X = X^l\partial/\partial x^l$, and

$$\alpha = \alpha_{i_1\ldots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k},$$

where $i_1<\cdots<i_k$, then the following formulas hold:

$$\begin{aligned}
\mathbf{d}\alpha &= \left(\frac{\partial\alpha_{i_1\ldots i_k}}{\partial x^l}\right)dx^l\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k},\\
\mathbf{i}_X\alpha &= X^l\alpha_{li_2\ldots i_k}dx^{i_2}\wedge\cdots\wedge dx^{i_k},\\
\pounds_X\alpha &= X^l\left(\frac{\partial\alpha_{i_1\ldots i_k}}{\partial x^l}\right)dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&\quad + \alpha_{li_2\ldots i_k}\left(\frac{\partial X^l}{\partial x^{i_1}}\right)dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}+\ldots.
\end{aligned}$$

Exercises

◇ **4.4-1.** Let Ω be a closed bounded region in $\mathbb{R}^2$. Use Green's theorem to show that the area of Ω equals the line integral

$$\frac{1}{2}\int_{\partial\Omega}(x\,dy - y\,dx).$$

◇ **4.4-2.** On $\mathbb{R}^2 \backslash \{(0,0)\}$ consider the one-form

$$\alpha = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

(a) Show that this form is closed.

(b) Using the angle θ as a variable on S^1, compute $i^*\alpha$, where $i : S^1 \to \mathbb{R}^2$ is the standard embedding.

(c) Show that α is not exact.

◇ **4.4-3** (The Magnetic Monopole)**.** Let $\mathbf{B} = g\mathbf{r}/r^3$ be a vector field on Euclidean three-space minus the origin where $r = \|\mathbf{r}\|$. Show that $\mathbf{B}$ cannot be written as the curl of something.