10
Poisson Manifolds

The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ carries a Poisson bracket given by

$$\{F, G\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle$$

for $\mu \in \mathfrak{g}^*$, a formula found by Lie, [1890, Section 75]. As we saw in the Introduction, this *Lie–Poisson bracket* plays an important role in the Hamiltonian description of many physical systems. This bracket is not the bracket associated with any symplectic structure on $\mathfrak{g}^*$, but is an example of the more general concept of a *Poisson manifold*. On the other hand, we do want to understand how this bracket *is* associated with a symplectic structure on coadjoint orbits and with the canonical symplectic structure on T^*G. These facts are developed in Chapters 13 and 14. Chapter 15 shows how this works in detail for the rigid body.

10.1 The Definition of Poisson Manifolds

This section generalizes the notion of a symplectic manifold by keeping just enough of the properties of Poisson brackets to describe Hamiltonian systems. The history of Poisson manifolds is complicated by the fact that the notion was rediscovered many times under different names; they occur in the works of Lie [1890], Dirac [1930,1964], Pauli [1953], Martin [1959], Jost [1964], Arens [1970], Hermann [1973], Sudarshan and Mukunda [1974], Vinogradov and Krasilshchik [1975], and Lichnerowicz [1975b]. The name

Poisson manifold was coined by Lichnerowicz. Further historical comments are given in §10.3.

Definition 10.1.1. *A* ***Poisson bracket*** *(or a* ***Poisson structure****) on a manifold P is a bilinear operation $\{\,,\}$ on $\mathcal{F}(P) = C^\infty(P)$ such that:*

(i) $(\mathcal{F}(P), \{\,,\})$ *is a Lie algebra; and*

(ii) $\{\,,\}$ *is a derivation in each factor, that is,*

$$\{FG, H\} = \{F, H\}\, G + F\, \{G, H\}$$

for all F, G, and $H \in \mathcal{F}(P)$.

A manifold P endowed with a Poisson bracket on $\mathcal{F}(P)$ is called a ***Poisson manifold****.*

A Poisson manifold is denoted by $(P, \{\,,\})$, or simply by P if there is no danger of confusion. Note that any manifold has the ***trivial Poisson structure***, which is defined by setting $\{F, G\} = 0$, for all $F, G \in \mathcal{F}(P)$. Occasionally, we consider two different Poisson brackets $\{\,,\}_1$ and $\{\,,\}_2$ on the same manifold; the two distinct Poisson manifolds are then denoted by $(P, \{\,,\}_1)$ and $(P, \{\,,\}_2)$. The notation $\{\,,\}_P$ for the bracket on P is also used when confusion might arise.

Examples

(a) Symplectic Bracket. *Any symplectic manifold is a Poisson manifold*. The Poisson bracket is defined by the symplectic form, as was shown in §5.5. Condition (ii) of the definition is satisfied as a consequence of the derivation property of vector fields:

$$\{FG, H\} = X_H[FG] = FX_H[G] + GX_H[F] = F\{G, H\} + G\{F, H\}. \quad \blacklozenge$$

(b) Lie–Poisson Bracket. If $\mathfrak{g}$ is a Lie algebra, then its dual $\mathfrak{g}^*$ is a Poisson manifold with respect to each of the ***Lie–Poisson brackets*** $\{\,,\}_+$ and $\{\,,\}_-$ defined by

$$\{F, G\}_\pm(\mu) = \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle \tag{10.1.1}$$

for $\mu \in \mathfrak{g}^*$ and $F, G \in \mathcal{F}(\mathfrak{g}^*)$. The properties of a Poisson bracket can be easily verified. Bilinearity and skew-symmetry are obvious. The derivation property of the bracket follows from the Leibniz rule for functional derivatives

$$\frac{\delta(FG)}{\delta\mu} = F(\mu)\frac{\delta G}{\delta \mu} + \frac{\delta F}{\delta \mu}G(\mu).$$

The Jacobi identity for the Lie–Poisson bracket follows from the Jacobi identity for the Lie algebra bracket and the formula

$$\pm\frac{\delta}{\delta\mu}\{F,G\}_\pm = \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right] - \mathbf{D}^2F(\mu)\left(\mathrm{ad}^*_{\delta G/\delta\mu}\,\mu,\cdot\right) + \mathbf{D}^2G(\mu)\left(\mathrm{ad}^*_{\delta F/\delta\mu}\,\mu,\cdot\right), \tag{10.1.2}$$

where we recall from the preceding chapter that for each $\xi \in \mathfrak{g}$, $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ denotes the map $\mathrm{ad}_\xi(\eta) = [\xi, \eta]$ and $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is its dual. We give a different proof that (10.1.1) is a Poisson bracket in Chapter 13. ♦

(c) Rigid-Body Bracket. Specializing Example (b) to the Lie algebra of the rotation group $\mathfrak{so}(3) \cong \mathbb{R}^3$ and identifying $\mathbb{R}^3$ and $(\mathbb{R}^3)^*$ via the standard inner product, we get the following Poisson structure on $\mathbb{R}^3$:

$$\{F,G\}_-(\mathbf{\Pi}) = -\mathbf{\Pi}\cdot(\nabla F \times \nabla G), \tag{10.1.3}$$

where $\mathbf{\Pi} \in \mathbb{R}^3$ and ∇F, the gradient of F, is evaluated at $\mathbf{\Pi}$. The Poisson bracket properties can be verified by direct computation in this case; see Exercise 1.2-1. We call (10.1.3) the ***rigid-body bracket***. ♦

(d) Ideal Fluid Bracket. Specialize the Lie–Poisson bracket to the Lie algebra $\mathfrak{X}_{\mathrm{div}}(\Omega)$ of divergence-free vector fields defined in a region Ω of $\mathbb{R}^3$ and tangent to $\partial\Omega$, with the Lie bracket being the *negative* of the Jacobi–Lie bracket. Identify $\mathfrak{X}^*_{\mathrm{div}}(\Omega)$ with $\mathfrak{X}_{\mathrm{div}}(\Omega)$ using the L^2 pairing

$$\langle \mathbf{v}, \mathbf{w}\rangle = \int_\Omega \mathbf{v}\cdot\mathbf{w}\, d^3x, \tag{10.1.4}$$

where $\mathbf{v}\cdot\mathbf{w}$ is the ordinary dot product in $\mathbb{R}^3$. Thus, the (+)-Lie–Poisson bracket is

$$\{F,G\}(\mathbf{v}) = -\int_\Omega \mathbf{v}\cdot\left[\frac{\delta F}{\delta\mathbf{v}}, \frac{\delta G}{\delta\mathbf{v}}\right] d^3x, \tag{10.1.5}$$

where the functional derivative $\delta F/\delta\mathbf{v}$ is the element of $\mathfrak{X}_{\mathrm{div}}(\Omega)$ defined by

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[F(\mathbf{v}+\varepsilon\delta\mathbf{v}) - F(\mathbf{v})\right] = \int_\Omega \frac{\delta F}{\delta\mathbf{v}}\cdot\delta\mathbf{v}\, d^3x.$$ ♦

(e) Poisson–Vlasov Bracket. Let $(P, \{\,,\}_P)$ be a Poisson manifold and let $\mathcal{F}(P)$ be the Lie algebra of functions under the Poisson bracket. Identify $\mathcal{F}(P)^*$ with densities f on P. Then the Lie–Poisson bracket has the expression

$$\{F,G\}(f) = \int_P f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}_P. \tag{10.1.6}$$

♦

(f) Frozen Lie–Poisson Bracket. Fix (or "freeze") $\nu \in \mathfrak{g}^*$ and define for any $F, G \in \mathcal{F}(\mathfrak{g}^*)$ the bracket

$$\{F, G\}_{\pm}^{\nu}(\mu) = \pm \left\langle \nu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle. \tag{10.1.7}$$

The properties of a Poisson bracket are verified as in the case of the Lie–Poisson bracket, the only difference being that (10.1.2) is replaced by

$$\pm \frac{\delta}{\delta \mu}\{F, G\}_{\pm}^{\nu} = -\mathbf{D}^2 F(\mu)\left(\operatorname{ad}^*_{\delta G/\delta \mu} \nu, \cdot\right) + \mathbf{D}^2 G(\mu)\left(\operatorname{ad}^*_{\delta F/\delta \mu} \nu, \cdot\right). \tag{10.1.8}$$

This bracket is useful in the description of the Lie–Poisson equations linearized at an equilibrium point.[1] ♦

(g) KdV Bracket. Let $S = [S^{ij}]$ be a symmetric matrix. On $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^n)$, set

$$\{F, G\}(u) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} S^{ij}\left[\frac{\delta F}{\delta u^i}\frac{d}{dx}\left(\frac{\delta G}{\delta u^j}\right) - \frac{d}{dx}\left(\frac{\delta G}{\delta u^j}\right)\frac{\delta F}{\delta u^i}\right] dx \tag{10.1.9}$$

for functions F, G satisfying $\delta F/\delta u$ and $\delta G/\delta u \to 0$ as $x \to \pm\infty$. This is a Poisson structure that is useful for the KdV equation and for gas dynamics (see Benjamin [1984]).[2] If S is invertible and $S^{-1} = [S_{ij}]$, then (10.1.9) is the Poisson bracket associated with the weak symplectic form

$$\begin{aligned}\Omega(u, v) = \frac{1}{2}\int_{-\infty}^{\infty} \sum_{i,j=l}^{n} S_{ij} \Bigg[& \left(\int_{-\infty}^{y} u^i(x)\, dx\right) v^j(y) \\ & - \left(\int_{-\infty}^{y} v^j(x)\, dx\right) u^i(y)\Bigg] dy.\end{aligned} \tag{10.1.10}$$

This is easily seen by noting that $X_H(u)$ is given by

$$X_H^i(u) = S^{ij}\frac{d}{dx}\frac{\delta H}{\delta u^j}.$$ ♦

(h) Toda Lattice Bracket. Let

$$P = \left\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n} \mid a^i > 0,\ i = 1, \ldots, n \right\}$$

[1]See, for example, Abarbanel, Holm, Marsden, and Ratiu [1986].

[2]This is a particular case of Example (f), the Lie algebra being the pseudo-differential operators on the line of order ≤ -1 and $\nu = dS/dx$.

and consider the bracket

$$\{F, G\}(\mathbf{a}, \mathbf{b}) = \left[\left(\frac{\partial F}{\partial \mathbf{a}} \right)^T, \left(\frac{\partial F}{\partial \mathbf{b}} \right)^T \right] \mathbf{W} \left[\begin{array}{c} \dfrac{\partial G}{\partial \mathbf{a}} \\ \dfrac{\partial G}{\partial \mathbf{b}} \end{array} \right], \tag{10.1.11}$$

where $(\partial F/\partial \mathbf{a})^T$ is the row vector $\left(\partial F/\partial a^1, \ldots, \partial F/\partial a^n\right)$, etc., and

$$\mathbf{W} = \begin{bmatrix} 0 & \mathbf{A} \\ -\mathbf{A} & 0 \end{bmatrix}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a^1 & & 0 \\ & \ddots & \\ 0 & & a^n \end{bmatrix}. \tag{10.1.12}$$

In terms of the coordinate functions a_i, b_j, the bracket (10.1.11) is given by

$$\begin{aligned} \left\{a^i, a^j\right\} &= 0, \\ \left\{b^i, b^j\right\} &= 0, \\ \left\{a^i, b^j\right\} &= 0 \qquad \text{if } i \neq j, \\ \left\{a^i, b^j\right\} &= a^i \qquad \text{if } i = j. \end{aligned} \tag{10.1.13}$$

This Poisson bracket is determined by the symplectic form

$$\Omega = -\sum_{i=1}^{n} \frac{1}{a^i} da^i \wedge db^i \tag{10.1.14}$$

as an easy verification shows. The mapping $(\mathbf{a}, \mathbf{b}) \mapsto (\log \mathbf{a}^{-1}, \mathbf{b})$ is a symplectic diffeomorphism of P with $\mathbb{R}^{2n}$ endowed with the canonical symplectic structure. This symplectic structure is known as the *first Poisson structure of the non-periodic Toda lattice*. We shall not study this example in any detail in this book, but we point out that its bracket is the restriction of a Lie–Poisson bracket to a certain coadjoint orbit of the group of lower triangular matrices; we refer the interested reader to §14.5 of Kostant [1979] and Symes [1980, 1982a, 1982b] for further information. ♦

Exercises

◇ **10.1-1.** If P_1 and P_2 are Poisson manifolds, show how to make $P_1 \times P_2$ into a Poisson manifold.

◇ **10.1-2.** Verify directly that the Lie–Poisson bracket satisfies Jacobi's identity.

◇ **10.1-3** (A Quadratic Bracket)**.** Let $A = \left[A^{ij}\right]$ be a skew-symmetric matrix. On $\mathbb{R}^n$, define $B^{ij} = A^{ij} x^i x^j$ (no sum). Show that the following defines a Poisson structure:

$$\{F, G\} = \sum_{i,j=1}^{n} B^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}.$$

⋄ **10.1-4** (A Cubic Bracket). For $\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$, put

$$\begin{aligned}\left\{x^1, x^2\right\} &= \|\mathbf{x}\|^2 x^3,\\ \left\{x^2, x^3\right\} &= \|\mathbf{x}\|^2 x^1,\\ \left\{x^3, x^1\right\} &= \|\mathbf{x}\|^2 x^2.\end{aligned}$$

Let $B^{ij} = \left\{x^i, x^j\right\}$, for $i < j$ and $i, j = 1, 2, 3$. Set $B^{ji} = -B^{ij}$ and define

$$\{F, G\} = \sum_{i,j=1}^{n} B^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}.$$

Check that this makes $\mathbb{R}^3$ into a Poisson manifold.

⋄ **10.1-5.** Let $\Phi : \mathfrak{g}^* \to \mathfrak{g}^*$ be a smooth map and define for $F, H : \mathfrak{g}^* \to \mathbb{R}$,

$$\{F, H\}_\Phi (\mu) = \left\langle \Phi(\mu), \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right]\right\rangle.$$

(a) Show that this rule defines a Poisson bracket on $\mathfrak{g}^*$ if and only if Φ satisfies the following identity:

$$\begin{aligned}\left\langle \mathbf{D}\Phi(\mu) \cdot \operatorname{ad}^*_\zeta(\mu), [\eta, \xi]\right\rangle &+ \left\langle \mathbf{D}\Phi(\mu) \cdot \operatorname{ad}^*_\eta \Phi(\mu), [\xi, \zeta]\right\rangle\\ &+ \left\langle \mathbf{D}\Phi(\mu) \cdot \operatorname{ad}^*_\xi \Phi(\mu), [\zeta, \eta]\right\rangle = 0,\end{aligned}$$

for all $\xi, \eta, \zeta \in \mathfrak{g}$ and all $\mu \in \mathfrak{g}^*$.

(b) Show that this relation holds if $\Phi(\mu) = \mu$ or $\Phi(\mu) = \nu$, a fixed element of $\mathfrak{g}^*$, thereby obtaining the Lie–Poisson structure (10.1.1) and the linearized Lie–Poisson structure (10.1.7) on $\mathfrak{g}^*$. Show that it also holds if $\Phi(\mu) = a\mu + \nu$ for fixed $a \in \mathbb{R}$ and $\nu \in \mathfrak{g}^*$.

(c) Assume that $\mathfrak{g}$ has a weakly nondegenerate bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Assume that κ is invariant under the Ad action and identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using κ. If $\Psi : \mathfrak{g} \to \mathfrak{g}$ is smooth, show that

$$\{F, H\}_\Psi (\xi) = \kappa(\Psi(\xi), [\nabla F(\xi), \nabla H(\xi)])$$

is a Poisson bracket if and only if

$$\begin{aligned}\kappa(\mathbf{D}\Psi(\lambda) \cdot [\Psi(\lambda), \zeta], [\eta, \xi]) &+ \kappa(\mathbf{D}\Psi(\lambda) \cdot [\Psi(\lambda), \eta], [\xi, \zeta])\\ &+ \kappa(\mathbf{D}\Psi(\lambda) \cdot [\Psi(\lambda), \xi], [\zeta, \eta]) = 0,\end{aligned}$$

for all $\lambda, \xi, \eta, \zeta \in \mathfrak{g}$. Here, $\nabla F(\xi), \nabla H(\xi) \in \mathfrak{g}$ are the gradients of F and H at $\xi \in \mathfrak{g}$ relative to κ.

Conclude as in (b) that this relation holds if $\Psi(\lambda) = a\lambda + \chi$ for $a \in \mathbb{R}$ and $\chi \in \mathfrak{g}$.

(d) Under the hypothesis of (c), let $\Psi(\lambda) = \nabla\psi(\lambda)$ for some smooth $\psi : \mathfrak{g} \to \mathbb{R}$. Show that $\{\ ,\ \}_\Psi$ is a Poisson bracket if and only if

$$\begin{aligned}&\mathbf{D}^2\psi(\lambda)([\nabla\psi(\lambda),\zeta],[\eta,\xi]) - \mathbf{D}^2\psi(\lambda)(\nabla\psi(\lambda),[\zeta,[\eta,\xi]])\\ &\quad+\mathbf{D}^2\psi(\lambda)([\nabla\psi(\lambda),\eta],[\xi,\zeta]) - \mathbf{D}^2\psi(\lambda)(\nabla\psi(\lambda),[\eta,[\xi,\zeta]])\\ &\quad+\mathbf{D}^2\psi(\lambda)([\nabla\psi(\lambda),\xi],[\zeta,\eta]) - \mathbf{D}^2\psi(\lambda)(\nabla\psi(\lambda),[\xi,[\zeta,\eta]]) = 0,\end{aligned}$$

for all $\lambda, \xi, \eta, \zeta \in \mathfrak{g}$. In particular, if $\mathbf{D}^2\psi(\lambda)$ is an invariant bilinear form for all λ, this condition holds. However, if $\mathfrak{g} = \mathfrak{so}(3)$ and ψ is arbitrary, then this condition also holds (see Exercise 1.3-2).

10.2 Hamiltonian Vector Fields and Casimir Functions

Hamiltonian Vector Fields. We begin by extending the notion of a Hamiltonian vector field from the symplectic to the Poisson context.

Proposition 10.2.1. *Let P be a Poisson manifold. If $H \in \mathcal{F}(P)$, then there is a unique vector field X_H on P such that*

$$X_H[G] = \{G, H\} \tag{10.2.1}$$

*for all $G \in \mathcal{F}(P)$. We call X_H the **Hamiltonian vector field** of H.*

Proof. This is a consequence of the fact that any derivation on $\mathcal{F}(P)$ is represented by a vector field. Fixing H, the map $G \mapsto \{G, H\}$ is a derivation, and so it uniquely determines X_H satisfying (10.2.1). (In infinite dimensions some technical conditions are needed for this proof, which are deliberately ignored here; see Abraham, Marsden, and Ratiu [1988, Section 4.2].) ■

Notice that (10.2.1) agrees with our definition of Poisson brackets in the symplectic case, so if the Poisson manifold P is symplectic, X_H defined here agrees with the definition in §5.5.

Proposition 10.2.2. *The map $H \mapsto X_H$ of $\mathcal{F}(P)$ to $\mathfrak{X}(P)$ is a Lie algebra antihomomorphism; that is,*

$$[X_H, X_K] = -X_{\{H,K\}}.$$

Proof. Using Jacobi's identity, we find that

$$\begin{aligned}[X_H, X_K][F] &= X_H[X_K[F]] - X_K[X_H[F]]\\ &= \{\{F,K\},H\} - \{\{F,H\},K\}\\ &= -\{F,\{H,K\}\}\\ &= -X_{\{H,K\}}[F].\end{aligned}$$

■

Equations of Motion in Poisson Bracket Form. Next, we establish the equation $\dot{F} = \{F, H\}$ in the Poisson context.

Proposition 10.2.3. *Let φ_t be a flow on a Poisson manifold P and let $H : P \to \mathbb{R}$ be a smooth function on P. Then*

(i) *for any $F \in \mathcal{F}(U)$, U open in P,*

$$\frac{d}{dt}(F \circ \varphi_t) = \{F, H\} \circ \varphi_t = \{F \circ \varphi_t, H\},$$

or, for short,

$$\dot{F} = \{F, H\}, \text{ for any } F \in \mathcal{F}(U),\ U \text{ open in } P,$$

if and only if φ_t is the flow of X_H.

(ii) *If φ_t is the flow of X_H, then $H \circ \varphi_t = H$.*

Proof. (i) Let $z \in P$. Then

$$\frac{d}{dt} F(\varphi_t(z)) = \mathbf{d}F(\varphi_t(z)) \cdot \frac{d}{dt}\varphi_t(z)$$

and

$$\{F, H\}(\varphi_t(z)) = \mathbf{d}F(\varphi_t(z)) \cdot X_H(\varphi_t(z)).$$

The two expressions are equal for any $F \in \mathcal{F}(U)$, U open in P, if and only if

$$\frac{d}{dt}\varphi_t(z) = X_H(\varphi_t(z)),$$

by the Hahn–Banach theorem. This is equivalent to $t \mapsto \varphi_t(z)$ being the integral curve of X_H with initial condition z, that is, φ_t is the flow of X_H.

On the other hand, if φ_t is the flow of X_H, then we have

$$X_H(\varphi_t(z)) = T_z\varphi_t(X_H(z)),$$

so that by the chain rule,

$$\begin{aligned} \frac{d}{dt} F(\varphi_t(z)) &= \mathbf{d}F(\varphi_t(z)) \cdot X_H(\varphi_t(z)) \\ &= \mathbf{d}F(\varphi_t(z)) \cdot T_z\varphi_t(X_H(z)) \\ &= \mathbf{d}(F \circ \varphi_t)(z) \cdot X_H(z) \\ &= \{F \circ \varphi_t, H\}(z). \end{aligned}$$

(ii) For the proof of (ii), let $H = F$ in (i). ■

Corollary 10.2.4. *Let $G, H \in \mathcal{F}(P)$. Then G is constant along the integral curves of X_H if and only if $\{G, H\} = 0$. Either statement is equivalent to H being constant along the integral curves of X_G.*

Among the elements of $\mathcal{F}(P)$ are functions C such that $\{C, F\} = 0$ for all $F \in \mathcal{F}(P)$, that is, C is constant along the flow of all Hamiltonian vector fields, or, equivalently, $X_C = 0$, that is, C generates trivial dynamics. Such functions are called ***Casimir functions*** of the Poisson structure. They form the center of the Poisson algebra.[3] This terminology is used in, for example, Sudarshan and Mukunda [1974]. H. B. G. Casimir is a prominent physicist who wrote his thesis (Casimir [1931]) on the quantum mechanics of the rigid body, under the direction of Paul Ehrenfest. Recall that it was Ehrenfest who, in *his* thesis, worked on the variational structure of ideal flow in Lagrangian or material representation.

Examples

(a) Symplectic Case. On a symplectic manifold P, any Casimir function is constant on connected components of P. This holds, since in the symplectic case, $X_C = 0$ implies $\mathbf{d}C = 0$, and hence C is locally constant. ♦

(b) Rigid-Body Casimirs. In the context of Example (c) of §10.1, let $C(\mathbf{\Pi}) = \|\mathbf{\Pi}\|^2/2$. Then $\nabla C(\mathbf{\Pi}) = \mathbf{\Pi}$, and by the properties of the triple product, we have for any $F \in \mathcal{F}(\mathbb{R}^3)$,

$$\begin{aligned} \{C, F\}(\mathbf{\Pi}) &= -\mathbf{\Pi} \cdot (\nabla C \times \nabla F) = -\mathbf{\Pi} \cdot (\mathbf{\Pi} \times \nabla F) \\ &= -\nabla F \cdot (\mathbf{\Pi} \times \mathbf{\Pi}) = 0. \end{aligned}$$

This shows that $C(\mathbf{\Pi}) = \|\mathbf{\Pi}\|^2/2$ is a Casimir function. A similar argument shows that

$$C_\Phi(\mathbf{\Pi}) = \Phi\left(\tfrac{1}{2}\|\mathbf{\Pi}\|^2\right) \tag{10.2.2}$$

is a Casimir function, where Φ is an arbitrary (differentiable) function of one variable; this is proved by noting that

$$\nabla C_\Phi(\mathbf{\Pi}) = \Phi'\left(\tfrac{1}{2}\|\mathbf{\Pi}\|^2\right)\mathbf{\Pi}.$$ ♦

(c) Helicity. In Example (d) of §10.1, the ***helicity***

$$C(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot (\nabla \times \mathbf{v})\, d^3x \tag{10.2.3}$$

can be checked to be a Casimir function if $\partial\Omega = \varnothing$. ♦

[3]The ***center*** of a group (or algebra) is the set of elements that commute with all elements of the group (or algebra).

(d) Poisson–Vlasov Casimirs. In Example (e) of §10.1, given a differentiable function $\Phi : \mathbb{R} \to \mathbb{R}$, the map $C : \mathcal{F}(P) \to \mathbb{R}$ defined by

$$C(f) = \int \Phi(f(q,p))\, dq\, dp \tag{10.2.4}$$

is a Casimir function. Here we choose P to be symplectic, have written $dq\, dp = dz$ for the Liouville measure, and have used it to identify functions and densities. ♦

Some History of Poisson Structures.[4] Following from the work of Lagrange and Poisson discussed at the end of §8.1, the general concept of a Poisson manifold should be credited to Sophus Lie in his treatise on transformation groups written around 1880 in the chapter on "function groups." Lie uses the word "group" for both "group" and "algebra." For example, a "function group" should really be translated as "function algebra."

On page 237, Lie defines what today is called a Poisson structure. The title of Chapter 19 is *The Coadjoint Group*, which is explicitly identified on page 334. Chapter 17, pages 294–298, defines a linear Poisson structure on the dual of a Lie algebra, today called the Lie–Poisson structure, and "Lie's third theorem" is proved for the set of regular elements. On page 349, together with a remark on page 367, it is shown that the Lie–Poisson structure naturally induces a symplectic structure on each coadjoint orbit. As we shall point out in §11.2, Lie also had many of the ideas of momentum maps. For many years this work appears to have been forgotten.

Because of the above history, Marsden and Weinstein [1983] coined the phrase "Lie–Poisson bracket" for this object, and this terminology is now in common use. However, it is not clear that Lie understood the fact that the Lie–Poisson bracket is obtained by a simple reduction process, namely, that it is induced from the canonical cotangent Poisson bracket on T^*G by passing to $\mathfrak{g}^*$ regarded as the quotient T^*G/G, as will be explained in Chapter 13. The link between the closedness of the symplectic form and the Jacobi identity is a little harder to trace explicitly; some comments in this direction are given in Souriau [1970], who gives credit to Maxwell.

Lie's work starts by taking functions $F_1, \ldots, F_r$ on a symplectic manifold M, with the property that there exist functions G_{ij} of r variables such that

$$\{F_i, F_j\} = G_{ij}(F_1, \ldots, F_r).$$

In Lie's time, all functions in sight are implicitly assumed to be analytic. The collection of all functions ϕ of $F_1, \ldots, F_r$ is the "function group"; it is

[4]We thank Hans Duistermaat and Alan Weinstein for their help with the comments in this section; the paper of Weinstein [1983a] should also be consulted by the interested reader.

provided with the bracket

$$[\phi, \psi] = \sum_{ij} G_{ij} \phi_i \psi_j, \tag{10.2.5}$$

where

$$\phi_i = \frac{\partial \phi}{\partial F_i} \quad \text{and} \quad \psi_j = \frac{\partial \psi}{\partial F_j}.$$

Considering $F = (F_1, \ldots, F_r)$ as a map from M to an r-dimensional space P, and ϕ and ψ as functions on P, one may formulate this as saying that $[\phi, \psi]$ is a Poisson structure on P, with the property that

$$F^*[\phi, \psi] = \{F^*\phi, F^*\psi\}.$$

Lie writes down the equations for the G_{ij} that follow from the antisymmetry and the Jacobi identity for the bracket $\{\,,\}$ on M. He continues with the question, If a given system of functions G_{ij} in r variables satisfies these equations, is it induced, as above, from a function group of functions of $2n$ variables? He shows that under suitable rank conditions the answer is yes. As we shall see below, this result is the precursor to many of the fundamental results about the geometry of Poisson manifolds.

It is obvious that if G_{ij} is a system that satisfies the equations that Lie writes down, then (10.2.5) is a Poisson structure in r-dimensional space. Conversely, for any Poisson structure $[\phi, \psi]$, the functions

$$G_{ij} = [F_i, F_j]$$

satisfy Lie's equations.

Lie continues with more remarks, that are not always stated as explicitly as one would like, on local normal forms of function groups (i.e., of Poisson structures) under suitable rank conditions. These amount to the following: A Poisson structure of constant rank is the same as a foliation with symplectic leaves. It is this characterization that Lie uses to get the symplectic form on the coadjoint orbits. On the other hand, Lie does not apply the symplectic form on the coadjoint orbits to representation theory.

Representation theory of Lie groups started only later with Schur on $\mathrm{GL}(n)$, and was continued by Elie Cartan with representations of semisimple Lie algebras, and in the 1930s by Weyl with the representation of compact Lie groups. The coadjoint orbit symplectic structure was connected with representation theory in the work of Kirillov and Kostant. On the other hand, Lie *did* apply the Poisson structure on the dual of the Lie algebra to prove that every abstract Lie algebra can be realized as a Lie algebra of Hamiltonian vector fields, or as a Lie subalgebra of the Poisson algebra of functions on some symplectic manifold. This is "Lie's third fundamental theorem" in the form given by Lie.

In geometry, people like Engel, Study, and, in particular, Elie Cartan studied Lie's work intensely and propagated it very actively. However, through the tainted glasses of retrospection, Lie's work on Poisson structures did not appear to receive as much attention in mechanics as it deserved; for example, even though Cartan himself did very important work in mechanics (such as Cartan [1923, 1928a, 1928b]), he did not seem to realize that the Lie–Poisson bracket was central to the Hamiltonian description of some of the rotating fluid systems he was studying. However, others, such as Hamel [1904, 1949], did study Lie intensively and used his work to make substantial contributions and extensions (such as to the study of nonholonomic systems, including rolling constraints), but many other active schools seem to have missed it. Even more surprising in this context is the contribution of Poincaré [1901b, 1910] to the Lagrangian side of the story, a tale to which we shall come in Chapter 13.

Exercises

◇ **10.2-1.** Verify the relation $[X_H, X_K] = -X_{\{H,K\}}$ directly for the rigid-body bracket.

◇ **10.2-2.** Verify that

$$C(f) = \int \Phi(f(q,p))\, dq\, dp,$$

defines a Casimir function for the Poisson–Vlasov bracket.

◇ **10.2-3.** Let P be a Poisson manifold and let $M \subset P$ be a connected submanifold with the property that for each $v \in T_x M$ there is a Hamiltonian vector field X_H on P such that $v = X_H(x)$; that is, $T_x M$ is spanned by Hamiltonian vector fields. Prove that any Casimir function is constant on M.

10.3 Properties of Hamiltonian Flows

Hamiltonian Flows Are Poisson. Now we establish the Poisson analogue of the symplectic nature of the flows of Hamiltonian vector fields.

Proposition 10.3.1. *If φ_t is the flow of X_H, then*

$$\varphi_t^* \{F, G\} = \{\varphi_t^* F, \varphi_t^* G\};$$

in other words,

$$\{F, G\} \circ \varphi_t = \{F \circ \varphi_t, G \circ \varphi_t\}.$$

Thus, the flows of Hamiltonian vector fields preserve the Poisson structure.

Proof. This is actually true even for time-dependent Hamiltonian systems (as we will see later), but here we will prove it only in the time-independent case. Let $F, K \in \mathcal{F}(P)$ and let φ_t be the flow of X_H. Let

$$u = \{F \circ \varphi_t, K \circ \varphi_t\} - \{F, K\} \circ \varphi_t.$$

Because of the bilinearity of the Poisson bracket,

$$\frac{du}{dt} = \left\{\frac{d}{dt} F \circ \varphi_t, K \circ \varphi_t\right\} + \left\{F \circ \varphi_t, \frac{d}{dt} K \circ \varphi_t\right\} - \frac{d}{dt} \{F, K\} \circ \varphi_t.$$

Using Proposition 10.2.3, this becomes

$$\frac{du}{dt} = \{\{F \circ \varphi_t, H\}, K \circ \varphi_t\} + \{F \circ \varphi_t, \{K \circ \varphi_t, H\}\} - \{\{F, K\} \circ \varphi_t, H\},$$

which, by Jacobi's identity, gives

$$\frac{du}{dt} = \{u, H\} = X_H[u].$$

The unique solution of this equation is $u_t = u_0 \circ \varphi_t$. Since $u_0 = 0$, we get $u = 0$, which is the result. ■

As in the symplectic case, with which this is, of course, consistent, this argument shows how Jacobi's identity plays a crucial role.

Poisson Maps. A smooth mapping $f : P_1 \to P_2$ between the two Poisson manifolds $(P_1, \{\,,\}_1)$ and $(P_2, \{\,,\}_2)$ is called ***canonical*** or ***Poisson*** if

$$f^* \{F, G\}_2 = \{f^* F, f^* G\}_1,$$

for all $F, G \in \mathcal{F}(P_2)$. Proposition 10.3.1 shows that flows of Hamiltonian vector fields are canonical maps. We saw already in Chapter 5 that if P_1 and P_2 are symplectic manifolds, a map $f : P_1 \to P_2$ is canonical if and only if it is symplectic.

Properties of Poisson Maps. The next proposition shows that Poisson maps push Hamiltonian flows to Hamiltonian flows.

Proposition 10.3.2. *Let $f : P_1 \to P_2$ be a Poisson map and let $H \in \mathcal{F}(P_2)$. If φ_t is the flow of X_H and ψ_t is the flow of $X_{H \circ f}$, then*

$$\varphi_t \circ f = f \circ \psi_t \quad \text{and} \quad Tf \circ X_{H \circ f} = X_H \circ f.$$

Conversely, if f is a map from P_1 to P_2 and for all $H \in \mathcal{F}(P_2)$ the Hamiltonian vector fields $X_{H \circ f} \in \mathfrak{X}(P_1)$ and $X_H \in \mathfrak{X}(P_2)$ are f-related, that is,

$$Tf \circ X_{H \circ f} = X_H \circ f,$$

then f is canonical.

Proof. For any $G \in \mathcal{F}(P_2)$ and $z \in P_1$, Proposition 10.2.3(i) and the definition of Poisson maps yield

$$\begin{aligned}\frac{d}{dt}G((f \circ \psi_t)(z)) &= \frac{d}{dt}(G \circ f)(\psi_t(z)) \\ &= \{G \circ f, H \circ f\}(\psi_t(z)) = \{G, H\}(f \circ \psi_t)(z),\end{aligned}$$

that is, $(f \circ \psi_t)(z)$ is an integral curve of X_H on P_2 through the point $f(z)$. Since $(\varphi_t \circ f)(z)$ is another such curve, uniqueness of integral curves implies that

$$(f \circ \psi_t)(z) = (\varphi_t \circ f)(z).$$

The relation $Tf \circ X_{H \circ f} = X_H \circ f$ follows from $f \circ \psi_t = \varphi_t \circ f$ by taking the time-derivative.

Conversely, assume that for any $H \in \mathcal{F}(P_2)$ we have $Tf \circ X_{H \circ f} = X_H \circ f$. Therefore, by the chain rule,

$$\begin{aligned}X_{H \circ f}[F \circ f](z) &= \mathbf{d}F(f(z)) \cdot T_z f(X_{H \circ f}(z)) \\ &= \mathbf{d}F(f(z)) \cdot X_H(f(z)) = X_H[F](f(z)),\end{aligned}$$

that is, $X_{H \circ f}[f^*F] = f^*(X_H[F])$. Thus, for $G \in \mathcal{F}(P_2)$,

$$\{G, H\} \circ f = f^*(X_H[G]) = X_{H \circ f}[f^*G] = \{G \circ f, H \circ f\},$$

and so f is canonical. ■

Exercises

◇ **10.3-1.** Verify directly that a rotation $R : \mathbb{R}^3 \to \mathbb{R}^3$ is a Poisson map for the rigid-body bracket.

◇ **10.3-2.** If P_1 and P_2 are Poisson manifolds, show that the projection $\pi_1 : P_1 \times P_2 \to P_1$ is a Poisson map. Is the corresponding statement true for symplectic maps?

10.4 The Poisson Tensor

Definition of the Poisson Tensor. By the derivation property of the Poisson bracket, the value of the bracket $\{F, G\}$ at $z \in P$ (and thus $X_F(z)$ as well) depends on F only through $\mathbf{d}F(z)$ (see Theorem 4.2.16 in Abraham, Marsden, and Ratiu [1988] for this type of argument). Thus, there is a contravariant antisymmetric two-tensor

$$B : T^*P \times T^*P \to \mathbb{R}$$

such that

$$B(z)(\alpha_z, \beta_z) = \{F, G\}(z),$$

where $\mathbf{d}F(z) = \alpha_z$ and $\mathbf{d}G(z) = \beta_z \in T_z^*P$. This tensor B is called a ***cosymplectic*** or ***Poisson structure***. In local coordinates $(z^1, \ldots, z^n)$, B is determined by its matrix elements $\left\{z^I, z^J\right\} = B^{IJ}(z)$, and the bracket becomes

$$\{F, G\} = B^{IJ}(z)\frac{\partial F}{\partial z^I}\frac{\partial G}{\partial z^J}. \tag{10.4.1}$$

Let $B^\sharp : T^*P \to TP$ be the vector bundle map associated to B, that is,

$$B(z)(\alpha_z, \beta_z) = \left\langle \alpha_z, B^\sharp(z)(\beta_z)\right\rangle.$$

Consistent with our conventions, $\dot{F} = \{F, H\}$, the Hamiltonian vector field, is given by $X_H(z) = B_z^\sharp \cdot \mathbf{d}H(z)$. Indeed, $\dot{F}(z) = \mathbf{d}F(z) \cdot X_H(z)$ and

$$\{F, H\}(z) = B(z)(\mathbf{d}F(z), \mathbf{d}H(z)) = \langle \mathbf{d}F(z), B^\sharp(z)(\mathbf{d}H(z))\rangle.$$

Comparing these expressions gives the stated result.

Coordinate Representation. A convenient way to specify a bracket in finite dimensions is by giving the coordinate relations $\left\{z^I, z^J\right\} = B^{IJ}(z)$. The Jacobi identity is then implied by the special cases

$$\left\{\left\{z^I, z^J\right\}, z^K\right\} + \left\{\left\{z^K, z^I\right\}, z^J\right\} + \left\{\left\{z^J, z^K\right\}, z^I\right\} = 0,$$

which are equivalent to the differential equations

$$B^{LI}\frac{\partial B^{JK}}{\partial z^L} + B^{LJ}\frac{\partial B^{KI}}{\partial z^L} + B^{LK}\frac{\partial B^{IJ}}{\partial z^L} = 0 \tag{10.4.2}$$

(the terms are cyclic in I, J, K). Writing $X_H[F] = \{F, H\}$ in coordinates gives

$$X_H^I\frac{\partial F}{\partial z^I} = B^{JK}\frac{\partial F}{\partial z^J}\frac{\partial H}{\partial z^K},$$

and so

$$X_H^I = B^{IJ}\frac{\partial H}{\partial z^J}. \tag{10.4.3}$$

This expression tells us that B^{IJ} should be thought of as the negative inverse of the symplectic matrix, which is literally correct in the nondegenerate case. Indeed, if we write out

$$\Omega(X_H, v) = \mathbf{d}H \cdot v$$

in coordinates, we get

$$\Omega_{IJ}X_H^I v^J = \frac{\partial H}{\partial z^J}v^J, \quad \text{i.e.,} \quad \Omega_{IJ}X_H^I = \frac{\partial H}{\partial z^J}.$$

If $[\Omega^{IJ}]$ denotes the inverse of $[\Omega_{IJ}]$, we get

$$X_H^I = \Omega^{JI} \frac{\partial H}{\partial z^J}, \tag{10.4.4}$$

so comparing (10.4.3) and (10.4.4), we see that

$$B^{IJ} = -\Omega^{IJ}.$$

Recalling that the matrix of $\Omega^\sharp$ is the inverse of that of $\Omega^\flat$ and that the matrix of $\Omega^\flat$ is the *negative* of that of Ω, we see that $B^\sharp = \Omega^\sharp$.

Let us prove this abstractly. The basic link between the Poisson tensor B and the symplectic form Ω is that they give the same Poisson bracket:

$$\{F, H\} = B(\mathbf{d}F, \mathbf{d}H) = \Omega(X_F, X_H),$$

that is,

$$\left\langle \mathbf{d}F, B^\sharp \mathbf{d}H \right\rangle = \langle \mathbf{d}F, X_H \rangle .$$

But

$$\Omega(X_H, v) = \mathbf{d}H \cdot v,$$

and so

$$\left\langle \Omega^\flat X_H, v \right\rangle = \langle \mathbf{d}H, v \rangle ,$$

whence

$$X_H = \Omega^\sharp \mathbf{d}H,$$

since $\Omega^\sharp = (\Omega^\flat)^{-1}$. Thus, $B^\sharp \mathbf{d}H = \Omega^\sharp \mathbf{d}H$, for all H, and thus

$$B^\sharp = \Omega^\sharp.$$

Coordinate Representation of Poisson Maps. We have seen that the matrix $[B^{IJ}]$ of the Poisson tensor B converts the differential

$$\mathbf{d}H = \frac{\partial H}{\partial z^I} dz^I$$

of a function to the corresponding Hamiltonian vector field; this is consistent with our treatment in the Introduction and Overview. Another basic concept, that of a Poisson map, is also worthwhile to work out in coordinates.

Let $f : P_1 \to P_2$ be a Poisson map, so $\{F \circ f, G \circ f\}_1 = \{F, G\}_2 \circ f$. In coordinates z^I on P_1 and w^K on P_2, and writing $w^K = w^K(z^I)$ for the map f, this reads

$$\frac{\partial}{\partial z^I}(F \circ f) \frac{\partial}{\partial z^J}(G \circ f) B_1^{IJ}(z) = \frac{\partial F}{\partial w^K} \frac{\partial G}{\partial w^L} B_2^{KL}(w).$$

By the chain rule, this is equivalent to

$$\frac{\partial F}{\partial w^K}\frac{\partial w^K}{\partial z^I}\frac{\partial G}{\partial w^L}\frac{\partial w^L}{\partial z^J}B_1^{IJ}(z) = \frac{\partial F}{\partial w^K}\frac{\partial G}{\partial w^L}B_2^{KL}(w).$$

Since F and G are arbitrary, *f is Poisson iff*

$$B_1^{IJ}(z)\frac{\partial w^K}{\partial z^I}\frac{\partial w^L}{\partial z^J} = B_2^{KL}(w).$$

Intrinsically, regarding $B_1(z)$ as a map $B_1(z) : T_z^*P_1 \times T_z^*P_1 \to \mathbb{R}$, this reads

$$B_1(z)(T_z^*f \cdot \alpha_w, T_z^*f \cdot \beta_w) = B_2(w)(\alpha_w, \beta_w), \tag{10.4.5}$$

where $\alpha_w, \beta_w \in T_w^*P_2$ and $f(z) = w$. In analogy with the case of vector fields, we shall say that if equation (10.4.5) holds, then B_1 and B_2 are ***f-related*** and denote it by $B_1 \sim_f B_2$. In other words, *f is Poisson iff*

$$B_1 \sim_f B_2. \tag{10.4.6}$$

Lie Derivative of the Poisson Tensor. The next proposition is equivalent to the fact that the flows of Hamiltonian vector fields are Poisson maps.

Proposition 10.4.1. *For any function $H \in \mathcal{F}(P)$, we have $\pounds_{X_H}B = 0$.*

Proof. By definition, we have

$$B(\mathbf{d}F, \mathbf{d}G) = \{F, G\} = X_G[F]$$

for any locally defined functions F and G on P. Therefore,

$$\pounds_{X_H}(B(\mathbf{d}F, \mathbf{d}G)) = \pounds_{X_H}\{F, G\} = \{\{F, G\}, H\}.$$

However, since the Lie derivative is a derivation,

$$\begin{aligned}
&\pounds_{X_H}(B(\mathbf{d}F, \mathbf{d}G)) \\
&\quad = (\pounds_{X_H}B)(\mathbf{d}F, \mathbf{d}G) + B(\pounds_{X_H}\mathbf{d}F, \mathbf{d}G) + B(\mathbf{d}F, \pounds_{X_H}\mathbf{d}G) \\
&\quad = (\pounds_{X_H}B)(\mathbf{d}F, \mathbf{d}G) + B(\mathbf{d}\{F, H\}, \mathbf{d}G) + B(\mathbf{d}F, \mathbf{d}\{G, H\}) \\
&\quad = (\pounds_{X_H}B)(\mathbf{d}F, \mathbf{d}G) + \{\{F, H\}, G\} + \{F, \{G, H\}\} \\
&\quad = (\pounds_{X_H}B)(\mathbf{d}F, \mathbf{d}G) + \{\{F, G\}, H\},
\end{aligned}$$

by the Jacobi identity. It follows that $(\pounds_{X_H}B)(\mathbf{d}F, \mathbf{d}G) = 0$ for any locally defined functions $F, G \in \mathcal{F}(U)$. Since any element of T_z^*P can be written as $\mathbf{d}F(z)$ for some $F \in \mathcal{F}(U)$, U open in P, it follows that $\pounds_{X_H}B = 0$. ■

Pauli–Jost Theorem. Suppose that the Poisson tensor B is strongly nondegenerate, that is, it defines an isomorphism $B^\sharp : \mathbf{d}F(z) \mapsto X_F(z)$ of T_z^*P with T_zP, for all $z \in P$. Then P is symplectic, and the symplectic form Ω is defined by the formula $\Omega(X_F, X_G) = \{F, G\}$ for any locally defined Hamiltonian vector fields X_F and X_G. One gets $\mathbf{d}\Omega = 0$ from Jacobi's identity—see Exercise 5.5-1. This is the ***Pauli–Jost theorem***, due to Pauli [1953] and Jost [1964].

One may be tempted to formulate the above nondegeneracy assumption in a slightly weaker form involving only the Poisson bracket: *Suppose that for every open subset V of P, if $F \in \mathcal{F}(V)$ and $\{F, G\} = 0$ for all $G \in \mathcal{F}(U)$ and all open subsets U of V, then $\mathbf{d}F = 0$ on V, that is, F is constant on the connected components of V.* This condition does *not* imply that P is symplectic, as the following counterexample shows. Let $P = \mathbb{R}^2$ with Poisson bracket

$$\{F, G\}(x, y) = y\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right).$$

If $\{F, G\} = 0$ for all G, then F must be constant on both the upper and lower half-planes, and hence by continuity it must be constant on $\mathbb{R}^2$. However, $\mathbb{R}^2$ with this Poisson structure is clearly not symplectic.

Characteristic Distribution. The subset $B^\sharp(T^*P)$ of TP is called the ***characteristic field*** or ***distribution*** of the Poisson structure; it need not be a subbundle of TP in general. Note that skew-symmetry of the tensor B is equivalent to $(B^\sharp)^* = -B^\sharp$, where $(B^\sharp)^* : T^*P \to TP$ is the dual of $B^\sharp$. If P is finite-dimensional, the ***rank*** of the Poisson structure at a point $z \in P$ is defined to be the rank of $B^\sharp(z) : T_z^*P \to T_zP$; in local coordinates, it is the rank of the matrix $\left[B^{IJ}(z)\right]$. Since the flows of Hamiltonian vector fields preserve the Poisson structure, the rank is constant along such a flow. A Poisson structure for which the rank is everywhere equal to the dimension of the manifold is nondegenerate and hence symplectic.

Poisson Immersions and Submanifolds. An injectively immersed submanifold $i : S \to P$ is called a ***Poisson immersion*** if any Hamiltonian vector field defined on an open subset of P containing $i(S)$ is in the range of T_zi at all points $i(z)$ for $z \in S$. This is equivalent to the following assertion:

Proposition 10.4.2. *An immersion $i : S \to P$ is Poisson iff it satisfies the following condition. If $F, G : V \subset S \to \mathbb{R}$, where V is open in S, and if $\overline{F}, \overline{G} : U \to \mathbb{R}$ are extensions of $F \circ i^{-1}, G \circ i^{-1} : i(V) \to \mathbb{R}$ to an open neighborhood U of $i(V)$ in P, then $\{\overline{F}, \overline{G}\}|i(V)$ is well-defined and independent of the extensions. The immersed submanifold S is thus endowed with an induced Poisson structure, and $i : S \to P$ becomes a Poisson map.*

Proof. If $i : S \to P$ is an injectively immersed Poisson manifold, then

$$\begin{aligned}\{\overline{F}, \overline{G}\}(i(z)) &= \mathbf{d}\overline{F}(i(z)) \cdot X_{\overline{G}}(i(z)) = \mathbf{d}\overline{F}(i(z)) \cdot T_z i(v) \\ &= \mathbf{d}(\overline{F} \circ i)(z) \cdot v = \mathbf{d}F(z) \cdot v,\end{aligned}$$

where $v \in T_z S$ is the unique vector satisfying $X_{\overline{G}}(i(z)) = T_z i(v)$. Thus, $\{\overline{F}, \overline{G}\}(i(z))$ is independent of the extension $\overline{F}$ of $F \circ i^{-1}$. By skew-symmetry of the bracket, it is also independent of the extension $\overline{G}$ of $G \circ i^{-1}$. Then one can define a Poisson structure on S by setting

$$\{F, G\} = \{\overline{F}, \overline{G}\} | i(V)$$

for any open subset V of S. In this way $i : S \to P$ becomes a Poisson map, since by the computation above we have $X_{\overline{G}}(i(z)) = T_z i(X_G)$.

Conversely, assume that the condition on the bracket stated above holds and let $H : U \to P$ be a Hamiltonian defined on an open subset U of P intersecting $i(S)$. Then by what was already shown, S is a Poisson manifold, and $i : S \to P$ is a Poisson map. Because i is Poisson, if $z \in S$ is such that $i(z) \in U$, we have

$$X_H(i(z)) = T_z i(X_{H \circ i}(z)),$$

and thus $X_H(i(z)) \in \operatorname{range} T_z i$, thereby showing that $i : S \to P$ is a Poisson immersion. ■

If $S \subset P$ is a submanifold of P and the inclusion i is Poisson, we say that S is a ***Poisson submanifold*** of P. Note that the only immersed Poisson submanifolds of a symplectic manifold are those whose range in P is open, since for any (weak) symplectic manifold P, we have

$$T_z P = \{\, X_H(z) \mid H \in \mathcal{F}(U),\ U \text{ open in } P \,\}.$$

Note that any Hamiltonian vector field must be tangent to a Poisson submanifold. Also note that the only Poisson submanifolds of a symplectic manifold P are its open sets.

Symplectic Stratifications. Now we come to an important result that states that every Poisson manifold is a union of symplectic manifolds, each of which is a Poisson submanifold.

Definition 10.4.3. *Let P be a Poisson manifold. We say that $z_1, z_2 \in P$ are **on the same symplectic leaf** of P if there is a piecewise smooth curve in P joining z_1 and z_2, each segment of which is a trajectory of a locally defined Hamiltonian vector field. This is clearly an equivalence relation, and an equivalence class is called a **symplectic leaf**. The symplectic leaf containing the point z is denoted by Σ_z.*

Theorem 10.4.4 (Symplectic Stratification Theorem)**.** *Let P be a finite-dimensional Poisson manifold. Then P is the disjoint union of its symplectic leaves. Each symplectic leaf in P is an injectively immersed Poisson* *submanifold, and the induced Poisson structure on the leaf is symplectic. The dimension of the leaf through a point z equals the rank of the Poisson structure at that point, and the tangent space to the leaf at z equals*

$$B^{\#}(z)(T_z^*P) = \{\, X_H(z) \mid H \in \mathcal{F}(U),\ U \text{ open in } P\,\}.$$

The picture one should have in mind is shown in Figure 10.4.1. Note in particular that the dimension of the symplectic leaf through a point can change dimension as the point varies.

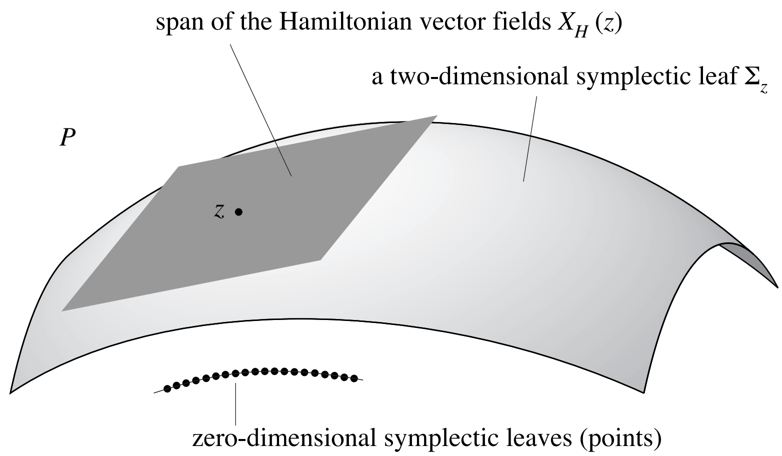

FIGURE 10.4.1. The symplectic leaves of a Poisson manifold.

The Poisson bracket on P can be alternatively described as follows.

> *To evaluate the Poisson bracket of F and G at $z \in P$, restrict F and G to the symplectic leaf Σ through z, take their bracket on Σ (in the sense of brackets on a symplectic manifold), and evaluate at z.*

Also note that since the Casimir functions have differentials that annihilate the characteristic field, they are constant on symplectic leaves.

To get a feeling for the geometric content of the symplectic stratification theorem, let us first prove it under the assumption that the characteristic field is a smooth vector subbundle of TP, which is the case considered originally by Lie [1890]. In finite dimensions, this is guaranteed if the rank of the Poisson structure is constant. Jacobi's identity shows that the characteristic field is involutive, and thus by the Frobenius theorem, it is integrable. Therefore, P is foliated by injectively immersed submanifolds whose tangent space at any point coincides with the subspace of all Hamiltonian vector fields evaluated at z. Thus, each such leaf Σ is an immersed Poisson submanifold of P. Define the two-form Ω on Σ by

$$\Omega(z)(X_F(z), X_G(z)) = \{F, G\}(z)$$

for any functions F, G defined on a neighborhood of z in P. Note that Ω is closed by the Jacobi identity (Exercise 5.5-1). Also, if

$$0 = \Omega(z)(X_F(z), X_G(z)) = \mathbf{d}F(z) \cdot X_G(z)$$

for all locally defined G, then

$$\mathbf{d}F(z)|T_z\Sigma = \mathbf{d}(F \circ i)(z) = 0$$

by the Hahn–Banach theorem. Therefore,

$$0 = X_{F\circ i}(z) = T_z i(X_F(z)) = X_F(z),$$

since Σ is a Poisson submanifold of P and the inclusion $i : \Sigma \to P$ is a Poisson map, thus showing that Ω is weakly nondegenerate and thereby proving the theorem for the constant-rank case.

The general case, proved by Kirillov [1976a], is more subtle, since for differentiable distributions that are not subbundles, integrability and involutivity are not equivalent. We shall prove this case in the Internet supplement.

Proposition 10.4.5. *If P is a Poisson manifold, $\Sigma \subset P$ is a symplectic leaf, and C is a Casimir function, then C is constant on Σ.*

Proof. If C were not locally constant on Σ, then there would exist a point $z \in \Sigma$ such that $\mathbf{d}C(z) \cdot v \neq 0$ for some $v \in T_z\Sigma$. But $T_z\Sigma$ is spanned by $X_k(z)$ for $k \in \mathcal{F}(P)$, and hence $\mathbf{d}C(z) \cdot X_k(z) = \{C, K\}(z) = 0$, which implies that $\mathbf{d}C(z) \cdot v = 0$, which is a contradiction. Thus C is locally constant on Σ and hence constant by connectedness of the leaf Σ. ■

Examples

(a) Let $P = \mathbb{R}^3$ with the rigid-body bracket. Then the symplectic leaves are spheres centered at the origin. The single point at the origin is the singular leaf in the sense that the Poisson structure has rank zero there. As we shall see later, it is true more generally that the symplectic leaves in $\mathfrak{g}^*$ with the Lie–Poisson bracket are the coadjoint orbits. ♦

(b) Symplectic leaves need not be submanifolds, and *one cannot conclude that if all the Casimir functions are constants then the Poisson structure is nondegenerate*. For example, consider the three torus $\mathbb{T}^3$ with a codimension 1 foliation with dense leaves, such as obtained by taking the leaves to be the product of $\mathbb{T}^1$ with a leaf of the irrational flow on $\mathbb{T}^2$. Put the usual area element on these leaves and define a Poisson structure on $\mathbb{T}^3$ by declaring these to be the symplectic leaves. Any Casimir function is constant, yet the Poisson structure is degenerate. ♦

Poisson–Darboux Theorem. Related to the stratification theorem is an analogue of Darboux' theorem. To state it, first recall from Exercise 10.3-2 that we define the product Poisson structure on $P_1 \times P_2$ where P_1, P_2 are Poisson manifolds by the requirements that the projections $\pi_1 : P_1 \times P_2 \to P$ and $\pi_2 : P_1 \times P_2 \to P_2$ be Poisson mappings, and $\pi_1^*(\mathcal{F}(P_1))$ and $\pi_2^*(\mathcal{F}(P_2))$ be commuting subalgebras of $\mathcal{F}(P_1 \times P_2)$. In terms of coordinates, if bracket relations $\left\{z^I, z^J\right\} = B^{IJ}(z)$ and $\left\{w^I, w^J\right\} = C^{IJ}(w)$ are given on P_1 and P_2, respectively, then these define a bracket on functions of z^I and w^J when augmented by the relations $\left\{z^I, w^J\right\} = 0$.

Theorem 10.4.6 (Lie–Weinstein)**.** *Let z_0 be a point in a Poisson manifold P. There is a neighborhood U of z_0 in P and an isomorphism $\varphi = \varphi_S \times \varphi_N : U \to S \times N$, where S is symplectic, N is Poisson, and the rank of N at $\varphi_N(z_0)$ is zero. The factors S and N are unique up to local isomorphism. Moreover, if the rank of the Poisson manifold is constant near z_0, there are coordinates $(q^1, \ldots, q^k, p_1, \ldots, p_k, y^1, \ldots, y^l)$ near z_0 satisfying the canonical bracket relations*

$$\left\{q^i, q^j\right\} = \{p_i, p_j\} = \left\{q^i, y^j\right\} = \left\{p_i, y^j\right\} = 0, \ \left\{q^i, p_j\right\} = \delta^i_j.$$

When one is proving this theorem, the manifold S can be taken to be the symplectic leaf of P through z_0, and N is, locally, any submanifold of P, transverse to S, and such that $S \cap N = \{z_0\}$. In many cases the transverse structure on N is of Lie–Poisson type. For the proof of this theorem and related results, see Weinstein [1983b]; the second part of the theorem is due to Lie [1890]. For the main examples in this book we shall not require a detailed local analysis of their Poisson structure, so we shall forgo a more detailed study of the local structure of Poisson manifolds.

Exercises

◇ **10.4-1.** If $H \in \mathcal{F}(P)$, where P is a Poisson manifold, show that the flow φ_t of X_H preserves the symplectic leaves of P.

◇ **10.4-2.** Let $(P, \{\ ,\ \})$ be a Poisson manifold with Poisson tensor $B \in \Omega_2(P)$. Let

$$B^\sharp : T^*P \to TP, \quad B^\sharp(\mathbf{d}H) = X_H,$$

be the induced bundle map. We shall denote by the same symbol $B^\sharp : \Omega^1(P) \to \mathfrak{X}(P)$ the induced map on the sections. The definitions give

$$B(\mathbf{d}F, \mathbf{d}H) = \left\langle \mathbf{d}F, B^\sharp(\mathbf{d}H)\right\rangle = \{F, H\}.$$

Define $\alpha^\sharp := B^\sharp(\alpha)$. Define for any $\alpha, \beta \in \Omega^1(P)$,

$$\{\alpha, \beta\} = -\pounds_{\alpha^\sharp}\beta + \pounds_{\beta^\sharp}\alpha - \mathbf{d}(B(\alpha, \beta)).$$

(a) Show that if the Poisson bracket on P is induced by a symplectic form Ω, that is, if $B^\sharp = \Omega^\sharp$, then

$$B(\alpha, \beta) = \Omega(\alpha^\sharp, \beta^\sharp).$$

(b) Show that for any $F, G \in \mathcal{F}(P)$, we have

$$\{F\alpha, G\beta\} = FG\,\{\alpha, \beta\} - F\alpha^\sharp[G]\beta + G\beta^\sharp[F]\alpha.$$

(c) Show that for any $F, G \in \mathcal{F}(P)$, we have

$$\mathbf{d}\,\{F, G\} = \{\mathbf{d}F, \mathbf{d}G\}\,.$$

(d) Show that if $\alpha, \beta \in \Omega^1(P)$ are closed, then $\{\alpha, \beta\} = \mathbf{d}(B(\alpha, \beta))$.

(e) Use $\pounds_{X_H} B = 0$ to show that $\{\alpha, \beta\}^\sharp = -[\alpha^\sharp, \beta^\sharp]$.

(f) Show that $(\Omega^1(P), \{\ ,\ \})$ is a Lie algebra; that is, prove Jacobi's identity.

◇ **10.4-3** (Weinstein [1983b])**.** Let P be a manifold and X, Y be two linearly independent commuting vector fields. Show that

$$\{F, K\} = X[F]Y[K] - Y[F]X[K]$$

defines a Poisson bracket on P. Show that

$$X_H = Y[H]X - X[H]Y.$$

Show that the symplectic leaves are two-dimensional and that their tangent spaces are spanned by X and Y. Show how to get Example (b) preceding Theorem 10.4.6 from this construction.

10.5 Quotients of Poisson Manifolds

Here we shall give the simplest version of a general construction of Poisson manifolds based on symmetry. This construction represents the first steps in a general procedure called ***reduction***.

Poisson Reduction Theorem. Suppose that G is a Lie group that acts on a Poisson manifold and that each map $\Phi_g : P \to P$ is a Poisson map. Let us also suppose that the action is free and proper, so that the quotient space P/G is a smooth manifold and the projection $\pi : P \to P/G$ is a submersion (see the discussion of this point in §9.3).

Theorem 10.5.1. *Under these hypotheses, there is a unique Poisson structure on P/G such that π is a Poisson map.* (*See Figure* 10.5.1.)

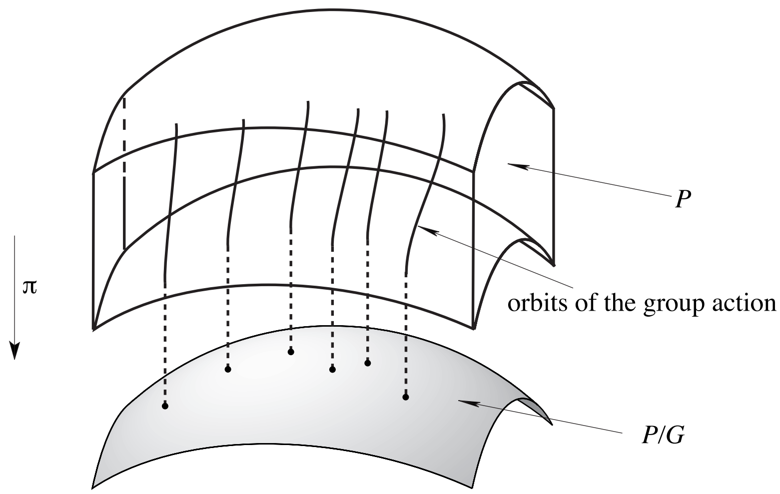

FIGURE 10.5.1. The quotient of a Poisson manifold by a group action is a Poisson manifold in a natural way.

Proof. Let us first assume that P/G is Poisson and show uniqueness. The condition that π be Poisson is that for two functions $f, k : P/G \to \mathbb{R}$,

$$\{f, k\} \circ \pi = \{f \circ \pi, k \circ \pi\}, \tag{10.5.1}$$

where the brackets are on P/G and P, respectively. The function $\overline{f} = f \circ \pi$ is the unique G-invariant function that projects to f. In other words, if $[z] \in P/G$ is an equivalence class, whereby $g_1 \cdot z$ and $g_2 \cdot z$ are equivalent, we let $\overline{f}(g \cdot z) = f([z])$ for all $g \in G$. Obviously, this defines $\overline{f}$ unambiguously, so that $\overline{f} = f \circ \pi$. We can also characterize this as saying that $\overline{f}$ assigns the value $f([z])$ to the whole orbit $G \cdot z$. We can write (10.5.1) as

$$\{f, k\} \circ \pi = \{\overline{f}, \overline{k}\}.$$

Since π is onto, this determines $\{f, k\}$ uniquely.

We can also use (10.5.1) to *define* $\{f, k\}$. First, note that

$$\begin{aligned} \{\overline{f}, \overline{k}\}(g \cdot z) &= \left(\{\overline{f}, \overline{k}\} \circ \Phi_g\right)(z) \\ &= \{\overline{f} \circ \Phi_g, \overline{k} \circ \Phi_g\}(z) \\ &= \{\overline{f}, \overline{k}\}(z), \end{aligned}$$

since Φ_g is Poisson and since $\overline{f}$ and $\overline{k}$ are constant on orbits. Thus, $\{\overline{f}, \overline{k}\}$ is constant on orbits, too, and so it defines $\{f, k\}$ uniquely.

It remains to show that $\{f, k\}$ so defined satisfies the properties of a Poisson structure. However, these all follow from their counterparts on P. For example, if we write Jacobi's identity on P, namely

$$0 = \{\{\overline{f}, \overline{k}\}, \overline{l}\} + \{\{\overline{l}, \overline{f}\}, \overline{k}\} + \{\{\overline{k}, \overline{l}\}, \overline{f}\},$$

it gives, by construction,

$$\begin{aligned} 0 &= \{\{f,k\}\circ\pi, l\circ\pi\} + \{\{l,f\}\circ\pi, k\circ\pi\} + \{\{k,l\}\circ\pi, f\circ\pi\} \\ &= \{\{f,k\},l\}\circ\pi + \{\{l,f\},k\}\circ\pi + \{\{k,l\},f\}\circ\pi, \end{aligned}$$

and thus by surjectivity of π, Jacobi's identity holds on P/G. ■

This construction is just one of many that produce new Poisson and symplectic manifolds from old ones. We refer to Marsden and Ratiu [1986] and Vaisman [1996] for generalizations of the construction here.

Reduction of Dynamics. If H is a G-invariant Hamiltonian on P, it defines a corresponding function h on P/G such that $H = h\circ\pi$. Since π is a Poisson map, it transforms X_H on P to X_h on P/G; that is, $T\pi\circ X_H = X_h\circ\pi$, or X_H and X_h are π-related. We say that the Hamiltonian system X_H on P ***reduces*** to that on P/G.

As we shall see in the next chapter, G-invariance of H may be associated with a conserved quantity $J : P \to \mathbb{R}$. If it is also G-invariant, the corresponding function j on P/G is conserved for X_h, since

$$\{h,j\}\circ\pi = \{H,J\} = 0$$

and so $\{h,j\} = 0$.

Example. Consider the differential equations on $\mathbb{C}^2$ given by

$$\begin{aligned} \dot z_1 &= -i\omega_1 z_1 + i\epsilon p\bar z_2 + iz_1(s_{11}|z_1|^2 + s_{12}|z_2|^2), \\ \dot z_2 &= -i\omega_2 z_2 + i\epsilon q\bar z_1 - iz_2(s_{21}|z_1|^2 + s_{22}|z_2|^2). \end{aligned} \tag{10.5.2}$$

Use the standard Hamiltonian structure obtained by taking the real and imaginary parts of z_i as conjugate variables. For example, we write $z_1 = q_1 + ip_1$ and require $\dot q_1 = \partial H/\partial p_1$ and $\dot p_1 = -\partial H/\partial q_1$. Recall from Chapter 5 that a useful trick in this regard that enables one to work in complex notation is to write Hamilton's equations as $\dot z_k = -2i\partial H/\partial \bar z_k$. Using this, one readily finds that (see Exercise 5.4-3) *the system* (10.5.2) *is Hamiltonian if and only if* $s_{12} = -s_{21}$ *and* $p = q$. In this case we can choose

$$\begin{aligned} H(z_1,z_2) = \frac{1}{2}(\omega_2|z_2|^2 + \omega_1|z_1|^2) - \epsilon p\,\mathrm{Re}(z_1z_2) - \frac{s_{11}}{4}|z_1|^4 \\ - \frac{s_{12}}{2}|z_1z_2|^2 + \frac{s_{22}}{4}|z_2|^4. \end{aligned} \tag{10.5.3}$$

Note that for equation (10.5.2) with $\epsilon = 0$ there are two copies of S^1 acting on z_1 and z_2 independently; corresponding conserved quantities are $|z_1|^2$ and $|z_2|^2$. However, for $\epsilon \neq 0$, the symmetry action is

$$(z_1,z_2) \mapsto (e^{i\theta}z_1, e^{-i\theta}z_2) \tag{10.5.4}$$

with the conserved quantity (Exercise 5.5-4)

$$J(z_1, z_2) = \tfrac{1}{2}(|z_1|^2 - |z_2|^2). \tag{10.5.5}$$

Let $\phi = (\pi/2) - \theta_1 - \theta_2$, where $z_1 = r_1 \exp(i\theta_1)$, $z_2 = r_2 \exp(i\theta_2)$. We know that the Hamiltonian structure for (10.5.2) on $\mathbb{C}^2$ described above induces one on $\mathbb{C}^2/S^1$ (exclude points where r_1 or r_2 vanishes), and that the two integrals (energy and the conserved quantity) descend to the quotient space, as does the Poisson bracket. The quotient space $\mathbb{C}^2/S^1$ is parametrized by (r_1, r_2, ϕ), and H and J can be dropped to the quotient; concretely, this means the following. If $F(z_1, z_2) = F(r_1, \theta_1, r_2, \theta_2)$ is S^1 invariant, then it can be written (uniquely) as a function f of (r_1, r_2, ϕ).

By Theorem 10.5.1, one can also drop the Poisson bracket to the quotient. Consequently, the equations in (r_1, r_2, ϕ) can be cast in Hamiltonian form $\dot{f} = \{f, h\}$ for the induced Poisson bracket. This bracket is obtained by using the chain rule to relate the complex variables and the polar coordinates. One finds that

$$\begin{aligned}&\{f, k\}(r_1, r_2, \phi)\\ &\quad = -\frac{1}{r_1}\left(\frac{\partial f}{\partial r_1}\frac{\partial k}{\partial \phi} - \frac{\partial f}{\partial \phi}\frac{\partial k}{\partial r_1}\right) - \frac{1}{r_2}\left(\frac{\partial f}{\partial r_2}\frac{\partial k}{\partial \phi} - \frac{\partial f}{\partial \phi}\frac{\partial k}{\partial r_2}\right).\end{aligned} \tag{10.5.6}$$

The (noncanonical) Poisson bracket (10.5.6) is, of course, the reduction of the original *canonical* Poisson bracket on the space of q and p variables, written in the new polar coordinate variables. Theorem 10.5.1 shows that Jacobi's identity is automatic for this reduced bracket. (See Knobloch, Mahalov, and Marsden [1994] for further examples of this type.) ◆

As we shall see in Chapter 13, a key example of the Poisson reduction given in 10.5.1 is that in which $P = T^*G$ and G acts on itself by left translations. Then $P/G \cong \mathfrak{g}^*$, and the reduced Poisson bracket is none other than the Lie–Poisson bracket!

Exercises

◇ **10.5-1.** Let $\mathbb{R}^3$ be equipped with the rigid-body bracket and let $G = S^1$ act on $P = \mathbb{R}^3 \backslash (z\text{-axis})$ by rotation about the z-axis. Compute the induced bracket on P/G.

◇ **10.5-2.** Compute explicitly the reduced Hamiltonian h in the example in the text and verify directly that the equations for $\dot{r}_1, \dot{r}_2, \dot{\phi}$ are Hamiltonian on $\mathbb{C}^2$ with Hamiltonian h. Also check that the function j induced by J is a constant of the motion.

10.6 The Schouten Bracket

The goal of this section is to express the Jacobi identity for a Poisson structure in geometric terms analogous to the condition $\mathbf{d}\Omega = 0$ for symplectic structures. This will be done in terms of a bracket defined on contravariant antisymmetric tensors generalizing the Lie bracket of vector fields (see, for example, Schouten [1940], Nijenhuis [1953], Lichnerowicz [1978], Olver [1984, 1986], Koszul [1985], Libermann and Marle [1987], Bhaskara and Viswanath [1988], Kosmann-Schwarzbach and Magri [1990], Vaisman [1994], and references therein).

Multivectors. A ***contravariant antisymmetric*** q***-tensor*** on a finite-dimensional vector space V is a q-linear map

$$A : V^* \times V^* \times \cdots \times V^* \,(q \text{ times}) \to \mathbb{R}$$

that is antisymmetric in each pair of arguments. The space of these tensors will be denoted by $\bigwedge_q(V)$. Thus, each element $\bigwedge_q(V)$ is a finite linear combination of terms of the form $v_1 \wedge \cdots \wedge v_q$, called a ***q-vector***, for $v_1, \ldots, v_q \in V$. If V is an infinite-dimensional Banach space, we define $\bigwedge_q(V)$ to be the span of all elements of the form $v_1 \wedge \cdots \wedge v_q$ with $v_1, \ldots, v_q \in V$, where the exterior product is defined in the usual manner relative to a weakly nondegenerate pairing $\langle\, , \rangle : V^* \times V \to \mathbb{R}$. Thus, $\bigwedge_0(V) = \mathbb{R}$ and $\bigwedge_1(V) = V$. If P is a smooth manifold, let

$$\bigwedge_q(P) = \bigcup_{z \in P} \bigwedge_q(T_z P),$$

a smooth vector bundle with fiber over $z \in P$ equal to $\bigwedge_q(T_zP)$. Let $\Omega_q(P)$ denote the smooth sections of $\bigwedge_q(P)$, that is, the elements of $\Omega_q(P)$ are smooth contravariant antisymmetric q-tensor fields on P. Let $\Omega_*(P)$ be the direct sum of the spaces $\Omega_q(P)$, where $\Omega_0(P) = \mathcal{F}(P)$. Note that

$$\Omega_q(P) = 0 \qquad \text{for } q > \dim(P),$$

and that

$$\Omega_1(P) = \mathfrak{X}(P).$$

If $X_1, \ldots, X_q \in \mathfrak{X}(P)$, then $X_1 \wedge \cdots \wedge X_q$ is called a ***q-vector field***, or a ***multivector field***.

On the manifold P, consider a $(q+p)$-form α and a contravariant antisymmetric q-tensor A. The ***interior product*** $\mathbf{i}_A\alpha$ of A with α is defined as follows. If $q = 0$, so $A \in \mathbb{R}$, let $\mathbf{i}_A\alpha = A\alpha$. If $q \geq 1$ and if $A = v_1 \wedge \cdots \wedge v_q$, where $v_i \in T_zP$, $i = 1, \ldots, q$, define $\mathbf{i}_A\alpha \in \Omega^p(P)$ by

$$(\mathbf{i}_A\alpha)(v_{q+1}, \ldots, v_{q+p}) = \alpha(v_1, \ldots, v_{q+p}) \qquad (10.6.1)$$

for arbitrary $v_{q+1}, \ldots, v_{q+p} \in T_z P$. One checks that the definition does not depend on the representation of A as a q-vector, so $\mathbf{i}_A \alpha$ is well-defined on $\bigwedge_q(P)$ by linear extension. In local coordinates, for finite-dimensional P,

$$(\mathbf{i}_A \alpha)_{i_{q+1} \ldots i_{q+p}} = A^{i_1 \ldots i_q} \alpha_{i_1 \ldots i_{q+p}}, \tag{10.6.2}$$

where all components are nonstrict; that is, the indices need not be written in ascending order. If P is finite-dimensional and $p = 0$, then (10.6.1) defines an isomorphism of $\Omega_q(P)$ with $\Omega^q(P)$. If P is a Banach manifold, then (10.6.1) defines a weakly nondegenerate pairing of $\Omega_q(P)$ with $\Omega^q(P)$. If $A \in \Omega_q(P)$, then q is called the ***degree*** of A and is denoted by $\deg A$. One checks that

$$\mathbf{i}_{A \wedge B} \alpha = \mathbf{i}_B \mathbf{i}_A \alpha. \tag{10.6.3}$$

The Lie derivative $\pounds_X$ is a derivation relative to $\wedge$, that is,

$$\pounds_X (A \wedge B) = (\pounds_X A) \wedge B + A \wedge (\pounds_X B)$$

for any $A, B \in \Omega_*(P)$.

The Schouten Bracket. The next theorem produces an interesting bracket on multivectors.

Theorem 10.6.1 (Schouten Bracket Theorem)**.** *There is a unique bilinear operation $[\,,]: \Omega_*(P) \times \Omega_*(P) \to \Omega_*(P)$ natural with respect to restriction to open sets*[5]*, called the **Schouten bracket**, that satisfies the following properties:*

(i) *It is a **biderivation of degree** -1, that is, it is bilinear,*

$$\deg[A, B] = \deg A + \deg B - 1, \tag{10.6.4}$$

and for $A, B, C \in \Omega_*(P)$,

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(\deg A + 1) \deg B} B \wedge [A, C]. \tag{10.6.5}$$

(ii) *It is determined on $\mathcal{F}(P)$ and $\mathfrak{X}(P)$ by*

  (a) *$[F, G] = 0$, for all $F, G \in \mathcal{F}(P)$;*

  (b) *$[X, F] = X[F]$, for all $F \in \mathcal{F}(P)$, $X \in \mathfrak{X}(P)$;*

  (c) *$[X, Y]$ for all $X, Y \in \mathfrak{X}(P)$ is the usual Jacobi–Lie bracket of vector fields.*

(iii) *$[A, B] = (-1)^{\deg A \deg B}[B, A]$.*

[5] "Natural with respect to restriction to open sets" means the same as it did in Proposition 4.2.4(v)

In addition, the Schouten bracket satisfies the ***graded Jacobi identity***

$$(-1)^{\deg A \deg C}[[A,B],C] + (-1)^{\deg B \deg A}[[B,C],A] \\ + (-1)^{\deg C \deg B}[[C,A],B] = 0. \qquad (10.6.6)$$

Proof. The proof proceeds in standard fashion and is similar to that characterizing the exterior or Lie derivative by its properties (see Abraham, Marsden, and Ratiu [1988]): On functions and vector fields it is given by (ii); then (i) and linear extension determine it on any skew-symmetric contravariant tensor in the second variable and a function and vector field in the first; (iii) tells how to switch such variables, and finally (i) again defines it on any pair of skew-symmetric contravariant tensors. The operation so defined satisfies (i), (ii), and (iii) by construction. Uniqueness is a consequence of the fact that the skew-symmetric contravariant tensors are generated as an exterior algebra locally by functions and vector fields, and (ii) gives these. The graded Jacobi identity is verified on an arbitrary triple of q-, p-, and r-vectors using (i), (ii), and (iii) and then invoking trilinearity of the identity. ■

Properties. The following formulas are useful in computing with the Schouten bracket. If $X \in \mathfrak{X}(P)$ and $A \in \Omega_p(P)$, induction on the degree of A and the use of property (i) show that

$$[X, A] = \pounds_X A. \qquad (10.6.7)$$

An immediate consequence of this formula and the graded Jacobi identity is the *derivation property of the Lie derivative relative to the Schouten bracket*, that is,

$$\pounds_X[A,B] = [\pounds_X A, B] + [A, \pounds_X B], \qquad (10.6.8)$$

for $A \in \Omega_p(P), B \in \Omega_q(P)$, and $X \in \mathfrak{X}(P)$. Using induction on the number of vector fields, (10.6.7), and the properties in Theorem 10.6.1, one can prove that

$$[X_1 \wedge \cdots \wedge X_r, A] = \sum_{i=1}^{r} (-1)^{i+1} X_1 \wedge \cdots \wedge \check{X}_i \wedge \cdots \wedge X_r \wedge (\pounds_{X_i} A), \qquad (10.6.9)$$

where $X_1, \ldots, X_r \in \mathfrak{X}(P)$ and $\check{X}_i$ means that X_i has been omitted. The last formula plus linear extension can be taken as the definition of the Schouten bracket, and one can deduce Theorem 10.6.1 from it; see Vaisman [1994] for this approach. If $A = Y_1 \wedge \cdots \wedge Y_s$ for $Y_1, \ldots, Y_s \in \mathfrak{X}(P)$, the formula above plus the derivation property of the Lie derivative give

$$\begin{aligned}[X_1 \wedge \cdots \wedge X_r, Y_1 \wedge \cdots \wedge Y_s] \\ = (-1)^{r+1} \sum_{i=1}^{r} \sum_{j=1}^{s} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \check{X}_i \wedge \cdots \\ \wedge X_r \wedge Y_1 \wedge \cdots \wedge \check{Y}_j \wedge \cdots \wedge Y_s. \quad (10.6.10)\end{aligned}$$

Finally, if $A \in \Omega_p(P)$, $B \in \Omega_q(P)$, and $\alpha \in \Omega^{p+q-1}(P)$, the formula

$$\mathbf{i}_{[A,B]}\alpha = (-1)^{q(p+1)} \mathbf{i}_A \mathbf{d}\, \mathbf{i}_B \alpha + (-1)^p \mathbf{i}_B \mathbf{d}\, \mathbf{i}_A \alpha - \mathbf{i}_B \mathbf{i}_A \mathbf{d}\alpha \qquad (10.6.11)$$

(which is a direct consequence of (10.6.10) and Cartan's formula for $\mathbf{d}\alpha$) can be taken as the definition of $[A, B] \in \Omega_{p+q-1}(P)$; this is the approach taken originally in Nijenhuis [1955].

Coordinate Formulas. In local coordinates, setting $\partial/\partial z^i = \partial_i$, the formulas (10.6.9) and (10.6.10) imply that

1. for any function f,

$$\left[f, \partial_{i_1} \wedge \cdots \wedge \partial_{i_p}\right] = \sum_{k=1}^{p} (-1)^{k-1} \left(\partial_{i_k} f\right) \partial_{i_1} \wedge \cdots \wedge \check{\partial}_{i_k} \wedge \cdots \wedge \partial_{i_p},$$

   where ˇ over a symbol means that it is omitted, and

2. $\left[\partial_{i_1} \wedge \cdots \wedge \partial_{i_p}, \partial_{j_1} \wedge \cdots \wedge \partial_{j_q}\right] = 0.$

Therefore, if

$$A = A^{i_1 \ldots i_p} \partial_{i_1} \wedge \cdots \wedge \partial_{i_p} \quad \text{and} \quad B = B^{j_1 \ldots j_q} \partial_{j_1} \wedge \cdots \wedge \partial_{j_q},$$

we get

$$\begin{aligned}[A, B] = A^{\ell i_1 \ldots i_{\ell-1} i_{\ell+1} \ldots i_p} \partial_\ell B^{j_1 \ldots j_q} \partial_{i_1} \wedge \cdots \wedge \partial_{i_{\ell-1}} \wedge \partial_{i_{\ell+1}} \\ \wedge \partial_{j_1} \wedge \cdots \wedge \partial_{j_q} \\ + (-1)^p B^{\ell j_1 \ldots j_{\ell-1} j_{\ell+1} \ldots j_q} \partial_\ell A^{i_1 \ldots i_p} \partial_{i_1} \wedge \cdots \wedge \partial_{i_p} \\ \wedge \partial_{j_1} \wedge \cdots \wedge \partial_{j_{\ell-1}} \wedge \partial_{j_{\ell+1}} \wedge \cdots \wedge \partial_{j_q} \qquad (10.6.12)\end{aligned}$$

or, more succinctly,

$$\begin{aligned}[A, B]^{k_2 \ldots k_{p+q}} = \varepsilon^{k_2 \ldots k_{p+q}}_{i_2 \ldots i_p j_1 \ldots j_q} A^{\ell i_2 \ldots i_p} \frac{\partial}{\partial x^\ell} B^{j_1 \ldots j_q} \\ + (-1)^p \varepsilon^{k_2 \ldots k_{p+q}}_{i_1 \ldots i_p j_2 \ldots j_q} B^{\ell j_2 \ldots j_p} \frac{\partial}{\partial x^\ell} A^{i_1 \ldots i_q}, \qquad (10.6.13)\end{aligned}$$

where all components are nonstrict. Here

$$\varepsilon^{i_1 \ldots i_{p+q}}_{j_1 \ldots j_{p+q}}$$

is the ***Kronecker symbol***: It is zero if $(i_1, \ldots, i_{p+q}) \neq (j_1, \ldots, j_{p+q})$, and is 1 (resp., -1) if $j_1, \ldots, j_{p+q}$ is an even (resp., odd) permutation of $i_1, \ldots, i_{p+q}$.

From §10.6 the Poisson tensor $B \in \Omega_2(P)$ defined by a Poisson bracket $\{\,,\}$ on P satisfies $B(\mathbf{d}F, \mathbf{d}G) = \{F, G\}$ for any $F, G \in \mathcal{F}(P)$. By (10.6.2), this can be written

$$\{F, G\} = \mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G), \tag{10.6.14}$$

or in local coordinates,

$$\{F, G\} = B^{IJ} \frac{\partial F}{\partial z^I} \frac{\partial G}{\partial z^J}.$$

Writing B locally as a sum of terms of the form $X \wedge Y$ for some $X, Y \in \mathfrak{X}(P)$ and taking $Z \in \mathfrak{X}(P)$ arbitrarily, by (10.6.1) we have for $F, G, H \in \mathcal{F}(P)$,

$$\begin{aligned}
&\mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H)(Z) \\
&\quad = (\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H)(X, Y, Z) \\
&\quad = \det \begin{bmatrix} \mathbf{d}F(X) & \mathbf{d}F(Y) & \mathbf{d}F(Z) \\ \mathbf{d}G(X) & \mathbf{d}G(Y) & \mathbf{d}G(Z) \\ \mathbf{d}H(X) & \mathbf{d}H(Y) & \mathbf{d}H(Z) \end{bmatrix} \\
&\quad = \det \begin{bmatrix} \mathbf{d}F(X) & \mathbf{d}F(Y) \\ \mathbf{d}G(X) & \mathbf{d}G(Y) \end{bmatrix} \mathbf{d}H(Z) + \det \begin{bmatrix} \mathbf{d}H(X) & \mathbf{d}H(Y) \\ \mathbf{d}F(X) & \mathbf{d}F(Y) \end{bmatrix} \mathbf{d}G(Z) \\
&\qquad + \det \begin{bmatrix} \mathbf{d}G(X) & \mathbf{d}G(Y) \\ \mathbf{d}H(X) & \mathbf{d}H(Y) \end{bmatrix} \mathbf{d}F(Z) \\
&\quad = \mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G)\mathbf{d}H(Z) + \mathbf{i}_B(\mathbf{d}H \wedge \mathbf{d}F)\mathbf{d}G(Z) + \mathbf{i}_B(\mathbf{d}G \wedge \mathbf{d}H)\mathbf{d}F(Z),
\end{aligned}$$

that is,

$$\begin{aligned}
&\mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H) \\
&\quad = \mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G)\mathbf{d}H + \mathbf{i}_B(\mathbf{d}H \wedge \mathbf{d}F)\mathbf{d}G + \mathbf{i}_B(\mathbf{d}G \wedge \mathbf{d}H)\mathbf{d}F.
\end{aligned} \tag{10.6.15}$$

The Jacobi–Schouten Identity. Equations (10.6.14) and (10.6.15) imply

$$\begin{aligned}
&\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} \\
&\qquad = \mathbf{i}_B(\mathbf{d}\{F, G\} \wedge \mathbf{d}H) + \mathbf{i}_B(\mathbf{d}\{H, F\} \wedge \mathbf{d}G) + \mathbf{i}_B(\mathbf{d}\{G, H\} \wedge \mathbf{d}F) \\
&\qquad = \mathbf{i}_B\mathbf{d}(\mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G)\mathbf{d}H + \mathbf{i}_B(\mathbf{d}H \wedge \mathbf{d}F)\mathbf{d}G + \mathbf{i}_B(\mathbf{d}G \wedge \mathbf{d}H)\mathbf{d}F) \\
&\qquad = \mathbf{i}_B\mathbf{d}\,\mathbf{i}_B(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H) \\
&\qquad = \tfrac{1}{2}\mathbf{i}_{[B,B]}(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H),
\end{aligned}$$

the last equality being a consequence of (10.6.11). We summarize what we have proved.

Theorem 10.6.2. *The following identity holds:*

$$\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = \frac{1}{2}\mathbf{i}_{[B,B]}(\mathbf{d}F \wedge \mathbf{d}G \wedge \mathbf{d}H). \tag{10.6.16}$$

This result shows that Jacobi's identity for $\{\ ,\ \}$ is equivalent to $[B, B] = 0$. Thus, *a Poisson structure is uniquely defined by a contravariant antisymmetric two-tensor whose Schouten bracket with itself vanishes*. The local formula (10.6.13) becomes

$$[B, B]^{IJK} = \sum_{L=1}^{n} \left(B^{LK} \frac{\partial B^{IJ}}{\partial z^L} + B^{LI} \frac{\partial B^{JK}}{\partial z^L} + B^{LJ} \frac{\partial B^{KI}}{\partial z^L} \right),$$

which coincides with our earlier expression (10.4.2).

The Lie–Schouten Identity. There is another interesting identity that gives the Lie derivative of the Poisson tensor along a Hamiltonian vector field.

Theorem 10.6.3. *The following identity holds:*

$$\pounds_{X_H} B = \mathbf{i}_{[B,B]} \mathbf{d}H. \tag{10.6.17}$$

Proof. In coordinates,

$$(\pounds_X B)^{IJ} = X^K \frac{\partial B^{IJ}}{\partial z^K} - B^{IK} \frac{\partial X^J}{\partial z^K} - B^{KJ} \frac{\partial X^I}{\partial z^K},$$

so if $X^I = B^{IJ}(\partial H / \partial z^J)$, this becomes

$$\begin{aligned}
(\pounds_{X_H} B)^{IJ} &= B^{KL} \frac{\partial B^{IJ}}{\partial z^K} \frac{\partial H}{\partial z^L} - B^{IK} \frac{\partial}{\partial z^K} \left(B^{JL} \frac{\partial H}{\partial z^L} \right) \\
&\quad + B^{JK} \frac{\partial}{\partial z^K} \left(B^{IL} \frac{\partial H}{\partial z^L} \right) \\
&= \left(B^{KL} \frac{\partial B^{IJ}}{\partial z^K} - B^{IK} \frac{\partial B^{JL}}{\partial z^K} - B^{KJ} \frac{\partial B^{IL}}{\partial z^K} \right) \frac{\partial H}{\partial z^L} \\
&= [B, B]^{LIJ} \frac{\partial H}{\partial z^L} = \left(\mathbf{i}_{[B,B]} \mathbf{d}H \right)^{IJ},
\end{aligned}$$

so (10.6.17) follows. ■

This identity shows how Jacobi's identity $[B, B] = 0$ *is directly used to show that the flow φ_t of a Hamiltonian vector field is Poisson.* The above derivation shows that *the flow of a time-dependent Hamiltonian vector field consists of Poisson maps*; indeed, even in this case,

$$\frac{d}{dt} (\varphi_t^* B) = \varphi_t^* (\pounds_{X_H} B) = \varphi_t^* \left(\mathbf{i}_{[B,B]} \mathbf{d}H \right) = 0$$

is valid.

Exercises

◇ **10.6-1.** Prove the following formulas by the method indicated in the text.

(a) If $A \in \Omega_q(P)$ and $X \in \mathfrak{X}(P)$, then $[X, A] = \pounds_X A$.

(b) If $A \in \Omega_q(P)$ and $X_1, \ldots, X_r \in \mathfrak{X}(P)$, then

$$[X_1 \wedge \cdots \wedge X_r, A] = \sum_{i=1}^{r} (-1)^{i+1} X_1 \wedge \cdots \wedge \check{X}_i \wedge \cdots \wedge X_r \wedge (\pounds_{X_i} A).$$

(c) If $X_1, \ldots, X_r, Y_1, \ldots, Y_s \in \mathfrak{X}(P)$, then

$$\begin{aligned}[X_1 \wedge \cdots \wedge X_r, Y_1 \wedge \cdots \wedge Y_s] \\ = (-1)^{r+1} \sum_{i=1}^{r} \sum_{j=1}^{s} (-1)^{i+j} [X_i, Y_i] \wedge X_1 \wedge \cdots \wedge \check{X}_i \\ \wedge \cdots \wedge X_r \wedge Y_1 \wedge \cdots \wedge \check{Y}_j \wedge \cdots \wedge Y_s.\end{aligned}$$

(d) If $A \in \Omega_p(P)$, $B \in \Omega_q(P)$, and $\alpha \in \Omega^{p+q-1}(P)$, then

$$\mathbf{i}_{[A,B]}\alpha = (-1)^{q(p+1)} \mathbf{i}_A \mathbf{d}\, \mathbf{i}_B \alpha + (-1)^p \mathbf{i}_B \mathbf{d}\, \mathbf{i}_A \alpha - \mathbf{i}_B \mathbf{i}_A \mathbf{d}\alpha.$$

◇ **10.6-2.** Let M be a finite-dimensional manifold. A ***k-vector field*** is a skew-symmetric contravariant tensor field $A(x) : T_x^*M \times \cdots \times T_x^*M \to \mathbb{R}$ (k copies of T_x^*M). Let $x_0 \in M$ be such that $A(x_0) = 0$.

(a) If $X \in \mathfrak{X}(M)$, show that $(\pounds_X A)(x_0)$ depends only on $X(x_0)$, thereby defining a map $\mathbf{d}_{x_0} A : T_{x_0}M \to T_{x_0}M \wedge \cdots \wedge T_{x_0}M$ (k times), called the ***intrinsic derivative*** of A at x_0.

(b) If $\alpha_1, \ldots, \alpha_k \in T_x^*M$, $v_1, \ldots, v_k \in T_xM$, show that

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k, v_1 \wedge \cdots \wedge v_k \rangle := \det [\langle \alpha_i, v_j \rangle]$$

defines a nondegenerate pairing between $T_x^*M \wedge \cdots \wedge T_x^*M$ and $T_xM \wedge \cdots \wedge T_xM$. Conclude that these two spaces are dual to each other, that the space $\Omega^k(M)$ of k-forms is dual to the space of k-contravariant skew-symmetric tensor fields $\Omega_k(M)$, and that the bases

$$\left\{ \mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_k} \;\middle|\; i_1 < \cdots < i_k \right\}$$

and

$$\left\{ \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \;\middle|\; i_1 < \cdots < i_k \right\}$$

are dual to each other.

(c) Show that the dual map

$$(\mathbf{d}_{x_0} A)^* : T^*_{x_0} M \wedge \cdots \wedge T^*_{x_0} M \to T^*_{x_0} M$$

is given by

$$(\mathbf{d}_{x_0} A)^*(\alpha_1 \wedge \cdots \wedge \alpha_k) = \mathbf{d}(A(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k))(x_0),$$

where $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k \in \Omega^1(M)$ are arbitrary one-forms whose values at x_0 are $\alpha_1, \ldots, \alpha_k$.

◇ **10.6-3** (Weinstein [1983b]). Let $(P, \{ \, , \})$ be a finite-dimensional Poisson manifold with Poisson tensor $B \in \Omega_2(P)$. Let $z_0 \in P$ be such that $B(z_0) = 0$. For $\alpha, \beta \in T^*_{z_0} P$, define

$$[\alpha, \beta]_B = (\mathbf{d}_{z_0} B)^*(\alpha \wedge \beta) = \mathbf{d}(B(\tilde{\alpha}, \tilde{\beta}))(z_0)$$

where $\mathbf{d}_{z_0} B$ is the intrinsic derivative of B and $\tilde{\alpha}, \tilde{\beta} \in \Omega^1(P)$ are such that $\tilde{\alpha}(z_0) = \alpha$, $\tilde{\beta}(z_0) = \beta$. (See Exercise 10.6-2.) Show that $(\alpha, \beta) \mapsto [\alpha, \beta]_B$ defines a bilinear skew-symmetric map $T^*_{z_0} P \times T^*_{z_0} P \to T^*_{z_0} P$. Show that the Jacobi identity for the Poisson bracket implies that $[\, ,]_B$ is a Lie bracket on $T^*_{z_0} P$. Since $(T^*_{z_0} P, [\, ,]_B)$ is a Lie algebra, its dual $T_{z_0} P$ naturally carries the induced Lie–Poisson structure, called the *linearization* of the given Poisson bracket at z_0. Show that the linearization in local coordinates has the expression

$$\{F, G\}(v) = \frac{\partial B^{ij}(z_0)}{\partial z^k} \frac{\partial F}{\partial v^i} \frac{\partial G}{\partial v^j} v^k,$$

for $F, G : T_{z_0} P \to \mathbb{R}$ and $v \in T_{z_0} P$.

◇ **10.6-4** (Magri–Weinstein). On the finite-dimensional manifold P, assume that one has a symplectic form Ω and a Poisson structure B. Define $K = B^\sharp \circ \Omega^\flat : TP \to TP$. Show that $(\Omega^\flat)^{-1} + B^\sharp : T^*P \to TP$ defines a new Poisson structure on P if and only if $\Omega^\flat \circ K^n$ induces a closed two-form (called a ***presymplectic form***) on P for all $n \in \mathbb{N}$.

10.7 Generalities on Lie–Poisson Structures

The Lie–Poisson Equations. We begin by working out Hamilton's equations for the Lie–Poisson bracket.

Proposition 10.7.1. *Let G be a Lie group. The equations of motion for the Hamiltonian H with respect to the $\pm$ Lie–Poisson brackets on $\mathfrak{g}^*$ are*

$$\frac{d\mu}{dt} = \mp \operatorname{ad}^*_{\delta H/\delta \mu} \mu. \qquad (10.7.1)$$

Proof. Let $F \in \mathcal{F}(\mathfrak{g}^*)$ be an arbitrary function. By the chain rule,

$$\frac{dF}{dt} = \mathbf{D}F(\mu) \cdot \dot{\mu} = \left\langle \dot{\mu}, \frac{\delta F}{\delta \mu} \right\rangle, \tag{10.7.2}$$

while

$$\begin{aligned} \{F, H\}_{\pm}(\mu) &= \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle = \pm \left\langle \mu, -\operatorname{ad}_{\delta H/\delta \mu} \frac{\delta F}{\delta \mu} \right\rangle \\ &= \mp \left\langle \operatorname{ad}^*_{\delta H/\delta \mu} \mu, \frac{\delta F}{\delta \mu} \right\rangle. \end{aligned} \tag{10.7.3}$$

Nondegeneracy of the pairing and arbitrariness of F imply the result. ■

Caution. In infinite dimensions, $\mathfrak{g}^*$ does not necessarily mean the literal functional-analytic dual of $\mathfrak{g}$, but rather a space in (nondegenerate) duality with $\mathfrak{g}$. In this case, care must be taken with the definition of $\delta F/\delta \mu$. ♦

Formula (10.7.1) says that on $\mathfrak{g}^*_{\pm}$, *the Hamiltonian vector field of* $H : \mathfrak{g}^* \to \mathbb{R}$ *is given by*

$$X_H(\mu) = \mp \operatorname{ad}^*_{\delta H/\delta \mu} \mu. \tag{10.7.4}$$

For example, for $G = \mathrm{SO}(3)$, formula (10.1.3) for the Lie–Poisson bracket gives

$$X_H(\mathbf{\Pi}) = \mathbf{\Pi} \times \nabla H. \tag{10.7.5}$$

Historical Note. Lagrange devoted a good deal of attention in Volume 2 of *Mécanique Analytique* to the study of rotational motion of mechanical systems. In fact, in equation A on page 212 he gives the reduced Lie–Poisson equations for SO(3) for a rather general Lagrangian. This equation is essentially the same as (10.7.5). His derivation was just how we would do it today—by reduction from material to spatial representation. Formula (10.7.5) actually hides a subtle point in that it identifies $\mathfrak{g}$ and $\mathfrak{g}^*$. Indeed, the way Lagrange wrote the equations, they are much more like their counterpart on $\mathfrak{g}$, which are called the *Euler–Poincaré equations*. We will come to these in Chapter 13, where additional historical information may be found.

Coordinate Formulas. In finite dimensions, if ξ_a, $a = 1, 2, \ldots, l$, is a basis for $\mathfrak{g}$, the structure constants C^d_{ab} are defined by

$$[\xi_a, \xi_b] = C^d_{ab} \xi_d \tag{10.7.6}$$

(a sum on "d" is understood). Thus, the Lie–Poisson bracket becomes

$$\{F, K\}_{\pm}(\mu) = \pm\mu_d \frac{\partial F}{\partial \mu_a}\frac{\partial K}{\partial \mu_b} C^d_{ab}, \tag{10.7.7}$$

where $\mu = \mu_a \xi^a$, $\{\xi^a\}$ is the basis of $\mathfrak{g}^*$ dual to $\{\xi_a\}$, and summation on repeated indices is understood. Taking F and K to be components of μ, (10.7.7) becomes

$$\{\mu_a, \mu_b\}_{\pm} = \pm C^d_{ab}\mu_d. \tag{10.7.8}$$

The equations of motion for a Hamiltonian H likewise become

$$\dot{\mu}_a = \mp\mu_d C^d_{ab}\frac{\partial H}{\partial \mu_b}. \tag{10.7.9}$$

Poisson Maps. In the Lie–Poisson reduction theorem in Chapter 13 we will show that the maps from T^*G to $\mathfrak{g}^*_-$ (resp., $\mathfrak{g}^*_+$) given by $\alpha_g \mapsto T^*_e L_g \cdot \alpha_g$ (resp., $\alpha_g \mapsto T^*_e R_g \cdot \alpha_g$) are Poisson maps. We will show in the next chapter that this is a general property of momentum maps. Here is another class of Poisson maps that will also turn out to be momentum maps.

Proposition 10.7.2. *Let G and H be Lie groups and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Let $\alpha : \mathfrak{g} \to \mathfrak{h}$ be a linear map. The map α is a homomorphism of Lie algebras if and only if its dual $\alpha^* : \mathfrak{h}^*_{\pm} \to \mathfrak{g}^*_{\pm}$ is a (linear) Poisson map.*

Proof. Let $F, K \in \mathcal{F}(\mathfrak{g}^*)$. To compute $\delta(F \circ \alpha^*)/\delta\mu$, we let $\nu = \alpha^*(\mu)$ and use the definition of the functional derivative and the chain rule to get

$$\begin{aligned}\left\langle \frac{\delta}{\delta\mu}(F \circ \alpha^*), \delta\mu \right\rangle &= \mathbf{D}(F \circ \alpha^*)(\mu) \cdot \delta\mu = \mathbf{D}F(\alpha^*(\mu)) \cdot \alpha^*(\delta\mu) \\ &= \left\langle \alpha^*(\delta\mu), \frac{\delta F}{\delta\nu} \right\rangle = \left\langle \delta\mu,\ \alpha \cdot \frac{\delta F}{\delta \nu} \right\rangle. \end{aligned}\tag{10.7.10}$$

Thus,

$$\frac{\delta}{\delta\mu}(F \circ \alpha^*) = \alpha \cdot \frac{\delta F}{\delta\nu}. \tag{10.7.11}$$

Hence,

$$\begin{aligned}\{F \circ \alpha^*, K \circ \alpha^*\}_{+}(\mu) &= \left\langle \mu, \left[\frac{\delta}{\delta\mu}(F \circ \alpha^*),\ \frac{\delta}{\delta\mu}(K \circ \alpha^*)\right]\right\rangle \\ &= \left\langle \mu, \left[\alpha \cdot \frac{\delta F}{\delta\nu},\ \alpha \cdot \frac{\delta K}{\delta\nu}\right]\right\rangle. \end{aligned}\tag{10.7.12}$$

The expression (10.7.12) equals

$$\left\langle \mu, \alpha \cdot \left[\frac{\delta F}{\delta\nu}, \frac{\delta G}{\delta\nu}\right]\right\rangle \tag{10.7.13}$$

for all F and K if and only if α is a Lie algebra homomorphism. ■

This theorem applies to the case $\alpha = T_e\sigma$ for $\sigma : G \to H$ a Lie group homomorphism, as one may see by studying the reduction diagram in Figure 10.7.1 (and being cautious that σ need not be a diffeomorphism).

$$
\begin{tikzcd}
T_g^*G \arrow[d, "\begin{array}{l}\text{right}\\ \text{translate to}\\ \text{identity}\end{array}"'] & T_{\sigma(g)}^*H \arrow[l, "T^*\sigma"'] \arrow[d, "\begin{array}{l}\text{right}\\ \text{translate to}\\ \text{identity}\end{array}"] \\
\mathfrak{g}_+^* & \mathfrak{h}_+^* \arrow[l, "\alpha^*"]
\end{tikzcd}
$$

FIGURE 10.7.1. Lie group homomorphisms induce Poisson maps.

Examples

(a) Plasma to Fluid Poisson Map for the Momentum Variables. Let G be the group of diffeomorphisms of a manifold Q and let H be the group of canonical transformations of $P = T^*Q$. We assume that the topology of Q is such that all locally Hamiltonian vector fields on T^*Q are globally Hamiltonian.[6] Thus, the Lie algebra $\mathfrak{h}$ consists of functions on T^*Q modulo constants. Its dual is identified with itself via the L^2-inner product relative to the Liouville measure $dq\,dp$ on T^*Q. Let $\sigma : G \to H$ be the map $\eta \mapsto T^*\eta^{-1}$, which is a group homomorphism, and let $\alpha = T_e\sigma : \mathfrak{g} \to \mathfrak{h}$. We claim that $\alpha^* : \mathcal{F}(T^*Q)/\mathbb{R} \to \mathfrak{g}^*$ is given by

$$
\alpha^*(F) = \int pf(q,p)\,dp, \tag{10.7.14}
$$

where we regard $\mathfrak{g}^*$ as the space of one-form densities on Q, and the integral denotes fiber integration for each fixed $q \in Q$. Indeed, α is the map taking vector fields X on Q to their lifts $X_{\mathcal{P}(X)}$ on T^*Q. Thus, as a map of $\mathfrak{X}(Q)$ to $\mathcal{F}(T^*Q)/\mathbb{R}$, α is given by $X \mapsto \mathcal{P}(X)$. Its dual is given by

$$
\begin{aligned}
\langle \alpha^*(f), X\rangle = \langle f, \alpha(X)\rangle &= \int_P f\mathcal{P}(X)\,dq\,dp \\
&= \int_P f(q,p)p \cdot X(q)\,dq\,dp,
\end{aligned} \tag{10.7.15}
$$

so $\alpha^*(F)$ is given by (10.7.14), as claimed. ♦

---

[6]For example, this holds if the first cohomology group $H^1(Q)$ is trivial.

(b) Plasma to Fluid Map for the Density Variable. Let $G = \mathcal{F}(Q)$ regarded as an abelian group and let the map $\sigma : G \to \mathrm{Diff}_{\mathrm{can}}(T^*Q)$ be given by $\sigma(\varphi) =$ fiber translation by $\mathbf{d}\varphi$. A computation similar to that above gives the Poisson map

$$\alpha^*(f)(q) = \int f(q,p)\, dp \tag{10.7.16}$$

from $\mathcal{F}(T^*Q)$ to $\mathrm{Den}(Q) = \mathcal{F}(Q)^*$. The integral in (10.7.16) denotes the fiber integration of $f(q,p)$ for fixed $q \in Q$. ♦

Linear Poisson Structures are Lie–Poisson. Next we characterize Lie–Poisson brackets as the linear ones. Let V^* and V be Banach spaces and let $\langle\, ,\rangle : V^* \times V \to \mathbb{R}$ be a weakly nondegenerate pairing of V^* with V. Think of elements of V as linear functionals on V^*. A Poisson bracket on V^* is called ***linear*** if the bracket of any two linear functionals on V^* is again linear. This condition is equivalent to the associated Poisson tensor $B(\mu) : V \to V^*$ being *linear* in $\mu \in V^*$.

Proposition 10.7.3. *Let $\langle\, ,\rangle : V^* \times V \to \mathbb{R}$ be a (weakly) nondegenerate pairing of the Banach spaces V^* and V, and let V^* have a linear Poisson bracket. Assume that the bracket of any two linear functionals on V^* is in the range of $\langle \mu, \cdot\rangle$ for all $\mu \in V^*$ (this condition is automatically satisfied if V is finite-dimensional). Then V is a Lie algebra, and the Poisson bracket on V^* is the corresponding Lie–Poisson bracket.*

Proof. If $x \in V$, we denote by x' the functional $x'(\mu) = \langle \mu, x\rangle$ on V^*. By hypothesis, the Poisson bracket $\{x', y'\}$ is a linear functional on V^*. By assumption this bracket is represented by an element that we denote by $[x,y]'$ in V, that is, we can write $\{x', y'\} = [x,y]'$. (The element $[x,y]$ is unique, since $\langle\, ,\rangle$ is weakly nondegenerate.) It is straightforward to check that the operation $[\, ,]$ on V so defined is a Lie algebra bracket. Thus, V is a Lie algebra, and one then checks that the given Poisson bracket is the Lie–Poisson bracket for this algebra. ■

Exercises

◇ **10.7-1.** Let $\sigma : \mathrm{SO}(3) \to \mathrm{GL}(3)$ be the inclusion map. Identify $\mathfrak{so}(3)^* = \mathbb{R}^3$ with the rigid-body bracket and identify $\mathfrak{gl}(3)^*$ with $\mathfrak{gl}(3)$ using $\langle A, B\rangle = \mathrm{trace}(AB^T)$. Compute the induced map $\alpha^* : \mathfrak{gl}(3) \to \mathbb{R}^3$ and verify directly that it is Poisson.