# 3

# An Introduction to Infinite-Dimensional Systems

A common choice of configuration space for classical field theory is an infinite-dimensional vector space of functions or tensor fields on space or spacetime, the elements of which are called ***fields***. Here we relate our treatment of infinite-dimensional Hamiltonian systems discussed in §2.1 to classical Lagrangian and Hamiltonian field theory and then give examples. Classical field theory is a large subject with many aspects not covered here; we treat only a few topics that are basic to subsequent developments; see Chapters 6 and 7 for additional information and references.

## 3.1 Lagrange's and Hamilton's Equations for Field Theory

As with finite-dimensional systems, one can begin with a Lagrangian and a variational principle, and then pass to the Hamiltonian via the Legendre transformation. At least formally, all the constructions we did in the finite-dimensional case go over to the infinite-dimensional one.

For instance, suppose we choose our configuration space $Q = \mathcal{F}(\mathbb{R}^3)$ to be the space of fields $\varphi$ on $\mathbb{R}^3$. Our Lagrangian will be a function $L(\varphi, \dot{\varphi})$ from $Q \times Q$ to $\mathbb{R}$. The variational principle is

$$\delta \int_a^b L(\varphi, \dot{\varphi})\, dt = 0, \tag{3.1.1}$$

which is equivalent to the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{\varphi}} = \frac{\delta L}{\delta \varphi} \tag{3.1.2}$$

in the usual way. Here,

$$\pi = \frac{\delta L}{\delta \dot{\varphi}} \tag{3.1.3}$$

is the conjugate momentum, which we regard as a density on $\mathbb{R}^3$ as in Chapter 2. The corresponding Hamiltonian is

$$H(\varphi, \pi) = \int \pi\dot{\varphi} - L(\varphi, \dot{\varphi}), \tag{3.1.4}$$

in accordance with our general theory. We also know that the Hamiltonian should generate the canonical Hamilton equations. We verify this now.

**Proposition 3.1.1.** *Let $Z = \mathcal{F}(\mathbb{R}^3) \times \mathrm{Den}(\mathbb{R}^3)$, with $\Omega$ defined as in Example* (b) *of* §2.2. *Then the Hamiltonian vector field $X_H : Z \to Z$ corresponding to a given energy function $H : Z \to \mathbb{R}$ is given by*

$$X_H = \left(\frac{\delta H}{\delta \pi}, -\frac{\delta H}{\delta \varphi}\right). \tag{3.1.5}$$

*Hamilton's equations on $Z$ are*

$$\frac{\partial \varphi}{\partial t} = \frac{\delta H}{\delta \pi}, \quad \frac{\partial \pi}{\partial t} = -\frac{\delta H}{\delta \varphi}. \tag{3.1.6}$$

**Remarks.**

**1.** The symbols $\mathcal{F}$ and Den stand for function spaces included in the space of all functions and densities, chosen to be appropriate to the functional-analytic needs of the particular problem. In practice this often means, among other things, that appropriate conditions at infinity are imposed to permit integration by parts.

**2.** The equations of motion for a curve $z(t) = (\varphi(t), \pi(t))$ written in the form $\Omega(dz/dt, \delta z) = \mathbf{d}H(z(t)) \cdot \delta z$ for all $\delta z \in Z$ with compact support are called the ***weak form of the equations of motion***. They can still be valid when there is not enough smoothness or decay at infinity to justify the literal equality $dz/dt = X_H(z)$; this situation can occur, for example, if one is considering shock waves. ♦

**Proof of Proposition 3.1.1.** To derive the partial functional derivatives, we use the natural pairing

$$\langle \, , \rangle : \mathcal{F}(\mathbb{R}^3) \times \mathrm{Den}(\mathbb{R}^3) \to \mathbb{R}, \quad \text{where} \quad \langle \varphi, \pi \rangle = \int \varphi \pi' \, d^3x, \tag{3.1.7}$$

where we write $\pi = \pi' d^3x \in \text{Den}$. Recalling that $\delta H/\delta\varphi$ is a density, let

$$X = \left( \frac{\delta H}{\delta \pi}, -\frac{\delta H}{\delta \varphi} \right).$$

We need to verify that $\Omega(X(\varphi, \pi), (\delta\varphi, \delta\pi)) = \mathbf{d}H(\varphi, \pi) \cdot (\delta\varphi, \delta\pi)$. Indeed,

$$\begin{aligned}
\Omega(X(\varphi, \pi), (\delta\varphi, \delta\pi)) &= \Omega\left( \left( \frac{\delta H}{\delta \pi}, -\frac{\delta H}{\delta \varphi} \right), (\delta\varphi, \delta\pi) \right) \\
&= \int \frac{\delta H}{\delta \pi} (\delta\pi)' d^3x + \int \delta\varphi \left( \frac{\delta H}{\delta \varphi} \right)' d^3x \\
&= \left\langle \frac{\delta H}{\delta \pi}, \delta\pi \right\rangle + \left\langle \delta\varphi, \frac{\delta H}{\delta \varphi} \right\rangle \\
&= \mathbf{D}_\pi H(\varphi, \pi) \cdot \delta\pi + \mathbf{D}_\varphi H(\varphi, \pi) \cdot \delta\varphi \\
&= \mathbf{d}H(\varphi, \pi) \cdot (\delta\varphi, \delta\pi).
\end{aligned}$$

■

## 3.2 Examples: Hamilton's Equations

**(a) The Wave Equation.** Consider $Z = \mathcal{F}(\mathbb{R}^3) \times \text{Den}(\mathbb{R}^3)$ as above. Let $\varphi$ denote the configuration variable, that is, the first component in the phase space $\mathcal{F}(\mathbb{R}^3) \times \text{Den}(\mathbb{R}^3)$, and interpret $\varphi$ as a measure of the displacement from equilibrium of a homogeneous elastic medium. Writing $\pi' = \rho\, d\varphi/dt$, where $\rho$ is the mass density, the ***kinetic energy*** is

$$T = \frac{1}{2} \int \frac{1}{\rho} [\pi']^2 \, d^3x.$$

For small displacements $\varphi$, one assumes a linear restoring force such as the one given by the ***potential energy***

$$\frac{k}{2} \int \|\nabla\varphi\|^2 \, d^3x,$$

for an (elastic) constant $k$.

Because we are considering a homogeneous medium, $\rho$ and $k$ are constants, so let us work in units in which they are unity. Nonlinear effects can be modeled in a naive way by introducing a nonlinear term, $U(\varphi)$, into the potential. However, for an elastic medium one really should use constitutive relations based on the principles of continuum mechanics; see Marsden and Hughes [1983]. For the naive model, the Hamiltonian $H : Z \to \mathbb{R}$ is the ***total energy***

$$H(\varphi, \pi) = \int \left[ \frac{1}{2} (\pi')^2 + \frac{1}{2} \|\nabla\varphi\|^2 + U(\varphi) \right] d^3x. \tag{3.2.1}$$

Using the definition of the functional derivative, we find that

$$\frac{\delta H}{\delta \pi} = \pi', \quad \frac{\delta H}{\delta \varphi} = (-\nabla^2 \varphi + U'(\varphi)) d^3 x. \tag{3.2.2}$$

Therefore, the equations of motion are

$$\frac{\partial \varphi}{\partial t} = \pi', \quad \frac{\partial \pi'}{\partial t} = \nabla^2 \varphi - U'(\varphi), \tag{3.2.3}$$

or, in second-order form,

$$\frac{\partial^2 \varphi}{\partial t^2} = \nabla^2 \varphi - U'(\varphi). \tag{3.2.4}$$

Various choices of $U$ correspond to various physical applications. When $U' = 0$, we get the linear wave equation, with unit propagation velocity. Another choice, $U(\varphi) = (1/2) m^2 \varphi^2 + \lambda \varphi^4$, occurs in the quantum theory of self-interacting mesons; the parameter $m$ is related to the meson mass, and $\varphi^4$ governs the nonlinear part of the interaction. When $\lambda = 0$, we get

$$\nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial t^2} = m^2 \varphi, \tag{3.2.5}$$

which is called the ***Klein–Gordon equation***.

♦

**Technical Aside.** For the wave equation, one appropriate choice of function space is $Z = H^1(\mathbb{R}^3) \times L^2_{\mathrm{Den}}(\mathbb{R}^3)$, where $H^1(\mathbb{R}^3)$ denotes the $H^1$-functions on $\mathbb{R}^3$, that is, functions that, along with their first derivatives are square integrable, and $L^2_{\mathrm{Den}}(\mathbb{R}^3)$ denotes the space of densities $\pi = \pi' \, d^3x$, where the function $\pi'$ on $\mathbb{R}^3$ is square integrable. Note that the Hamiltonian vector field

$$X_H(\varphi, \pi) = (\pi', (\nabla^2 \varphi - U'(\varphi)) d^3 x)$$

is defined only on the dense subspace $H^2(\mathbb{R}^3) \times H^1_{\mathrm{Den}}(\mathbb{R}^3)$ of $Z$. This is a common occurrence in the study of Hamiltonian partial differential equations; we return to this in §3.3. ♦

In the preceding example, $\Omega$ was given by the canonical form with the result that the equations of motion were in the standard form (3.1.5). In addition, the Hamiltonian function was given by the actual energy of the system under consideration. We now give examples in which these statements require reinterpretation but that nevertheless fall into the framework of the general theory developed so far.

**(b) The Schrödinger Equation.** Let $\mathcal{H}$ be a complex Hilbert space, for example, the space of complex-valued functions $\psi$ on $\mathbb{R}^3$ with the Hermitian inner product

$$\langle \psi_1, \psi_2 \rangle = \int \psi_1(x)\overline{\psi}_2(x)\, d^3x,$$

where the overbar denotes complex conjugation. For a self-adjoint complex-linear operator $H_{\rm op} : \mathcal{H} \to \mathcal{H}$, the Schrödinger equation is

$$i\hbar \frac{\partial \psi}{\partial t} = H_{\rm op}\psi, \tag{3.2.6}$$

where $\hbar$ is Planck's constant. Define

$$A = \frac{-i}{\hbar} H_{\rm op},$$

so that the Schrödinger equation becomes

$$\frac{\partial \psi}{\partial t} = A\psi. \tag{3.2.7}$$

The symplectic form on $\mathcal{H}$ is given by $\Omega(\psi_1, \psi_2) = -2\hbar \operatorname{Im} \langle \psi_1, \psi_2 \rangle$. Self-adjointness of $H_{\rm op}$ is a condition stronger than symmetry and is essential for proving well-posedness of the initial-value problem for (3.2.6); for an exposition, see, for instance, Abraham, Marsden, and Ratiu [1988]. Historically, it was Kato [1950] who established self-adjointness for important problems such as the hydrogen atom.

From §2.5 we know that since $H_{\rm op}$ is symmetric, $A$ is Hamiltonian. The Hamiltonian is

$$H(\psi) = \hbar \langle iA\psi, \psi \rangle = \langle H_{\rm op}\psi, \psi \rangle, \tag{3.2.8}$$

which is the ***expectation value*** of $H_{\rm op}$ at $\psi$, defined by $\langle H_{\rm op} \rangle (\psi) = \langle H_{\rm op}\psi, \psi \rangle$. ♦

**(c) The Korteweg–de Vries (KdV) Equation.** Denote by $Z$ the vector subspace $\mathcal{F}(\mathbb{R})$ consisting of those functions $u$ with $|u(x)|$ decreasing sufficiently fast as $x \to \pm\infty$ that the integrals we will write are defined and integration by parts is justified. As we shall see later, the Poisson brackets for the KdV equation are quite simple, and historically they were found before the symplectic structure (see Gardner [1971] and Zakharov [1971, 1974]). To be consistent with our exposition, we begin with the somewhat more complicated symplectic structure. Pair $Z$ with itself using the $L^2$ inner product. Let the KdV symplectic structure $\Omega$ be defined by

$$\Omega(u_1, u_2) = \frac{1}{2} \left( \int_{-\infty}^{\infty} [\hat{u}_1(x)u_2(x) - \hat{u}_2(x)u_1(x)]\, dx \right), \tag{3.2.9}$$

where $\hat{u}$ denotes a primitive of $u$, that is,

$$\hat{u} = \int_{-\infty}^{x} u(y)\, dy.$$

In §8.5 we shall see a way to *construct* this form. The form $\Omega$ is clearly skew-symmetric. Note that if $u_1 = \partial v/\partial x$ for some $v \in Z$, then

$$\begin{aligned}
&\int_{-\infty}^{\infty} \hat{u}_2(x) u_1(x)\, dx \\
&\quad = \int_{-\infty}^{\infty} \hat{u}_2(x) \frac{\partial \hat{u}_1(x)}{\partial x}\, dx \\
&\quad = \hat{u}_1(x)\hat{u}_2(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \hat{u}_1(x) u_2(x)\, dx \\
&\quad = \left(\int_{-\infty}^{\infty} \frac{\partial v(x)}{\partial x}\, dx\right)\left(\int_{-\infty}^{\infty} u_2(x)\, dx\right) - \int_{-\infty}^{\infty} \hat{u}_1(x) u_2(x)\, dx \\
&\quad = \left( v(x)\Big|_{-\infty}^{\infty}\right)\left(\int_{-\infty}^{\infty} u_2(x)\, dx\right) - \int_{-\infty}^{\infty} \hat{u}_1(x) u_2(x)\, dx \\
&\quad = -\int_{-\infty}^{\infty} \hat{u}_1(x) u_2(x)\, dx.
\end{aligned}$$

Thus, if $u_1(x) = \partial v(x)/\partial x$, then $\Omega$ can be written as

$$\Omega(u_1, u_2) = \int_{-\infty}^{\infty} \hat{u}_1(x) u_2(x)\, dx = \int_{-\infty}^{\infty} v(x) u_2(x)\, dx. \tag{3.2.10}$$

To prove weak nondegeneracy of $\Omega$, we check that if $v \neq 0$, there is a $w$ such that $\Omega(w, v) \neq 0$. Indeed, if $v \neq 0$ and we let $w = \partial v/\partial x$, then $w \neq 0$ because $v(x) \to 0$ as $|x| \to \infty$. Hence by (3.2.10),

$$\Omega(w, v) = \Omega\left(\frac{\partial v}{\partial x}, v\right) = \int_{-\infty}^{\infty} (v(x))^2\, dx \neq 0.$$

Suppose that a Hamiltonian $H : Z \to \mathbb{R}$ is given. We claim that the corresponding Hamiltonian vector field $X_H$ is given by

$$X_H(u) = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u}\right). \tag{3.2.11}$$

Indeed, by (3.2.10),

$$\Omega(X_H(v), w) = \int_{-\infty}^{\infty} \frac{\delta H}{\delta v}(x) w(x)\, dx = \mathbf{d}H(v) \cdot w.$$

It follows from (3.2.11) that the corresponding Hamilton equations are

$$u_t = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u}\right), \tag{3.2.12}$$

where, in (3.2.12) and in the following, subscripts denote derivatives with respect to the subscripted variable. As a special case, consider the function

$$H_1(u) = -\frac{1}{6}\int_{-\infty}^{\infty} u^3 \, dx.$$

Then

$$\frac{\partial}{\partial x}\frac{\delta H_1}{\delta u} = -uu_x,$$

and so (3.2.12) becomes the ***one-dimensional transport equation***

$$u_t + uu_x = 0. \tag{3.2.13}$$

Next, let

$$H_2(u) = \int_{-\infty}^{\infty}\left(\frac{1}{2}u_x^2 - u^3\right) dx; \tag{3.2.14}$$

then (3.2.12) becomes

$$u_t + 6uu_x + u_{xxx} = 0. \tag{3.2.15}$$

This is the ***Korteweg–de Vries (KdV) equation*** that describes shallow water waves. For a concise presentation of its famous complete set of integrals, see Abraham and Marsden [1978], §6.5, and for more information, see Newell [1985]. The first few of its integrals are given in Exercise 3.3-1. We will return to this example from time to time in the text, but for now we will find traveling wave solutions of the KdV equation.

**Traveling Waves.** If we look for traveling wave solutions of (3.2.15), that is, $u(x,t) = \varphi(x - ct)$, for a constant $c > 0$ and a positive function $\varphi$, we see that $u$ satisfies the KdV equation if and only if $\varphi$ satisfies

$$c\varphi' - 6\varphi\varphi' - \varphi''' = 0. \tag{3.2.16}$$

Integrating once gives

$$c\varphi - 3\varphi^2 - \varphi'' = C, \tag{3.2.17}$$

where $C$ is a constant. This equation is Hamiltonian in the canonical variables $(\varphi, \varphi')$ with Hamiltonian function

$$h(\varphi, \varphi') = \frac{1}{2}(\varphi')^2 - \frac{c}{2}\varphi^2 + \varphi^3 + C\varphi. \tag{3.2.18}$$

From conservation of energy, $h(\varphi, \varphi') = D$, it follows that

$$\varphi' = \pm\sqrt{c\varphi^2 - 2\varphi^3 - 2C\varphi + 2D}, \tag{3.2.19}$$

or, writing $s = x - ct$, we get

$$s = \pm \int \frac{d\varphi}{\sqrt{c\varphi^2 - 2\varphi^3 - 2C\varphi + 2D}}. \tag{3.2.20}$$

We seek solutions that together with their derivatives vanish at $\pm\infty$. Then (3.2.17) and (3.2.19) give $C = D = 0$, so

$$s = \pm \int \frac{d\varphi}{\sqrt{c\varphi^2 - 2\varphi^3}} = \pm \frac{1}{\sqrt{c}} \log \left| \frac{\sqrt{c - 2\varphi} - \sqrt{c}}{\sqrt{c - 2\varphi} + \sqrt{c}} \right| + K \tag{3.2.21}$$

for some constant $K$ that will be determined below.

For $C = D = 0$, the Hamiltonian (3.2.18) becomes

$$h(\varphi, \varphi') = \frac{1}{2}(\varphi')^2 - \frac{c}{2}\varphi^2 + \varphi^3, \tag{3.2.22}$$

and thus the two equilibria given by $\partial h/\partial\varphi = 0$ and $\partial h/\partial\varphi' = 0$ are $(0, 0)$ and $(c/3, 0)$. The matrix of the linearized Hamiltonian system at these equilibria is

$$\begin{bmatrix} 0 & 1 \\ \pm c & 0 \end{bmatrix},$$

which shows that $(0, 0)$ is a saddle and $(c/3, 0)$ is spectrally stable. The second variation criterion on the potential energy (see §1.10) $-c\varphi^2/2 + \varphi^3$ at $(c/3, 0)$ shows that this equilibrium is stable. Thus, if $(\varphi(s), \varphi'(s))$ is a homoclinic orbit emanating and ending at $(0, 0)$, the value of the Hamiltonian function (3.2.22) on it is $H(0, 0) = 0$. From (3.2.22) it follows that $(c/2, 0)$ is a point on this homoclinic orbit, and thus (3.2.20) for $C = D = 0$ is its expression. Taking the initial condition of this orbit at $s = 0$ to be $\varphi(0) = c/2$, $\varphi'(0) = 0$, (3.2.21) forces $K = 0$, and so

$$\left| \frac{\sqrt{c - 2\varphi} - \sqrt{c}}{\sqrt{c - 2\varphi} + \sqrt{c}} \right| = e^{\pm\sqrt{c}s}.$$

Since $\varphi \geq 0$ by hypothesis, the expression in the absolute value is negative, and thus

$$\frac{\sqrt{c - 2\varphi} - \sqrt{c}}{\sqrt{c - 2\varphi} + \sqrt{c}} = -e^{\pm\sqrt{c}s},$$

whose solution is

$$\varphi(s) = \frac{2ce^{\pm\sqrt{c}s}}{(1 + e^{\pm\sqrt{c}s})^2} = \frac{c}{2\cosh^2(\sqrt{c}s/2)}.$$

This produces the ***soliton solution***

$$u(x, t) = \frac{c}{2} \operatorname{sech}^2 \left[ \frac{\sqrt{c}}{2}(x - ct) \right]. \qquad \blacklozenge$$

**(d) Sine–Gordon Equation.** For functions $u(x, t)$, where $x$ and $t$ are real variables, the ***sine–Gordon equation*** is $u_{tt} = u_{xx} + \sin u$. Equation (3.2.4) shows that it is Hamiltonian with the momentum density $\pi = u_t\, dx$ (and associated function $\pi' = u_t$),

$$H(u) = \int_{-\infty}^{\infty} \left( \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \cos u \right) dx, \tag{3.2.23}$$

and the canonical bracket structure, as in the wave equation. This equation also has a complete set of integrals; see again Newell [1985]. ♦

**(e) Abstract Wave Equation.** Let $\mathcal{H}$ be a real Hilbert space and $B : \mathcal{H} \to \mathcal{H}$ a linear operator. On $\mathcal{H} \times \mathcal{H}$ put the symplectic structure $\Omega$ given by (2.2.6). One can check that:

(i) $A = \begin{bmatrix} 0 & I \\ -B & 0 \end{bmatrix}$ is $\Omega$-skew if and only if $B$ is a symmetric operator on $\mathcal{H}$; and

(ii) if $B$ is symmetric, then a Hamiltonian for $A$ is

$$H(x, y) = \frac{1}{2}(\|y\|^2 + \langle Bx, x \rangle). \tag{3.2.24}$$

The equations of motion (2.4.10) give the ***abstract wave equation***

$$\ddot{x} + Bx = 0.$$ ♦

**(f) Linear Elastodynamics.** On $\mathbb{R}^3$ consider the equations

$$\rho \mathbf{u}_{tt} = \operatorname{div}(\mathbf{c} \cdot \nabla \mathbf{u}),$$

that is,

$$\rho u_{tt}^i = \frac{\partial}{\partial x^j} \left[ c^{ijkl} \frac{\partial u^k}{\partial x^l} \right], \tag{3.2.25}$$

where $\rho$ is a positive function and $\mathbf{c}$ is a fourth-order tensor field (the ***elasticity tensor***) on $\mathbb{R}^3$ with the symmetries $c^{ijkl} = c^{klij} = c^{jikl}$.

On $\mathcal{F}(\mathbb{R}^3; \mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3; \mathbb{R}^3)$ (or, more precisely, on

$$H^1(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$$

with suitable decay properties at infinity) define

$$\Omega((\mathbf{u}, \dot{\mathbf{u}}), (\mathbf{v}, \dot{\mathbf{v}})) = \int_{\mathbb{R}^3} \rho(\dot{\mathbf{v}} \cdot \mathbf{u} - \dot{\mathbf{u}} \cdot \mathbf{v})\, d^3x. \tag{3.2.26}$$

The form $\Omega$ is the canonical symplectic form (2.2.3) for fields $\mathbf{u}$ and their conjugate momenta $\pi = \rho\dot{\mathbf{u}}$.

On the space of functions $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$, consider the $\rho$-weighted $L^2$-inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_\rho = \int_{\mathbb{R}^3} \rho \mathbf{u} \cdot \mathbf{v} \, d^3x. \tag{3.2.27}$$

Then the operator $B\mathbf{u} = -(1/\rho)\,\mathrm{div}(\mathbf{c} \cdot \nabla \mathbf{u})$ is symmetric with respect to this inner product, and thus by Example (e) above, the operator $A(\mathbf{u}, \dot{\mathbf{u}}) = (\dot{\mathbf{u}}, (1/\rho)\,\mathrm{div}(\mathbf{c} \cdot \nabla \mathbf{u}))$ is $\Omega$-skew.

The equations (3.2.25) of linear elastodynamics are checked to be Hamiltonian with respect to $\Omega$ given by (3.2.26), and with energy

$$H(\mathbf{u}, \dot{\mathbf{u}}) = \frac{1}{2} \int \rho \|\dot{\mathbf{u}}\|^2 \, d^3x + \frac{1}{2} \int c^{ijkl} e_{ij} e_{kl} \, d^3x, \tag{3.2.28}$$

where

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right). \qquad \blacklozenge$$

## Exercises

◇ **3.2-1.**

(a) Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$. Show directly that the sine–Gordon equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi + \sin \varphi = 0$$

is the Euler–Lagrange equation of a suitable Lagrangian.

(b) Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{C}$. Write the nonlinear Schrödinger equation

$$i \frac{\partial \varphi}{\partial t} + \nabla^2 \varphi + \beta \varphi |\varphi|^2 = 0$$

as a Hamiltonian system.

◇ **3.2-2.** Find a "soliton" solution for the sine–Gordon equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + \sin \varphi = 0$$

in one spatial dimension.

◇ **3.2-3.** Consider the complex nonlinear Schrödinger equation in one spatial dimension:

$$i\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} + \beta\varphi|\varphi|^2 = 0, \quad \beta \neq 0.$$

(a) Show that the function $\psi : \mathbb{R} \to \mathbb{C}$ defining the traveling wave solution $\varphi(x,t) = \psi(x - ct)$ for $c > 0$ satisfies a second-order complex differential equation equivalent to a Hamiltonian system in $\mathbb{R}^4$ relative to the noncanonical symplectic form whose matrix is given by

$$\mathbb{J}_c = \begin{bmatrix} 0 & c & 1 & 0 \\ -c & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

(See Exercise 2.4-1.)

(b) Analyze the equilibria of the resulting Hamiltonian system in $\mathbb{R}^4$ and determine their linear stability properties.

(c) Let $\psi(s) = e^{ics/2}a(s)$ for a real function $a(s)$ and determine a second-order equation for $a(s)$. Show that the resulting equation is Hamiltonian and has heteroclinic orbits for $\beta < 0$. Find them.

(d) Find "soliton" solutions for the complex nonlinear Schrödinger equation.

## 3.3 Examples: Poisson Brackets and Conserved Quantities

Before proceeding with infinite-dimensional examples, it is first useful to recall some basic facts about angular momentum of particles in $\mathbb{R}^3$. (The reader should supply a corresponding discussion for linear momentum.) Consider a particle moving in $\mathbb{R}^3$ under the influence of a potential $V$. Let the position coordinate be denoted by $\mathbf{q}$, so that Newton's second law reads

$$m\ddot{\mathbf{q}} = -\nabla V(\mathbf{q}).$$

Let $\mathbf{p} = m\dot{\mathbf{q}}$ be the linear momentum and $\mathbf{J} = \mathbf{q} \times \mathbf{p}$ be the angular momentum. Then

$$\frac{d}{dt}\mathbf{J} = \dot{\mathbf{q}} \times \mathbf{p} + \mathbf{q} \times \dot{\mathbf{p}} = -\mathbf{q} \times \nabla V(\mathbf{q}).$$

If $V$ is radially symmetric, it is a function of $\|\mathbf{q}\|$ alone: assume

$$V(\mathbf{q}) = f(\|\mathbf{q}\|^2),$$

where $f$ is a smooth function (exclude $\mathbf{q} = \mathbf{0}$ if necessary). Then

$$\nabla V(\mathbf{q}) = 2f'(\|\mathbf{q}\|^2)\mathbf{q},$$

so that $\mathbf{q} \times \nabla V(\mathbf{q}) = 0$. Thus, in this case, $d\mathbf{J}/dt = 0$, so $\mathbf{J}$ is conserved.

Alternatively, with

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2m}\|\mathbf{p}\|^2 + V(\mathbf{q}),$$

we can check directly that $\{H, J_l\} = 0$ for $l = 1, 2, 3$, where $\mathbf{J} = (J_1, J_2, J_3)$. This also shows that each component $J_l$ is conserved by the Hamiltonian dynamics determined by $H$.

Additional insight is gained by looking at the components of $\mathbf{J}$ more closely. For example, consider the scalar function

$$F(\mathbf{q}, \mathbf{p}) = \mathbf{J}(\mathbf{q}, \mathbf{p}) \cdot \omega\mathbf{k},$$

where $\omega$ is a constant and $\mathbf{k} = (0, 0, 1)$. We find that

$$F(\mathbf{q}, \mathbf{p}) = \omega(q^1 p_2 - p_1 q^2).$$

The Hamiltonian vector field of $F$ is

$$\begin{aligned} X_F(\mathbf{q}, \mathbf{p}) &= \left( \frac{\partial F}{\partial p_1}, \frac{\partial F}{\partial p_2}, \frac{\partial F}{\partial p_3}, -\frac{\partial F}{\partial q^1}, -\frac{\partial F}{\partial q^2}, -\frac{\partial F}{\partial q^3} \right) \\ &= (-\omega q^2, \omega q^1, 0, -\omega p_2, \omega p_1, 0). \end{aligned}$$

Note that $X_F$ is just the vector field corresponding to the flow in the $(q^1, q^2)$ plane and the $(p_1, p_2)$ plane given by rotations about the origin with angular velocity $\omega$. More generally, the Hamiltonian vector field associated with the scalar function defined by $J_{\boldsymbol{\omega}} := \mathbf{J} \cdot \boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is a vector in $\mathbb{R}^3$, has a flow consisting of rotations about the axis $\boldsymbol{\omega}$. As we shall see in Chapters 11 and 12, this is the basis for understanding the link between conservation laws and symmetry more generally.

Another identity is worth noting. Namely, for two vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$,

$$\{J_{\boldsymbol{\omega}_1}, J_{\boldsymbol{\omega}_2}\} = J_{\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2},$$

which, as we shall see later, is an important link between the Poisson bracket structure and the structure of the Lie algebra of the rotation group.

**(a) The Schrödinger Bracket.** In Example (b) of §3.2, we saw that if $H_{\mathrm{op}}$ is a self-adjoint complex linear operator on a Hilbert space $\mathcal{H}$, then $A = H_{\mathrm{op}}/(i\hbar)$ is Hamiltonian, and the corresponding energy function $H_A$ is the expectation value $\langle H_{\mathrm{op}} \rangle$ of $H_{\mathrm{op}}$. Letting $H_{\mathrm{op}}$ and $K_{\mathrm{op}}$ be two such operators, and applying the Poisson bracket–commutator correspondence (2.7.10), or a direct calculation, we get

$$\{\langle H_{\mathrm{op}} \rangle, \langle K_{\mathrm{op}} \rangle\} = \langle [H_{\mathrm{op}}, K_{\mathrm{op}}] \rangle . \tag{3.3.1}$$

In other words, *the expectation value of the commutator is the Poisson bracket of the expectation values.*

Results like this lead one to statements like "Commutators in quantum mechanics are not only *analogous* to Poisson brackets, they *are* Poisson brackets." Even more striking are *true statements* like this: "Don't tell me that quantum mechanics is right and classical mechanics is wrong—after all, quantum mechanics is a *special case* of classical mechanics."

Notice that if we take $K_{\rm op}\psi = \psi$, the identity operator, the corresponding Hamiltonian function is $p(\psi) = \|\psi\|^2$, and from (3.3.1) we see that $p$ is a conserved quantity for any choice of $H_{\rm op}$, a fact that is central to the probabilistic interpretation of quantum mechanics. Later, we shall see that $p$ is the conserved quantity associated to the ***phase symmetry*** $\psi \mapsto e^{i\theta}\psi$.

More generally, if $F$ and $G$ are two functions on $\mathcal{H}$ with $\delta F/\delta\psi = \nabla F$, the gradient of $F$ taken relative to the real inner product $\operatorname{Re}\langle\, ,\rangle$ on $H$, one finds that

$$X_F = \frac{1}{2i\hbar}\nabla F \tag{3.3.2}$$

and

$$\{F, G\} = -\frac{1}{2\hbar}\operatorname{Im}\langle\nabla F, \nabla G\rangle\,. \tag{3.3.3}$$

Notice that (3.3.2), (3.3.3), and $\operatorname{Im} z = -\operatorname{Re}(iz)$ give

$$\begin{aligned}\mathbf{d}F\cdot X_G = \operatorname{Re}\langle\nabla F, X_G\rangle &= \frac{1}{2\hbar}\operatorname{Re}\langle\nabla F, -i\nabla G\rangle\\ &= \frac{1}{2\hbar}\operatorname{Re}\langle i\nabla F, \nabla G\rangle\\ &= -\frac{1}{2\hbar}\operatorname{Im}\langle\nabla F, \nabla G\rangle\\ &= \{F, G\}\end{aligned}$$

as expected. ♦

**(b) KdV Bracket.** Using the definition of the bracket (2.7.1), the symplectic structure, and the Hamiltonian vector field formula from Example (c) of §3.2, one finds that

$$\{F, G\} = \int_{-\infty}^{\infty}\frac{\delta F}{\delta u}\frac{\partial}{\partial x}\left(\frac{\delta G}{\delta u}\right)dx \tag{3.3.4}$$

for functions $F, G$ of $u$ having functional derivatives that vanish at $\pm\infty$. ♦

**(c) Linear and Angular Momentum for the Wave Equation.** The wave equation on $\mathbb{R}^3$ discussed in Example (a) of §3.2 has the Hamiltonian

$$H(\varphi, \pi) = \int_{\mathbb{R}^3}\left[\frac{1}{2}(\pi')^2 + \frac{1}{2}\|\nabla\varphi\|^2 + U(\varphi)\right]d^3x. \tag{3.3.5}$$

Define the ***linear momentum*** in the $x$-direction by

$$P_x(\varphi, \pi) = \int \pi' \frac{\partial \varphi}{\partial x}\, d^3x. \tag{3.3.6}$$

By (3.3.6), $\delta P_x/\delta\pi = \partial\varphi/\partial x$, and $\delta P_x/\delta\varphi = (-\partial\pi'/\partial x)\, d^3x$, so we get from (3.2.2)

$$\begin{aligned}\{H, P_x\}(\varphi, \pi) &= \int_{\mathbb{R}^3} \left( \frac{\delta P_x}{\delta \pi}\frac{\delta H}{\delta \varphi} - \frac{\delta H}{\delta \pi}\frac{\delta P_x}{\delta \varphi}\right) \\ &= \int_{\mathbb{R}^3} \left[ \frac{\partial \varphi}{\partial x}(-\nabla^2\varphi + U'(\varphi)) + \pi' \frac{\partial \pi'}{\partial x}\right] d^3x \\ &= \int_{\mathbb{R}^3} \left[ -\nabla^2 \varphi \frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial x}\left( U(\varphi) + \frac{1}{2}(\pi')^2\right)\right] d^3x \\ &= 0, \end{aligned} \tag{3.3.7}$$

assuming that the fields and $U$ vanish appropriately at $\infty$. (The first term vanishes because it switches sign under integration by parts.) Thus, $P_x$ is conserved. The conservation of $P_x$ is connected with invariance of $H$ under translations in the $x$-direction. Deeper insights into this connection are explored later. Of course, similar conservation laws hold in the $y$- and $z$-directions.

Likewise, the angular momenta $\mathbf{J} = (J_x, J_y, J_z)$, where, for example,

$$J_z(\varphi) = \int_{\mathbb{R}^3} \pi' \left( x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \varphi\, d^3x, \tag{3.3.8}$$

are constants of the motion. This is proved in an analogous way. (For precise function spaces in which these operations can be justified, see Chernoff and Marsden [1974].) ♦

**(d) Linear and Angular Momentum: The Schrödinger Equation.**

**Linear Momentum.** In Example (b) of §3.2, assume that $\mathcal{H}$ is the space of complex-valued $L^2$-functions on $\mathbb{R}^3$ and that the self-adjoint linear operator $H_{\rm op}\colon \mathcal{H} \to \mathcal{H}$ commutes with infinitesimal translations of the argument by a fixed vector $\xi \in \mathbb{R}^3$, that is, $H_{\rm op}(\mathbf{D}\psi(\cdot)\cdot\xi) = \mathbf{D}(H_{\rm op}\psi(\cdot))\cdot\xi$ for any $\psi$ whose derivative is in $\mathcal{H}$. One checks, using (3.3.1), that

$$P_\xi(\psi) = \left\langle \frac{i}{\hbar}\mathbf{D}\psi\cdot\xi, \psi \right\rangle \tag{3.3.9}$$

Poisson commutes with $\langle H_{\rm op}\rangle$. If $\xi$ is the unit vector along the $x$-axis, the corresponding conserved quantity is

$$P_x(\psi) = \left\langle \frac{i}{\hbar}\frac{\partial\psi}{\partial x}, \psi \right\rangle.$$

**Angular Momentum.** Assume that $H_{\rm op}\colon \mathcal{H} \to \mathcal{H}$ commutes with infinitesimal rotations by a fixed skew-symmetric $3 \times 3$ matrix $\hat{\omega}$, that is,

$$H_{\rm op}(\mathbf{D}\psi(x) \cdot \hat{\omega}x) = \mathbf{D}((H_{\rm op}\psi)(x)) \cdot \hat{\omega}x \qquad (3.3.10)$$

for every $\psi$ whose derivative is in $\mathcal{H}$, where on the left-hand side, $H_{\rm op}$ is thought of as acting on the function $x \mapsto \mathbf{D}\psi(x) \cdot \hat{\omega}x$. Then the angular momentum function

$$\mathbf{J}(\hat{\omega}) : x \mapsto \langle i\mathbf{D}\psi(x) \cdot \hat{\omega}(x)/\hbar, \psi(x)\rangle \qquad (3.3.11)$$

Poisson commutes with $\mathcal{H}$ so is a conserved quantity. If we choose $\omega = (0, 0, 1)$; that is,

$$\hat{\omega} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

this corresponds to an infinitesimal rotation around the $z$-axis. Explicitly, the angular momentum around the $x^l$-axis is given by

$$J_l(\psi) = \left\langle \frac{i}{\hbar}\left( x^j \frac{\partial \psi}{\partial x^k} - x^k \frac{\partial \psi}{\partial x^j} \right), \psi \right\rangle,$$

where $(j, k, l)$ is a cyclic permutation of $(1, 2, 3)$. ♦

**(e) Linear and Angular Momentum for Linear Elastodynamics.** Consider again the equations of linear elastodynamics; see Example (f) of §3.2. Observe that the Hamiltonian is invariant under translations if the elasticity tensor $\mathbf{c}$ is homogeneous (independent of $(x, y, z)$); the corresponding conserved linear momentum in the $x$-direction is

$$P_x = \int_{\mathbb{R}^3} \rho \dot{\mathbf{u}} \cdot \frac{\partial \mathbf{u}}{\partial x}\, d^3x. \qquad (3.3.12)$$

Likewise, the Hamiltonian is invariant under rotations if $\mathbf{c}$ is isotropic, that is, invariant under rotations, which is equivalent to $\mathbf{c}$ having the form

$$c^{ijkl} = \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) + \lambda\delta^{ij}\delta^{kl},$$

where $\mu$ and $\lambda$ are constants (see Marsden and Hughes [1983, Section 4.3] for the proof). The conserved angular momentum about the $z$-axis is

$$J = \int_{\mathbb{R}^3} \rho \dot{\mathbf{u}} \cdot \left( x \frac{\partial \mathbf{u}}{\partial y} - y \frac{\partial \mathbf{u}}{\partial x} \right) d^3x.$$ ♦

In Chapter 11, we will gain a deeper insight into the significance and construction of these conserved quantities.

**Some Technicalities for Infinite-Dimensional Systems.** In general, unless the symplectic form on the Banach space $Z$ is strong, the Hamiltonian vector field $X_H$ is *not* defined on the whole of $Z$ but only on a dense subspace. For example, in the case of the wave equation $\partial^2\varphi/\partial t^2 = \nabla^2\varphi - U'(\varphi)$, a possible choice of phase space is $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, but $X_H$ is defined only on the dense subspace $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. It can also happen that the Hamiltonian $H$ is not even defined on the whole of $Z$. For example, if $H_{\rm op} = \nabla^2 + V$ for the Schrödinger equation on $L^2(\mathbb{R}^3)$, then $H$ could have domain containing $H^2(\mathbb{R}^3)$, that coincides with the domain of the Hamiltonian vector field $iH_{\rm op}$. If $V$ is singular, the domain need not be exactly $H^2(\mathbb{R}^3)$. As a quadratic form, $H$ might be extendable to $H^1(\mathbb{R}^3)$. See Reed and Simon [1974, Volume II] or Kato [1984] for details.

The problem of existence and even uniqueness of solutions can be quite delicate. For linear systems one often appeals to Stone's theorem for the Schrödinger and wave equations, and to the Hille–Yosida theorem in the case of more general linear systems. We refer to Marsden and Hughes [1983, Chapter 6], for the theory and examples. In the case of nonlinear Hamiltonian systems, the theorems of Segal [1962], Kato [1975], and Hughes, Kato, and Marsden [1977] are relevant.

For infinite-dimensional nonlinear Hamiltonian systems, technical differentiability conditions on their flows $\varphi_t$ are needed to ensure that each $\varphi_t$ is a symplectic map; see Chernoff and Marsden [1974], and especially Marsden and Hughes [1983, Chapter 6]. These technicalities are needed in many interesting examples. ♦

## Exercises

◇ **3.3-1.** Show that $\{F_i, F_j\} = 0,\ i,j = 0,1,2,3$, where the Poisson bracket is the KdV bracket and where

$$F_0(u) = \int_{-\infty}^{\infty} u\,dx,$$

$$F_1(u) = \int_{-\infty}^{\infty} \frac{1}{2}u^2\,dx,$$

$$F_2(u) = \int_{-\infty}^{\infty} \left(-u^3 + \frac{1}{2}(u_x)^2\right) dx \qquad \text{(the KdV Hamiltonian)},$$

$$F_3(u) = \int_{-\infty}^{\infty} \left(\frac{5}{2}u^4 - 5uu_x^2 + \frac{1}{2}(u_{xx})^2\right) dx.$$