11
Momentum Maps

In this chapter we show how to obtain conserved quantities for Lagrangian and Hamiltonian systems with symmetries. This is done using the concept of a momentum mapping, which is a geometric generalization of the classical linear and angular momentum. This concept is more than a mathematical reformulation of a concept that simply describes the well-known Noether theorem. Rather, it is a rich concept that is ubiquitous in the modern developments of geometric mechanics. It has led to surprising insights into many areas of mechanics and geometry.

11.1 Canonical Actions and Their Infinitesimal Generators

Canonical Actions. Let P be a Poisson manifold, let G be a Lie group, and let $\Phi : G \times P \to P$ be a smooth left action of G on P by canonical transformations. If we denote the action by $g \cdot z = \Phi_g(z)$, so that $\Phi_g : P \to P$, then the action being ***canonical*** means that

$$\Phi_g^* \{F_1, F_2\} = \left\{\Phi_g^* F_1, \Phi_g^* F_2\right\} \tag{11.1.1}$$

for any $F_1, F_2 \in \mathcal{F}(P)$ and any $g \in G$. If P is a symplectic manifold with symplectic form Ω, then the action is canonical if and only if it is symplectic, that is, $\Phi_g^*\Omega = \Omega$ for all $g \in G$.

Infinitesimal Generators. Recall from Chapter 9 on Lie groups that the ***infinitesimal generator*** of the action corresponding to a Lie algebra element $\xi \in \mathfrak{g}$ is the vector field ξ_P on P obtained by differentiating the action with respect to g at the identity in the direction ξ. By the chain rule,

$$\xi_P(z) = \frac{d}{dt} \left[\exp(t\xi) \cdot z\right]\bigg|_{t=0}. \tag{11.1.2}$$

We will need two general identities, both of which were proved in Chapter 9. First, the flow of the vector field ξ_P is

$$\varphi_t = \Phi_{\exp t\xi}. \tag{11.1.3}$$

Second, we have

$$\Phi_{g^{-1}}^* \xi_P = (\mathrm{Ad}_g\ \xi)_P \tag{11.1.4}$$

and its differentiated companion

$$[\xi_P, \eta_P] = -\left[\xi, \eta\right]_P. \tag{11.1.5}$$

The Rotation Group. To illustrate these identities, consider the action of SO(3) on $\mathbb{R}^3$. As was explained in Chapter 9, the Lie algebra $\mathfrak{so}(3)$ of SO(3) is identified with $\mathbb{R}^3$, and the Lie bracket is identified with the cross product. For the action of SO(3) on $\mathbb{R}^3$ given by rotations, the infinitesimal generator of $\boldsymbol{\omega} \in \mathbb{R}^3$ is

$$\boldsymbol{\omega}_{\mathbb{R}^3}(\mathbf{x}) = \boldsymbol{\omega} \times \mathbf{x} = \hat{\boldsymbol{\omega}}(\mathbf{x}). \tag{11.1.6}$$

Then (11.1.4) becomes the identity

$$(\mathbf{A}\boldsymbol{\omega} \times \mathbf{x}) = \mathbf{A}(\boldsymbol{\omega} \times \mathbf{A}^{-1}\mathbf{x}) \tag{11.1.7}$$

for $\mathbf{A} \in \mathrm{SO}(3)$, while (11.1.5) becomes the Jacobi identity for the vector product.

Poisson Automorphisms. Returning to the general case, differentiate (11.1.1) with respect to g in the direction ξ, to give

$$\xi_P[\{F_1, F_2\}] = \{\xi_P[F_1], F_2\} + \{F_1, \xi_P[F_2]\}. \tag{11.1.8}$$

In the symplectic case, differentiating $\Phi_g^*\Omega = \Omega$ gives

$$\pounds_{\xi_P}\Omega = 0, \tag{11.1.9}$$

that is, ξ_P is ***locally Hamiltonian***. For Poisson manifolds, a vector field satisfying (11.1.8) is called an ***infinitesimal Poisson automorphism***. Such a vector field need not be locally Hamiltonian (that is, locally of the form X_H). For example, consider the Poisson structure

$$\{F, H\} = x\left(\frac{\partial F}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial x}\frac{\partial F}{\partial y}\right) \tag{11.1.10}$$

on $\mathbb{R}^2$ and $X = \partial/\partial y$ in a neighborhood of a point of the y-axis.

We are interested in the case in which ξ_P is globally Hamiltonian, a condition stronger than (11.1.8). Thus, *assume that there is a global Hamiltonian* $J(\xi) \in \mathcal{F}(P)$ for ξ_P, that is,

$$X_{J(\xi)} = \xi_P. \tag{11.1.11}$$

Does this equation determine $J(\xi)$? Obviously not, for if $J_1(\xi)$ and $J_2(\xi)$ both satisfy (11.1.11), then

$$X_{J_1(\xi)-J_2(\xi)} = 0; \quad \text{i.e.,} \quad J_1(\xi) - J_2(\xi) \in \mathcal{C}(P),$$

the space of Casimir functions on P. If P is symplectic and connected, then $J(\xi)$ is determined by (11.1.11) up to a constant.

Exercises

◇ **11.1-1.** Verify (11.1.4), namely, $\Phi^*_{g^{-1}}\xi_P = (\mathrm{Ad}_g\, \xi)_P$ and its differentiated companion (11.1.5) $[\xi_P, \eta_P] = -[\xi, \eta]_P$, for the action of $\mathrm{GL}(n)$ on itself by conjugation.

◇ **11.1-2.** Let S^1 act on S^2 by rotations about the z-axis. Compute $J(\xi)$.

11.2 Momentum Maps

Since the right-hand side of (11.1.11) is linear in ξ, by using a basis in the finite-dimensional case *we can modify any given* $J(\xi)$ *so it too is linear in* ξ*, and still retain condition* (11.1.11). Indeed, if $e_1, \ldots, e_r$ is a basis of $\mathfrak{g}$, let the new map $\tilde{J}$ be defined by $\tilde{J}(\xi) = \xi^a J(e_a)$.

In equation (11.1.11), we can replace the assumption of a left *Lie group* action by a canonical left *Lie algebra* action $\xi \mapsto \xi_P$. In the Poisson manifold context, canonical means that (11.1.8) is satisfied and, in the symplectic manifold context, that (11.1.9) is satisfied. (Recall that for a left Lie algebra action, the map $\xi \in \mathfrak{g} \mapsto \xi_P \in \mathfrak{X}(P)$ is a Lie algebra antihomomorphism.) Thus, we make the following definition:

Definition 11.2.1. *Let a Lie algebra* $\mathfrak{g}$ *act canonically* (*on the left*) *on the Poisson manifold* P. *Suppose there is a linear map* $J : \mathfrak{g} \to \mathcal{F}(P)$ *such that*

$$X_{J(\xi)} = \xi_P \tag{11.2.1}$$

for all $\xi \in \mathfrak{g}$. *The map* $\mathbf{J} : P \to \mathfrak{g}^*$ *defined by*

$$\langle \mathbf{J}(z), \xi \rangle = J(\xi)(z) \tag{11.2.2}$$

for all $\xi \in \mathfrak{g}$ *and* $z \in P$ *is called a* ***momentum mapping*** *of the action.*

Angular Momentum. Consider the angular momentum function for a particle in Euclidean three-space, $\mathbf{J}(z) = \mathbf{q} \times \mathbf{p}$, where $z = (\mathbf{q}, \mathbf{p})$. Let $\xi \in \mathbb{R}^3$ and consider the component of $\mathbf{J}$ around the axis ξ, namely, $\langle \mathbf{J}(z), \xi \rangle = \xi \cdot (\mathbf{q} \times \mathbf{p})$. One checks that Hamilton's equations determined by this function of $\mathbf{q}$ and $\mathbf{p}$ describe infinitesimal rotations about the axis ξ. This statement was checked explicitly for ξ a vector in $\mathbb{R}^3$ that is of the form $\omega \mathbf{k}$ in §3.3, preceding example (a). The defining condition (11.2.1) is a generalization of this elementary statement about angular momentum.

Momentum Maps and Poisson Brackets. Recalling that $X_H[F] = \{F, H\}$, we see that (11.2.1) can be phrased in terms of the Poisson bracket as follows: *For any function F on P and any $\xi \in \mathfrak{g}$,*

$$\{F, J(\xi)\} = \xi_P[F]. \tag{11.2.3}$$

Equation (11.2.2) defines an isomorphism between the space of smooth maps $\mathbf{J}$ from P to $\mathfrak{g}^*$ and the space of linear maps J from $\mathfrak{g}$ to $\mathcal{F}(P)$. We think of the collection of functions $J(\xi)$ as ξ varies in $\mathfrak{g}$ as the components of $\mathbf{J}$. Denote by

$$\mathcal{H}(P) = \{ X_F \in \mathfrak{X}(P) \mid F \in \mathcal{F}(P) \} \tag{11.2.4}$$

the Lie algebra of Hamiltonian vector fields on P and by

$$\mathcal{P}(P) = \{ X \in \mathfrak{X}(P) \mid X[\{F_1, F_2\}] = \{X[F_1], F_2\} + \{F_1, X[F_2]\} \} \tag{11.2.5}$$

the Lie algebra of infinitesimal Poisson automorphisms of P. By (11.1.8), for any $\xi \in \mathfrak{g}$ we have $\xi_P \in \mathcal{P}(P)$. Therefore, giving a momentum map $\mathbf{J}$ is equivalent to specifying a linear map $J : \mathfrak{g} \to \mathcal{F}(P)$ making the diagram in Figure 11.2.1 commute.

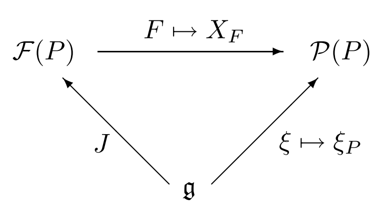

FIGURE 11.2.1. The commutative diagram defining a momentum map.

Since both $\xi \mapsto \xi_P$ and $F \mapsto X_F$ are Lie algebra antihomomorphisms, for $\xi, \eta \in \mathfrak{g}$ we get

$$X_{J([\xi,\eta])} = [\xi, \eta]_P = -[\xi_P, \eta_P] = -\left[X_{J(\xi)}, X_{J(\eta)}\right] = X_{\{J(\xi), J(\eta)\}}, \tag{11.2.6}$$

and so we have the basic identity

$$X_{J([\xi,\eta])} = X_{\{J(\xi),J(\eta)\}}. \tag{11.2.7}$$

The preceding development *defines* momentum maps but does not tell us how to *compute* them in examples. We shall concentrate on that aspect in Chapter 12.

Building on the above commutative diagram, §11.3 discusses an alternative approach to the definition of the momentum map, but it will not be used subsequently in the main text. Rather, we shall give the formulas that will be most important for later applications; the interested reader is referred to Souriau [1970], Weinstein [1977], Abraham and Marsden [1978], Guillemin and Sternberg [1984], and Libermann and Marle [1987] for more information.

Some History of the Momentum Map The momentum map can be found in the second volume of Lie [1890], where it appears in the context of homogeneous canonical transformations, in which case its expression is given as the contraction of the canonical one-form with the infinitesimal generator of the action. On page 300 it is shown that the momentum map is canonical and on page 329 that it is equivariant with respect to some linear action whose generators are identified on page 331. On page 338 it is proved that if the momentum map has constant rank (a hypothesis that seems to be implicit in all of Lie's work in this area), its image is Ad^*-invariant, and on page 343, actions are classified by Ad^*-invariant submanifolds.

We now present the modern history of the momentum map based on information and references provided to us by B. Kostant and J.-M. Souriau. We would like to thank them for all their help.

In Kostant's 1965 Phillips lectures at Haverford (the notes of which were written by Dale Husemoller), and in the 1965 U.S.–Japan Seminar (see Kostant [1966]), Kostant introduced the momentum map to generalize a theorem of Wang and thereby classified all homogeneous symplectic manifolds; this is called today "Kostant's coadjoint orbit covering theorem." These lectures also contained the key points of geometric quantization. Souriau introduced the momentum map in his 1965 Marseille lecture notes and put it in print in Souriau [1966]. The momentum map finally got its formal definition and its name, based on its physical interpretation, in Souriau [1967]. Souriau also studied its properties of equivariance, and formulated the coadjoint orbit theorem. The momentum map appeared as a key tool in Kostant's quantization lectures (see, e.g., Theorem 5.4.1 in Kostant [1970]), and Souriau [1970] discussed it at length in his book. Kostant and Souriau realized its importance for linear representations, a fact apparently not foreseen by Lie (Weinstein [1983a]). Independently, work on the momentum map and the coadjoint orbit covering theorem was done by A. Kirillov. This is described in Kirillov [1976b]. This book was first published in 1972 and states that his work on the classification theorem was done about five years earlier (page 301). The modern formulation of the momentum map was developed in the context of classical mechanics in the work of Smale [1970], who applied it extensively in his topological program for the planar n-body problem. Marsden and Weinstein [1974] and other authors quickly seized on the treasures of these ideas.

Exercises

◇ **11.2-1.** Verify that Hamilton's equations determined by the function $\langle \mathbf{J}(z), \xi \rangle = \xi \cdot (\mathbf{q} \times \mathbf{p})$ give the infinitesimal generator of rotations about the ξ-axis.

◇ **11.2-2.** Verify that $J([\xi, \eta]) = \{J(\xi), J(\eta)\}$ for angular momentum.

◇ **11.2-3.**

(a) Let P be a symplectic manifold and G a Lie group acting canonically on P, with an associated momentum map $\mathbf{J} : P \longrightarrow \mathfrak{g}^*$. Let S be a symplectic submanifold of P that is invariant under the G-action. Show that the G-action on S admits a momentum map given by $\mathbf{J}|_S$.

(b) Generalize (a) to the case in which P is a Poisson manifold and S is an immersed G-invariant Poisson submanifold.

11.3 An Algebraic Definition of the Momentum Map

This section gives an optional approach to momentum maps and may be skipped on a first reading.[1] The point of departure is the commutative diagram in Figure 11.2.1 plus the observation that the following sequence is ***exact*** (that is, the range of each map equals the kernel of the following one):

$$0 \longrightarrow \mathcal{C}(P) \xrightarrow{\ i\ } \mathcal{F}(P) \xrightarrow{\ \mathcal{H}\ } \mathcal{P}(P) \xrightarrow{\ \pi\ } \mathcal{P}(P)/\mathcal{H}(P) \longrightarrow 0.$$

Here, i is the inclusion, π the projection, $\mathcal{H}(F) = X_F$, and $\mathcal{H}(P)$ denotes the Lie algebra of globally Hamiltonian vector fields on P. Let us investigate conditions under which a left Lie algebra action, that is, an antihomomorphism $\rho : \mathfrak{g} \to \mathcal{P}(P)$, lifts through $\mathcal{H}$ to a linear map $J : \mathfrak{g} \to \mathcal{F}(P)$. As we have already seen, this is equivalent to $\mathbf{J}$ being a momentum map. (The requirement that J be a Lie algebra homomorphism will be discussed later.)

[1] This section assumes that the reader knows some topology and a little more Lie theory than we have actually covered; this material is *not* needed later on.

If $\mathcal{H} \circ J = \rho$, then $\pi \circ \rho = \pi \circ \mathcal{H} \circ J = 0$. Conversely, if $\pi \circ \rho = 0$, then $\rho(\mathfrak{g}) \subset \mathcal{H}(P)$, so there is a linear map $J : \mathfrak{g} \to \mathcal{F}(P)$ such that $\mathcal{H} \circ J = \rho$. Thus, the obstruction to the existence of J is $\pi \circ \rho = 0$. If P is symplectic, then $\mathcal{P}(P)$ coincides with the Lie algebra of locally Hamiltonian vector fields and thus $\mathcal{P}(P)/\mathcal{H}(P)$ is isomorphic to the first cohomology space $H^1(P)$ regarded as an abelian group. *Thus, in the symplectic case, $\pi \circ \rho = 0$ if and only if the induced mapping $\rho' : \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to H^1(P)$ vanishes.* Here is a list of cases that guarantee that $\pi \circ \rho = 0$:

1. *P is symplectic and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 0$.* By the first Whitehead lemma, this is the case whenever $\mathfrak{g}$ is semisimple (see Jacobson [1962] and Guillemin and Sternberg [1984]).
2. $\mathcal{P}(P)/\mathcal{H}(P) = 0$. If P is symplectic, this is equivalent to the vanishing of the first cohomology group $H^1(P)$.
3. If P is exact symplectic, that is, $\Omega = -\mathbf{d}\Theta$, and Θ is invariant under the $\mathfrak{g}$ action, that is,
$$\pounds_{\xi_P} \Theta = 0. \tag{11.3.1}$$

Case 3 occurs, for example, when $P = T^*Q$ and the action is a lift. In Case 3, there is an explicit formula for the momentum map. Since

$$0 = \pounds_{\xi_P} \Theta = \mathbf{d}\mathbf{i}_{\xi_P} \Theta + \mathbf{i}_{\xi_P} \mathbf{d}\Theta, \tag{11.3.2}$$

it follows that

$$\mathbf{d}(\mathbf{i}_{\xi_P} \Theta) = \mathbf{i}_{\xi_P} \Omega, \tag{11.3.3}$$

that is, the interior product of ξ_P with Θ satisfies (11.2.1), and hence the momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ is given by

$$\langle \mathbf{J}(z), \xi \rangle = (\mathbf{i}_{\xi_P} \Theta)(z). \tag{11.3.4}$$

In coordinates, write $\Theta = p_i \, dq^i$ and define $A^j{}_a$ and B_{aj} by

$$\xi_P = \xi^a A^j{}_a \frac{\partial}{\partial q^j} + \xi^a B_{aj} \frac{\partial}{\partial p_j}. \tag{11.3.5}$$

Then (11.3.4) reads

$$J_a(q, p) = p_i A^i{}_a(q, p). \tag{11.3.6}$$

The following example shows that ρ' does not always vanish. Consider the phase space $P = S^1 \times S^1$, with the symplectic form $\Omega = d\theta_1 \wedge d\theta_2$, the Lie algebra $\mathfrak{g} = \mathbb{R}^2$, and the action

$$\rho(x_1, x_2) = x_1 \frac{\partial}{\partial \theta_1} + x_2 \frac{\partial}{\partial \theta_2}. \tag{11.3.7}$$

In this case $[\mathfrak{g}, \mathfrak{g}] = 0$ and $\rho' : \mathbb{R}^2 \to H^1(S^1 \times S^1)$ is an isomorphism, as can be easily checked.

11.4 Conservation of Momentum Maps

One reason that momentum maps are important in mechanics is that they are conserved quantities.

Theorem 11.4.1 (Hamiltonian Version of Noether's Theorem). *If the Lie algebra* $\mathfrak{g}$ *acts canonically on the Poisson manifold* P *and admits a momentum mapping* $\mathbf{J} : P \to \mathfrak{g}^*$*, and if* $H \in \mathcal{F}(P)$ *is* $\mathfrak{g}$*-invariant, that is,* $\xi_P[H] = 0$ *for all* $\xi \in \mathfrak{g}$*, then* $\mathbf{J}$ *is a constant of the motion for* H*, that is,*

$$\mathbf{J} \circ \varphi_t = \mathbf{J},$$

where φ_t *is the flow of* X_H*. If the Lie algebra action comes from a canonical left Lie group action* Φ*, then the invariance hypothesis on* H *is implied by the invariance condition* $H \circ \Phi_g = H$ *for all* $g \in G$*.*

Proof. The condition $\xi_P[H] = 0$ implies that the Poisson bracket of $J(\xi)$, the Hamiltonian function for ξ_P, and H vanishes: $\{J(\xi), H\} = 0$. This implies that for each Lie algebra element ξ, $J(\xi)$ is a conserved quantity along the flow of X_H. This means that the values of the corresponding $\mathfrak{g}^*$-valued momentum map $\mathbf{J}$ are conserved. The last assertion of the theorem follows by differentiating the condition $H \circ \Phi_g = H$ with respect to g at the identity e in the direction ξ to obtain $\xi_P[H] = 0$. ■

We dedicate the rest of this section to a list of concrete examples of momentum maps.

Examples

(a) The Hamiltonian. On a Poisson manifold P, consider the $\mathbb{R}$-action given by the flow of a complete Hamiltonian vector field X_H. A corresponding momentum map $\mathbf{J} : P \to \mathbb{R}$ (where we identify $\mathbb{R}^*$ with $\mathbb{R}$ via the usual dot product) equals H. ♦

(b) Linear Momentum. In §6.4 we discussed the N-particle system and constructed the cotangent lift of the $\mathbb{R}^3$-action on $\mathbb{R}^{3N}$ (translation on every factor) to be the action on $T^*\mathbb{R}^{3N} \cong \mathbb{R}^{6N}$ given by

$$\mathbf{x} \cdot (\mathbf{q}_i, \mathbf{p}^j) = (\mathbf{q}_j + \mathbf{x}, \mathbf{p}^j), \quad j = 1, \ldots, N. \tag{11.4.1}$$

We show that this action has a momentum map and compute it from the definition. In the next chapter, we shall recompute it more easily utilizing further developments of the theory. Let $\xi \in \mathfrak{g} = \mathbb{R}^3$; the infinitesimal generator ξ_P at a point $(\mathbf{q}_j, \mathbf{p}^j) \in \mathbb{R}^{6N} = P$ is given by differentiating (11.4.1) with respect to $\mathbf{x}$ in the direction ξ:

$$\xi_P(\mathbf{q}_j, \mathbf{p}^j) = (\xi, \xi, \ldots, \xi, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}). \tag{11.4.2}$$

On the other hand, by definition of the canonical symplectic structure Ω on P, any candidate $J(\xi)$ has a Hamiltonian vector field given by

$$X_{J(\xi)}(\mathbf{q}_j, \mathbf{p}^j) = \left(\frac{\partial J(\xi)}{\partial \mathbf{p}^j}, -\frac{\partial J(\xi)}{\partial \mathbf{q}_j} \right). \tag{11.4.3}$$

Then, $X_{J(\xi)} = \xi_P$ implies that

$$\frac{\partial J(\xi)}{\partial \mathbf{p}^j} = \xi \quad \text{and} \quad \frac{\partial J(\xi)}{\partial \mathbf{q}_j} = 0, \quad 1 \leq j \leq N. \tag{11.4.4}$$

Solving these equations and choosing constants such that J is linear, we get

$$J(\xi)(\mathbf{q}_j, \mathbf{p}^j) = \left(\sum_{j=1}^{N} \mathbf{p}^j \right) \cdot \xi, \quad \text{i.e.,} \quad \mathbf{J}(\mathbf{q}_j, \mathbf{p}^j) = \sum_{j=1}^{N} \mathbf{p}^j. \tag{11.4.5}$$

This expression is called the ***total linear momentum*** of the N-particle system. In this example, Noether's theorem can be deduced directly as follows. Denote by $J_\alpha, q_j^\alpha, p_\alpha^j$, the αth components of $\mathbf{J}$, $\mathbf{q}_j$, and $\mathbf{p}^j$, $\alpha = 1, 2, 3$. Given a Hamiltonian H, determining the evolution of the N-particle system by Hamilton's equations, we get

$$\frac{dJ_\alpha}{dt} = \sum_{j=1}^{N} \frac{dp_\alpha^j}{dt} = -\sum_{j=1}^{N} \frac{\partial H}{\partial q_\alpha^j} = -\left[\sum_{j=1}^{N} \frac{\partial}{\partial q_\alpha^j} \right] H. \tag{11.4.6}$$

The bracket on the right is an operator that evaluates the variation of the scalar function H under a spatial translation, that is, under the action of the translation group $\mathbb{R}^3$ on each of the N coordinate directions. Obviously, J_α is conserved if H is translation-invariant, which is exactly the statement of Noether's theorem. ♦

(c) Angular Momentum. Let SO(3) act on the configuration space $Q = \mathbb{R}^3$ by $\Phi(\mathbf{A}, \mathbf{q}) = \mathbf{A}\mathbf{q}$. We show that the lifted action to $P = T^*\mathbb{R}^3$ has a momentum map and compute it. First note that if $(\mathbf{q}, \mathbf{v}) \in T_\mathrm{q}\mathbb{R}^3$, then $T_\mathrm{q}\Phi_\mathrm{A}(\mathbf{q}, \mathbf{v}) = (\mathbf{A}\mathbf{q}, \mathbf{A}\mathbf{v})$. Let $\mathbf{A} \cdot (\mathbf{q}, \mathbf{p}) = T^*_{\mathrm{Aq}}\Phi_{\mathrm{A}^{-1}}(\mathbf{q}, \mathbf{p})$ denote the lift of the SO(3) action to P, and identify covectors with vectors using the Euclidean inner product. If $(\mathbf{q}, \mathbf{p}) \in T^*_\mathrm{q}\mathbb{R}^3$, then $(\mathbf{A}\mathbf{q}, \mathbf{v}) \in T_{\mathrm{Aq}}\mathbb{R}^3$, so

$$\begin{aligned} \langle \mathbf{A} \cdot (\mathbf{q}, \mathbf{p}), (\mathbf{A}\mathbf{q}, \mathbf{v}) \rangle &= \left\langle (\mathbf{q}, \mathbf{p}), \mathbf{A}^{-1} \cdot (\mathbf{A}\mathbf{q}, \mathbf{v}) \right\rangle \\ &= \left\langle \mathbf{p}, \mathbf{A}^{-1}\mathbf{v} \right\rangle \\ &= \langle \mathbf{A}\mathbf{p}, \mathbf{v} \rangle = \langle (\mathbf{A}\mathbf{q}, \mathbf{A}\mathbf{p}), (\mathbf{A}\mathbf{q}, \mathbf{v}) \rangle, \end{aligned}$$

that is,

$$\mathbf{A} \cdot (\mathbf{q}, \mathbf{p}) = (\mathbf{A}\mathbf{q}, \mathbf{A}\mathbf{p}). \tag{11.4.7}$$

Differentiating with respect to **A**, we find that the infinitesimal generator corresponding to $\xi = \hat{\omega} \in \mathfrak{so}(3)$ is

$$\hat{\omega}_P(\mathbf{q}, \mathbf{p}) = (\xi\mathbf{q}, \xi\mathbf{p}) = (\omega \times \mathbf{q}, \omega \times \mathbf{p})\,. \tag{11.4.8}$$

As in the previous example, to find the momentum map, we solve

$$\frac{\partial J(\xi)}{\partial \mathbf{p}} = \xi\mathbf{q} \quad \text{and} \quad -\frac{\partial J(\xi)}{\partial \mathbf{q}} = \xi\mathbf{p}, \tag{11.4.9}$$

such that $J(\xi)$ is linear in ξ. A solution is given by

$$J(\xi)(\mathbf{q}, \mathbf{p}) = (\xi\mathbf{q}) \cdot \mathbf{p} = (\omega \times \mathbf{q}) \cdot \mathbf{p} = (\mathbf{q} \times \mathbf{p}) \cdot \omega,$$

so that

$$\mathbf{J}(\mathbf{q}, \mathbf{p}) = \mathbf{q} \times \mathbf{p}. \tag{11.4.10}$$

Of course, (11.4.10) is the standard formula for the *angular momentum* of a particle.

In this case, Noether's theorem states that a Hamiltonian that is rotationally invariant has the three components of **J** as constants of the motion. This example can be generalized as follows. ♦

(d) Momentum for Matrix Groups. Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a subgroup of the general linear group of $\mathbb{R}^n$. We let G act on $\mathbb{R}^n$ by matrix multiplication on the left, that is, $\Phi_{\mathrm{A}}(\mathbf{q}) = \mathbf{A}\mathbf{q}$. As in the previous example, the induced action on $P = T^*\mathbb{R}^n$ is given by

$$\mathbf{A} \cdot (\mathbf{q}, \mathbf{p}) = (\mathbf{A}\mathbf{q}, (\mathbf{A}^T)^{-1}\mathbf{p}) \tag{11.4.11}$$

and the infinitesimal generator corresponding to $\xi \in \mathfrak{g}$ by

$$\xi_P(\mathbf{q}, \mathbf{p}) = (\xi\mathbf{q}, -\xi^T\mathbf{p}). \tag{11.4.12}$$

To find the momentum map, we solve

$$\frac{\partial J(\xi)}{\partial \mathbf{p}} = \xi\mathbf{q} \quad \text{and} \quad \frac{\partial J(\xi)}{\partial \mathbf{q}} = \xi^T\mathbf{p}, \tag{11.4.13}$$

which we can do by choosing $J(\xi)(\mathbf{q}, \mathbf{p}) = (\xi\mathbf{q}) \cdot \mathbf{p}$, that is,

$$\langle \mathbf{J}(\mathbf{q}, \mathbf{p}), \xi \rangle = (\xi\mathbf{q}) \cdot \mathbf{p}. \tag{11.4.14}$$

If $n = 3$ and $G = \mathrm{SO}(3)$, (11.4.14) is equivalent to (11.4.10). In coordinates, $(\xi\mathbf{q}) \cdot \mathbf{p} = \xi^i{}_j q^j p_i$, so

$$[\mathbf{J}\,(\mathbf{q}, \mathbf{p})]^i_j = q^i p_j.$$

If we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using $\langle A, B \rangle = \operatorname{trace}\left(AB^T\right)$, then $\mathbf{J}(\mathbf{q}, \mathbf{p})$ is the projection of the matrix $q^j p_i$ onto the subspace $\mathfrak{g}$. ♦

(e) Canonical Momentum on $\mathfrak{g}^*$. Let the Lie group G with Lie algebra $\mathfrak{g}$ act by the coadjoint action on $\mathfrak{g}^*$ endowed with the $\pm$ Lie–Poisson structure. Since $\mathrm{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra isomorphism, its dual $\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$ is a canonical map by Proposition 10.7.2. Let us prove this fact directly. A computation shows that

$$\frac{\delta F}{\delta(\mathrm{Ad}^*_{g^{-1}}\,\mu)} = \mathrm{Ad}_g \frac{\delta\left(F \circ \mathrm{Ad}^*_{g^{-1}}\right)}{\delta\mu}, \tag{11.4.15}$$

whence

$$\begin{aligned}
&\{F, H\}_{\pm}\left(\mathrm{Ad}^*_{g^{-1}}\,\mu\right)\\
&\quad= \pm\left\langle \mathrm{Ad}^*_{g^{-1}}\,\mu, \left[\frac{\delta F}{\delta\left(\mathrm{Ad}^*_{g^{-1}}\,\mu\right)}, \frac{\delta H}{\delta\left(\mathrm{Ad}^*_{g^{-1}}\,\mu\right)}\right]\right\rangle\\
&\quad= \pm\left\langle \mathrm{Ad}^*_{g^{-1}}\,\mu, \left[\mathrm{Ad}_g \frac{\delta\left(F \circ \mathrm{Ad}^*_{g^{-1}}\right)}{\delta\mu}, \mathrm{Ad}_g \frac{\delta\left(H \circ \mathrm{Ad}^*_{g^{-1}}\right)}{\delta\mu}\right]\right\rangle\\
&\quad= \pm\left\langle \mu, \left[\frac{\delta\left(F \circ \mathrm{Ad}^*_{g^{-1}}\right)}{\delta\mu}, \frac{\delta\left(H \circ \mathrm{Ad}^*_{g^{-1}}\right)}{\delta\mu}\right]\right\rangle\\
&\quad= \left\{F \circ \mathrm{Ad}^*_{g^{-1}}, H \circ \mathrm{Ad}^*_{g^{-1}}\right\}_{\pm}(\mu),
\end{aligned}$$

that is, the coadjoint action of G on $\mathfrak{g}^*$ is canonical. From Proposition 10.7.1, the Hamiltonian vector field for $H \in \mathcal{F}(\mathfrak{g}^*)$ is given by

$$X_H(\mu) = \mp\,\mathrm{ad}^*_{(\delta H/\delta\mu)}\,\mu. \tag{11.4.16}$$

Since the infinitesimal generator of the coadjoint action corresponding to $\xi \in \mathfrak{g}$ is given by $\xi_{\mathfrak{g}^*} = -\,\mathrm{ad}^*_\xi$, it follows that the momentum map of the coadjoint action, if it exists, must satisfy

$$\mp\,\mathrm{ad}^*_{(\delta J(\xi)/\delta\mu)}\,\mu = -\,\mathrm{ad}^*_\xi\,\mu \tag{11.4.17}$$

for every $\mu \in \mathfrak{g}^*$, that is, $J(\xi)(\mu) = \pm\langle\mu, \xi\rangle$, which means that

$$\mathbf{J} = \pm \text{ identity on } \mathfrak{g}^*. \qquad \blacklozenge$$

(f) Dual of a Lie Algebra Homomorphism. The plasma to fluid map and averaging over a symmetry group in fluid flows are duals of Lie algebra homomorphisms and provide examples of interesting Poisson maps (see §1.7). Let us now show that all such maps are momentum maps.

Let H and G be Lie groups, let $A : H \to G$ be a Lie group homomorphism, and suppose that $\alpha : \mathfrak{h} \to \mathfrak{g}$ is the induced Lie algebra homomorphism, so its dual $\alpha^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map. We assert that α^* *is also a momentum map.* Let H act on $\mathfrak{g}^*_+$ by

$$h \cdot \mu = \mathrm{Ad}^*_{A(h)^{-1}}\,\mu,$$

that is,

$$\langle h \cdot \mu, \xi \rangle = \left\langle \mu, \operatorname{Ad}_{A(h)^{-1}} \xi \right\rangle . \tag{11.4.18}$$

Differentiating (11.4.18) with respect to h at e in the direction $\eta \in \mathfrak{h}$ gives the infinitesimal generator

$$\langle \eta_{\mathfrak{g}^*}(\mu), \xi \rangle = - \left\langle \mu, \operatorname{ad}_{\alpha(\eta)} \xi \right\rangle = - \left\langle \operatorname{ad}^*_{\alpha(\eta)} \mu, \xi \right\rangle . \tag{11.4.19}$$

Setting $\mathbf{J}(\mu) = \alpha^*(\mu)$, that is,

$$J(\eta)(\mu) = \langle \mathbf{J}(\mu), \eta \rangle = \langle \alpha^*(\mu), \eta \rangle = \langle \mu, \alpha(\eta) \rangle , \tag{11.4.20}$$

we get

$$\frac{\delta J(\eta)}{\delta \mu} = \alpha(\eta),$$

and so on $\mathfrak{g}^*_+$,

$$X_{J(\eta)}(\mu) = - \operatorname{ad}^*_{\delta J(\eta)/\delta\mu} \mu = - \operatorname{ad}^*_{\alpha(\eta)} \mu = \eta_{\mathfrak{g}^*}(\mu), \tag{11.4.21}$$

so we have proved the assertion. ♦

(g) Momentum Maps for Subalgebras. Assume that $\mathbf{J}_{\mathfrak{g}} : P \to \mathfrak{g}^*$ is a momentum map of a canonical left Lie algebra action of $\mathfrak{g}$ on the Poisson manifold P and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Then $\mathfrak{h}$ also acts canonically on P, and this action admits a momentum map $\mathbf{J}_{\mathfrak{h}} : P \to \mathfrak{h}^*$ given by

$$\mathbf{J}_{\mathfrak{h}}(z) = \mathbf{J}_{\mathfrak{g}}(z)|\mathfrak{h}. \tag{11.4.22}$$

Indeed, if $\eta \in \mathfrak{h}$, we have $\eta_P = X_{J_{\mathfrak{g}}(\eta)}$, since the $\mathfrak{g}$-action admits the momentum map $\mathbf{J}_{\mathfrak{g}}$ and $\eta \in \mathfrak{g}$. Therefore, $J_{\mathfrak{h}}(\eta) = J_{\mathfrak{g}}(\eta)$ for all $\eta \in \mathfrak{h}$ defines the induced $\mathfrak{h}$-momentum map on P. This is equivalent to

$$\langle \mathbf{J}_{\mathfrak{h}}(z), \eta \rangle = \langle \mathbf{J}_{\mathfrak{g}}(z), \eta \rangle$$

for all $z \in P$ and $\eta \in \mathfrak{g}$, which proves formula (11.4.22) . ♦

(h) Momentum Maps for Projective Representations. This example deals with the momentum map for an action of a finite-dimensional Lie group G on projective space that is induced by a unitary representation on the underlying Hilbert space. Recall from §5.3 that the unitary group $\mathrm{U}(\mathcal{H})$ acts on $\mathbb{P}\mathcal{H}$ by symplectomorphisms. Due to the difficulties in defining the Lie algebra of $\mathrm{U}(\mathcal{H})$ (see Example (d) at the end of §9.3), we cannot define the momentum map for the whole unitary group.

Let $\rho : G \to \mathrm{U}(\mathcal{H})$ be a unitary representation of G. We can define the infinitesimal action of its Lie algebra $\mathfrak{g}$ on $\mathbb{P}\mathcal{D}_G$, the essential G-smooth part of $\mathbb{P}\mathcal{H}$, by

$$\xi_{\mathbb{P}\mathcal{H}}([\psi]) = \frac{d}{dt}[(\exp(tA(\xi)))\psi]\bigg|_{t=0} = T_\psi \pi(A(\xi)\psi), \tag{11.4.23}$$

where the infinitesimal generator $A(\xi)$ was defined in §9.3, where $[\psi] \in \mathbb{P}\mathcal{D}_G$, and where the projection is denoted by $\pi : \mathcal{H} \setminus \{0\} \to \mathbb{P}\mathcal{H}$. Let $\varphi \in (\mathbb{C}\psi)^\perp$ and $\|\psi\| = 1$. Since $A(\xi)\psi - \langle A(\xi)\psi, \psi\rangle\psi \in (\mathbb{C}\psi)^\perp$, we have

$$
\begin{aligned}
(\mathbf{i}_{\xi_{\mathbb{P}\mathcal{H}}}\Omega)(T_\psi\pi(\varphi)) &= -2\hbar\,\mathrm{Im}\langle A(\xi)\psi - \langle A(\xi)\psi, \psi\rangle\psi, \varphi\rangle \\
&= -2\hbar\,\mathrm{Im}\langle A(\xi)\psi, \varphi\rangle.
\end{aligned}
$$

On the other hand, if $\mathbf{J} : \mathbb{P}\mathcal{D}_G \to \mathfrak{g}^*$ is defined by

$$
\langle \mathbf{J}([\psi]), \xi\rangle = J(\xi)([\psi]) = -i\hbar\frac{\langle \psi, A(\xi)\psi\rangle}{\|\psi\|^2}, \tag{11.4.24}
$$

then for $\varphi \in (\mathbb{C}\psi)^\perp$ and $\|\psi\| = 1$, a short computation gives

$$
\begin{aligned}
\mathbf{d}(J(\xi))([\psi])(T_\psi\pi(\varphi)) &= \left.\frac{d}{dt}J(\xi)([\psi + t\varphi])\right|_{t=0} \\
&= -2\hbar\,\mathrm{Im}\langle A(\xi)\psi, \varphi\rangle.
\end{aligned}
$$

This shows that the map $\mathbf{J}$ defined in (11.4.24) is the momentum map of the G-action on $\mathbb{P}\mathcal{H}$. We caution that this momentum map is defined only on a dense subset of the symplectic manifold. Recall that a similar thing happened when we discussed the angular momentum for quantum mechanics in §3.3. ♦

Exercises

◇ **11.4-1.** For the action of S^1 on $\mathbb{C}^2$ given by

$$
e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{-i\theta}z_2),
$$

show that the momentum map is $J = (|z_1|^2 - |z_2|^2)/2$. Show that the Hamiltonian given in equation (10.5.3) is invariant under S^1, so that Theorem 11.4.1 applies.

◇ **11.4-2** (Momentum Maps Induced by Subgroups)**.** Consider a Poisson action of a Lie group G on the Poisson manifold P with a momentum map $\mathbf{J}$ and let H be a Lie subgroup of G. Denote by $i : \mathfrak{h} \to \mathfrak{g}$ the inclusion between the corresponding Lie algebras and $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ the dual map. Check that the induced H-action on P has a momentum map given by $\mathbf{K} = i^* \circ \mathbf{J}$, that is, $K = J|\mathfrak{h}$.

◇ **11.4-3** (Euclidean Group in the Plane)**.** The special Euclidean group SE(2) consists of all transformations of $\mathbb{R}^2$ of the form $\mathbf{Az} + \mathbf{a}$, where $\mathbf{z}, \mathbf{a} \in \mathbb{R}^2$, and

$$
\mathbf{A} \in \mathrm{SO}(2) = \left\{\text{matrices of the form} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right\}. \tag{11.4.25}
$$

This group is three-dimensional, with the composition law

$$(\mathbf{A},\mathbf{a})\cdot(\mathbf{B},\mathbf{b})=(\mathbf{AB},\mathbf{Ab}+\mathbf{a}), \tag{11.4.26}$$

identity element $(\mathbf{I},\mathbf{0})$, and inverse $(\mathbf{A},\mathbf{a})^{-1}=\left(\mathbf{A}^{-1},-\mathbf{A}^{-1}\mathbf{a}\right)$. We let SE(2) act on $\mathbb{R}^2$ by $(\mathbf{A},\mathbf{a})\cdot\mathbf{z}=\mathbf{Az}+\mathbf{a}$. Let $\mathbf{z}=(q,p)$ denote coordinates on $\mathbb{R}^2$. Since det $\mathbf{A}=1$, we get $\Phi^*_{(\mathrm{A},\mathrm{a})}(dq\wedge dp)=dq\wedge dp$, that is, SE(2) acts canonically on the symplectic manifold $\mathbb{R}^2$. Show that this action has a momentum map given by $\mathbf{J}(q,p)=\left(-\frac{1}{2}(q^2+p^2),p,-q\right)$.

11.5 Equivariance of Momentum Maps

Infinitesimal Equivariance. Return to the commutative diagram in §11.2 and the relations (11.2.6). Since two of the maps in the diagram are Lie algebra antihomomorphisms, it is natural to ask whether J is a Lie algebra homomorphism. Equivalently, since $X_{J[\xi,\eta]}=X_{\{J(\xi),J(\eta)\}}$, it follows that

$$J([\xi,\eta])-\{J(\xi),J(\eta)\}=:\Sigma(\xi,\eta)$$

is a Casimir function on P and hence is constant on every symplectic leaf of P. As a function on $\mathfrak{g}\times\mathfrak{g}$ with values in the vector space $\mathcal{C}(P)$ of Casimir functions on P, Σ is bilinear, antisymmetric, and satisfies

$$\Sigma(\xi,[\eta,\zeta])+\Sigma(\eta,[\zeta,\xi])+\Sigma(\zeta,[\xi,\eta])=0 \tag{11.5.1}$$

for all $\xi,\eta,\zeta\in\mathfrak{g}$. One says that Σ is a ***$\mathcal{C}(P)$-valued 2-cocycle*** of $\mathfrak{g}$; see Souriau [1970] and Guillemin and Sternberg [1984, p. 170], for more information.

It is natural to ask when $\Sigma(\xi,\eta)=0$ for all $\xi,\eta\in\mathfrak{g}$. In general, this does not happen, and one is led to the study of this invariant. We shall derive an equivalent condition for $J:\mathfrak{g}\to\mathcal{F}(P)$ to be a Lie algebra homomorphism, that is, for $\Sigma=0$, or, in other words, for the following ***commutation relations*** to hold:

$$J([\xi,\eta])=\{J(\xi),J(\eta)\}. \tag{11.5.2}$$

Differentiating relation (11.2.2) with respect to z in the direction $v_z\in T_zP$, we get

$$\mathbf{d}(J(\xi))(z)\cdot v_z=\langle T_z\mathbf{J}\cdot v_z,\xi\rangle \tag{11.5.3}$$

for all $z\in P$, $v_z\in T_zP$, and $\xi\in\mathfrak{g}$. Thus, for $\xi,\eta\in\mathfrak{g}$,

$$\begin{aligned}\{J(\xi),J(\eta)\}(z)&=X_{J(\eta)}\left[J(\xi)\right](z)=\mathbf{d}(J(\xi))(z)\cdot X_{J(\eta)}(z)\\&=\left\langle T_z\mathbf{J}\cdot X_{J(\eta)}(z),\xi\right\rangle=\langle T_z\mathbf{J}\cdot\eta_P(z),\xi\rangle.\end{aligned} \tag{11.5.4}$$

Note that

$$J([\xi,\eta])(z) = \langle \mathbf{J}(z), [\xi,\eta] \rangle = -\langle \mathbf{J}(z), \mathrm{ad}_\eta \, \xi \rangle = -\left\langle \mathrm{ad}_\eta^* \, \mathbf{J}(z), \xi \right\rangle. \quad (11.5.5)$$

Consequently, J is a Lie algebra homomorphism if and only if

$$T_z\mathbf{J} \cdot \eta_P(z) = -\,\mathrm{ad}_\eta^* \, \mathbf{J}(z) \quad (11.5.6)$$

for all $\eta \in \mathfrak{g}$, that is, (11.5.2) and (11.5.6) are equivalent. Momentum maps satisfying (11.5.2) (or (11.5.6)) are called ***infinitesimally equivariant momentum maps***, and canonical (left) Lie algebra actions admitting infinitesimally equivariant momentum maps are called ***Hamiltonian actions***. With this terminology, we have proved the following theorem:

Theorem 11.5.1. *A canonical left Lie algebra action is Hamiltonian if and only if there is a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathcal{F}(P)$ such that $X_{\psi(\xi)} = \xi_P$ for all $\xi \in \mathfrak{g}$. If ψ exists, an infinitesimally equivariant momentum map $\mathbf{J}$ is determined by $J = \psi$. Conversely, if $\mathbf{J}$ is infinitesimally equivariant, we can take $\psi = J$.*

Equivariance. Let us justify the terminology "infinitesimally equivariant momentum map." Suppose the canonical left Lie algebra action of $\mathfrak{g}$ on P arises from a canonical left Lie group action of G on P, where $\mathfrak{g}$ is the Lie algebra of G. We say that $\mathbf{J}$ is ***equivariant*** if

$$\mathrm{Ad}_{g^{-1}}^* \circ \mathbf{J} = \mathbf{J} \circ \Phi_g \quad (11.5.7)$$

for all $g \in G$, that is, the diagram in Figure 11.5.1 commutes.

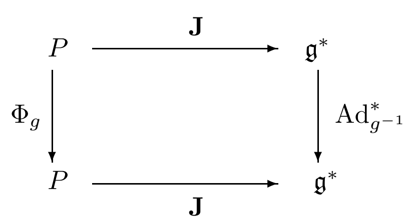

FIGURE 11.5.1. Equivariance of momentum maps.

Equivariance can be reformulated as the identity

$$J(\mathrm{Ad}_g \, \xi)(g \cdot z) = J(\xi)(z) \quad (11.5.8)$$

for all $g \in G$, $\xi \in \mathfrak{g}$, and $z \in P$. A (left) canonical Lie group action is called ***globally Hamiltonian*** if it has an equivariant momentum map. Differentiating (11.5.7) with respect to g at $g = e$ in the direction $\eta \in \mathfrak{g}$ shows that *equivariance implies infinitesimal equivariance*. We shall see shortly that all the preceding examples (except the one in Exercise 11.4-3) have equivariant momentum maps. Another case of interest occurs in Yang–Mills theory, where the 2-cocycle Σ is related to the ***anomaly*** (see Bao and Nair [1985] and references therein). The converse question, "When does infinitesimal equivariance imply equivariance?" is treated in §12.4.

Momentum Maps for Compact Groups. In the next chapter we shall see that many momentum maps that occur in examples are equivariant. The next result shows that for *compact* groups one can *always* choose them to be equivariant.[2]

Theorem 11.5.2. *Let G be a compact Lie group acting in a canonical fashion on the Poisson manifold P and having a momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. Then $\mathbf{J}$ can be changed by addition of an element of $L(\mathfrak{g}, \mathcal{C}(P))$ such that the resulting map is an equivariant momentum map. In particular, if P is symplectic, then $\mathbf{J}$ can be changed by the addition of an element of $\mathfrak{g}^*$ on each connected component so that the resulting map is an equivariant momentum map.*

Proof. For each $g \in G$ define $\mathbf{J}^g(z) = \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(g^{-1} \cdot z)$ or, equivalently, $J^g(\xi) = J(\mathrm{Ad}_{g^{-1}} \xi) \circ \Phi_{g^{-1}}$. Then $\mathbf{J}^g$ is also a momentum map for the G-action on P. Indeed, if $z \in P$, $\xi \in \mathfrak{g}$, and $F : P \to \mathbb{R}$, we have

$$\begin{aligned}
\{F, J^g(\xi)\}(z) &= -\mathbf{d}J^g(\xi)(z) \cdot X_F(z) \\
&= -\mathbf{d}J(\mathrm{Ad}_{g^{-1}} \xi)(g^{-1} \cdot z) \cdot T_z\Phi_{g^{-1}} \cdot X_F(z) \\
&= -\mathbf{d}J(\mathrm{Ad}_{g^{-1}} \xi)(g^{-1} \cdot z) \cdot (\Phi_g^* X_F)(g^{-1} \cdot z) \\
&= -\mathbf{d}J(\mathrm{Ad}_{g^{-1}} \xi)(g^{-1} \cdot z) \cdot X_{\Phi_g^* F}(g^{-1} \cdot z) \\
&= \{\Phi_g^* F, J(\mathrm{Ad}_{g^{-1}} \xi)\}(g^{-1} \cdot z) \\
&= (\mathrm{Ad}_{g^{-1}} \xi)_P[\Phi_g^* F](g^{-1} \cdot z) \\
&= (\Phi_g^* \xi_P)[\Phi_g^* F](g^{-1} \cdot z) \\
&= \mathbf{d}F(z) \cdot \xi_P(z) = \{F, J(\xi)\}(z).
\end{aligned}$$

Therefore, $\{F, J^g(\xi) - J(\xi)\} = 0$ for every $F : P \to \mathbb{R}$, that is, $J^g(\xi) - J(\xi)$ is a Casimir function on P for every $g \in G$ and every $\xi \in \mathfrak{g}$. Now define

$$\langle \mathbf{J} \rangle = \int_G \mathbf{J}^g \, dg,$$

where dg denotes the Haar measure on G normalized such that the total volume of G is 1. Equivalently, this definition states that

$$\langle J \rangle(\xi) = \int_G J^g(\xi) \, dg$$

[2] A fairly general context in which nonequivariant momentum maps are unavoidable is given in Marsden, Misiolek, Perlmutter, and Ratiu [1998].

for every $\xi \in \mathfrak{g}$. By linearity of the Poisson bracket in each factor, it follows that

$$\{F, \langle J\rangle(\xi)\} = \int_G \{F, J^g(\xi)\}\, dg = \int_G \{F, J(\xi)\}\, dg = \{F, J(\xi)\}.$$

Thus $\langle \mathbf{J}\rangle$ is also a momentum map for the G-action on P, and $\langle J\rangle(\xi) - J(\xi)$ is a Casimir function on P for every $\xi \in \mathfrak{g}$, that is, $\langle \mathbf{J}\rangle - \mathbf{J} \in L(\mathfrak{g}, \mathcal{C}(P))$.

The momentum map $\langle \mathbf{J}\rangle$ is equivariant. Indeed, noting that

$$\mathbf{J}^g(h \cdot z) = \mathrm{Ad}^*_{h^{-1}} \mathbf{J}^{h^{-1}g}(z)$$

and using invariance of the Haar measure on G under translations and inversion, for any $h \in G$ we have, after changing variables $g = hk$ in the third equality below,

$$\begin{aligned}\langle \mathbf{J}\rangle(h \cdot z) &= \int_G \mathrm{Ad}^*_{h^{-1}} \mathbf{J}^{h^{-1}g}(z)\, dg = \mathrm{Ad}^*_{h^{-1}} \int_G \mathbf{J}^{h^{-1}g}(z)\, dg \\ &= \mathrm{Ad}^*_{h^{-1}} \int_G \mathbf{J}^k(z)\, dk = \mathrm{Ad}^*_{h^{-1}} \langle \mathbf{J}\rangle(z). \qquad \blacksquare\end{aligned}$$

Exercises

◇ **11.5-1.** Show that the map $J : S^2 \to \mathbb{R}$ given by $(x, y, z) \mapsto z$ is a momentum map.

◇ **11.5-2.** Check directly that angular momentum is an equivariant momentum map, whereas the momentum map in Exercise 11.4-3 is *not* equivariant.

◇ **11.5-3.** Prove that the momentum map determined by (11.3.4), namely,

$$\langle \mathbf{J}(z), \xi\rangle = \left(\mathbf{i}_{\xi_P}\Theta\right)(z)\,,$$

is equivariant.

◇ **11.5-4.** Let $V(n, k)$ denote the vector space of complex $n \times k$ matrices (n rows, k columns). If $A \in V(n, k)$, we denote by $A^\dagger$ its adjoint (transpose conjugate).

(i) Show that

$$\langle A, B\rangle = \mathrm{trace}(AB^\dagger)$$

is a Hermitian inner product on $V(n, k)$.

(ii) Conclude that $V(n, k)$, viewed as a real vector space, is a symplectic vector space and determine the symplectic form.

(iii) Show that the action

$$(U, V) \cdot A = UAV^{-1}$$

of $U(n) \times U(k)$ on $V(n,k)$ is a canonical action.

(iv) Compute the infinitesimal generators of this action.

(v) Show that $\mathbf{J} : V(n,k) \to \mathfrak{u}(n)^* \times \mathfrak{u}(k)^*$ given by

$$\langle \mathbf{J}(A), (\xi, \eta) \rangle = \frac{1}{2} \operatorname{trace}(AA^\dagger \xi) - \frac{1}{2} \operatorname{trace}(A^\dagger A \eta)$$

is the momentum map of this action. Identify $\mathfrak{u}(n)^*$ with $\mathfrak{u}(n)$ by the pairing

$$\langle \xi_1, \xi_2 \rangle = -\operatorname{Re}[\operatorname{trace}(\xi_1 \xi_2)] = -\operatorname{trace}(\xi_1 \xi_2),$$

and similarly, for $\mathfrak{u}(k)^* \cong \mathfrak{u}(k)$; conclude that

$$\mathbf{J}(A) = \frac{1}{2}(-iAA^\dagger, iA^\dagger A) \in \mathfrak{u}(n) \times \mathfrak{u}(k).$$

(vi) Show that $\mathbf{J}$ is equivariant.