Differential Geometry of Curves and Surfaces

Abbreviated lecture notes

1. Curves

1. If $U \subset \mathbb{R}^n$ is an open set then a **smooth map** (or a **differentiable map**) $\mathbf{F} : U \to \mathbb{R}^m$ is a C^∞ map. If $D \subset \mathbb{R}^n$ is any set then $\mathbf{F} : D \to \mathbb{R}^m$ is **smooth** if there exist an open set $U \supset D$ and a smooth map $\mathbf{G} : U \to \mathbb{R}^m$ such that $\mathbf{G}|_D = \mathbf{F}$.
2. A **curve** in $\mathbb{R}^n$ is a smooth map $\mathbf{c} : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval. The curve is called **regular** if $\dot{\mathbf{c}}(t) \neq \mathbf{0}$ for all $t \in I$.
3. If $\mathbf{c} : I \to \mathbb{R}^n$ is a curve and $t_0 \in I$ then the **arclength** measured from t_0 is
$$s(t) = \int_{t_0}^{t} \|\dot{\mathbf{c}}(u)\| du.$$
If $\mathbf{c}$ is regular then $s(t)$ is invertible, and we write $\mathbf{c}(s) = \mathbf{c}(t(s))$ (slightly abusing the notation). In this case we have $\|\mathbf{c}'(s)\| = 1$.
4. If $\mathbf{c} : I \to \mathbb{R}^2$ is a regular curve parameterized by arclength, we define the positive orthonormal frame $\{\mathbf{e}_1(s), \mathbf{e}_2(s)\}$ by taking $\mathbf{e}_1(s) = \mathbf{c}'(s)$ (tangent to the curve) and $\mathbf{e}_2(s) = R_{\frac{\pi}{2}}\mathbf{e}_1(s)$, where $R_{\frac{\pi}{2}} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is a rotation by 90° in the positive direction. The **curvature** of $\mathbf{c}$ is the smooth function $k : I \to \mathbb{R}$ such that $\mathbf{c}''(s) = k(s)\mathbf{e}_2(s)$. We have
$$\begin{bmatrix} \mathbf{e}_1'(s) \\ \mathbf{e}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k(s) \\ -k(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1(s) \\ \mathbf{e}_2(s) \end{bmatrix}.$$
5. If $k(s_0) \neq 0$ then $r(s_0) = \frac{1}{|k(s_0)|}$ is the radius of the circle that approximates $\mathbf{c}(s)$ to second order at s_0 (**radius of curvature**). We have
$$\ddot{\mathbf{c}}(t) = \ddot{s}(t)\mathbf{e}_1(s(t)) \pm \frac{\dot{s}^2(t)}{r(s(t))}\mathbf{e}_2(s(t))$$
6. A **positive isometry** of $\mathbb{R}^2$ is a map $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $\mathbf{F}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where $A \in SO(2)$ is a **rotation matrix**, that is, $A = \left(\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right)$ for some $\alpha \in \mathbb{R}$.
7. Two regular plane curves parameterized by arclength are related by a positive isometry if and only if their curvatures coincide.

8. If $\mathbf{c} : I \to \mathbb{R}^2$ is a curve (not necessarily parameterized by its arclength) then its curvature is given by
$$k(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\left[(\dot{x}(t))^2 + (\dot{y}(t))^2\right]^{\frac{3}{2}}},$$
where $\mathbf{c}(t) = (x(t), y(t))$.
9. A regular plane curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$ is said to be **closed** if $\mathbf{c}(a) = \mathbf{c}(b)$ and moreover $\mathbf{c}^{(n)}(a) = \mathbf{c}^{(n)}(b)$ for any $n \in \mathbb{N}$ (so that it can be extended to a periodic curve $\mathbf{c} : \mathbb{R} \to \mathbb{R}^2$). A closed curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$ is said to be **simple** if its restriction to the interval $[a, b)$ is injective. A simple closed curve is said to be **convex** if it bounds a convex set. A **vertex** of a simple closed curve is a critical point (maximum, minimum or inflection point) of its curvature.
10. **Four Vertex Theorem:** Every simple closed plane curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$ has at least four vertices on $[a, b)$ (in fact, at least two minima and two maxima).
11. If $\mathbf{c} : [a, b] \to \mathbb{R}^2$ is a plane curve parameterized by arclength and we write its unit tangent vector as $\mathbf{c}'(s) = (\cos(\theta(s)), \sin(\theta(s)))$ then its curvature is $k(s) = \theta'(s)$.
12. The **rotation index** of a closed plane curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$, parameterized by its arclength, with curvature $k : [a, b] \to \mathbb{R}$, is the integer
$$m = \frac{1}{2\pi} \int_a^b k(s) ds.$$
13. A **(free) homotopy by closed regular curves** bewteen two closed regular plane curves $\mathbf{c}_0, \mathbf{c}_1 : [a, b] \to \mathbb{R}^2$ is a smooth map $\mathbf{H} : [a, b] \times [0, 1] \to \mathbb{R}^2$ such that:
    (i) $\mathbf{H}(t, 0) = \mathbf{c}_0(t)$ for all $t \in [a, b]$;
    (ii) $\mathbf{H}(t, 1) = \mathbf{c}_1(t)$ for all $t \in [a, b]$;
    (iii) $\mathbf{c}_u(t) = \mathbf{H}(t, u)$ is a closed regular curve for all $u \in [0, 1]$.
14. If two closed regular plane curves are homotopic by closed regular curves then they have the same rotation index.
15. The **total curvature** of a closed plane curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$, parameterized by its arclength, with curvature $k : [a, b] \to \mathbb{R}$, is
$$\mu = \int_a^b |k(s)| ds.$$
16. The total curvature μ of a closed regular curve satisfies $\mu \geq 2\pi$, and $\mu = 2\pi$ if and only if the curve is convex.
17. **Isoperimetric inequality:** If $\mathbf{c}$ is a simple closed curve with of minimal length enclosing a region of fixed area A then $\mathbf{c}$ parameterizes a circle of radius $r = \sqrt{\frac{A}{\pi}}$. Conversely, if $\mathbf{c}$ is a simple closed curve of fixed length l enclosing a region of maximal area then $\mathbf{c}$ parameterizes a circle of radius $r = \frac{l}{2\pi}$.
18. The **curvature** of a space curve $\mathbf{c} : I \to \mathbb{R}^3$ parameterized by arclength is
$$k(s) = \|\mathbf{c}''(s)\| \geq 0.$$

If $k(s) \neq 0$ we define the **normal vector** as

$$\mathbf{e}_2(s) = \frac{1}{k(s)}\mathbf{c}''(s),$$

and the **binormal vector** as

$$\mathbf{e}_3(s) = \mathbf{e}_1(s) \times \mathbf{e}_2(s),$$

where

$$\mathbf{e}_1(s) = \mathbf{c}'(s)$$

is the unit tangent vector.

19. **Frenet-Serret formulas:**

$$\begin{bmatrix} \mathbf{e}_1'(s) \\ \mathbf{e}_2'(s) \\ \mathbf{e}_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1(s) \\ \mathbf{e}_2(s) \\ \mathbf{e}_3(s) \end{bmatrix},$$

where the function $\tau(s)$ is called the **torsion** of the curve.

20. A regular space curve $\mathbf{c} : I \to \mathbb{R}^3$ with nonvanishing curvature has zero torsion if and only if it lies on a plane.
21. A **positive isometry** of $\mathbb{R}^3$ is a map $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $\mathbf{F}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where $A \in SO(3)$ is a **rotation matrix**, that is, $A^t A = I$ and $\det A = 1$.
22. Two regular space curves with nonvanishing curvature are related by a positive isometry if and only if their curvatures and torsions coincide.
23. **Frenchel's Theorem:** Let $\mathbf{c} : [a, b] \to \mathbb{R}^3$ be a closed regular space curve parameterized by arclength, and let $k(s) = \|\mathbf{c}''(s)\|$ be its curvature. Then

$$\int_a^b k(s)ds \geq 2\pi,$$

and the equality holds if and only if $\mathbf{c}$ is a plane convex curve.

24. A simple closed regular curve in $\mathbb{R}^3$ is called a **knot**. Two knots are called **equivalent** if they are homotopic (up to reparameterization) by simple closed regular curves. A knot is called **trivial** if it is equivalent to the circle.
25. Let $\mathbf{c} : [a, b] \to \mathbb{R}^3$ be a nontrivial knot parameterized by arclength, and let $k(s) = \|\mathbf{c}''(s)\|$ be its curvature. Then

$$\int_a^b k(s)ds \geq 4\pi.$$

2. Differentiable manifolds

1. A set $M \subset \mathbb{R}^n$ is said to be a **differentiable manifold of dimension** $m \in \{1, \ldots, n-1\}$ if for any point $\mathbf{a} \in M$ there exists an open neighborhood $U \ni \mathbf{a}$ and a smooth function $\mathbf{f} : V \subset \mathbb{R}^m \to \mathbb{R}^{n-m}$ such that

$$M \cap U = \mathrm{Graph}(\mathbf{f}) \cap U$$

for some ordering of the Cartesian coordinates of $\mathbb{R}^n$. We also define a manifold of dimension 0 as a set of isolated points, and a manifold of dimension n as an open set.

2. $M \subset \mathbb{R}^n$ is a differentiable manifold of dimension m if and only if for each point $\mathbf{a} \in M$ there exists an open set $U \ni \mathbf{a}$ and a smooth function $\mathbf{F} : U \to \mathbb{R}^{n-m}$ such that:
   (i) $M \cap U = \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = \mathbf{0}\}$;
   (ii) $\operatorname{rank} D\mathbf{F}(\mathbf{a}) = n - m$.
3. A vector $\mathbf{v} \in \mathbb{R}^n$ is said to be **tangent** to a set $M \subset \mathbb{R}^n$ at the point $\mathbf{a} \in M$ if there exists a smooth curve $\mathbf{c} : \mathbb{R} \to M$ such that $\mathbf{c}(0) = \mathbf{a}$ and $\dot{\mathbf{c}}(0) = \mathbf{v}$. A vector $\mathbf{v} \in \mathbb{R}^n$ is said to be **orthogonal** to M at the point $\mathbf{a}$ if it is orthogonal to all vectors tangent to M at $\mathbf{a}$.
4. If $M \subset \mathbb{R}^n$ is a manifold of dimension m then the set $T_{\mathbf{a}}M$ of all vectors tangent to M at the point $\mathbf{a} \in M$ is a vector space of dimension m, called the **tangent space** to M at $\mathbf{a}$. Its orthogonal complement $T_{\mathbf{a}}^{\perp}M$ is a vector space of dimension $(n-m)$, called the **normal space** to M at $\mathbf{a}$.
5. Let $M \subset \mathbb{R}^n$ be an m-manifold, $\mathbf{a} \in M$, $U \ni \mathbf{a}$ an open set and $\mathbf{F} : U \to \mathbb{R}^{n-m}$ such that $M \cap U = \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = \mathbf{0}\}$ with $\operatorname{rank} D\mathbf{F}(\mathbf{a}) = n - m$. Then $T_{\mathbf{a}}M = \ker D\mathbf{F}(\mathbf{a})$.
6. A **parameterization** of a given m-manifold $M \subset \mathbb{R}^n$ is a smooth injective map $\mathbf{g} : U \to M$, with $U \subset \mathbb{R}^m$ open, such that $\operatorname{rank} D\mathbf{g}(\mathbf{t}) = m$ for all $\mathbf{t} \in U$. We have
$$T_{\mathbf{g}(\mathbf{t})}M = \operatorname{span}\left\{\frac{\partial \mathbf{g}}{\partial t^1}(\mathbf{t}), \ldots, \frac{\partial \mathbf{g}}{\partial t^m}(\mathbf{t})\right\}.$$
7. Given a smooth map $\mathbf{g} : U \to \mathbb{R}^n$, with $U \subset \mathbb{R}^m$ open, such that $\operatorname{rank} D\mathbf{g}(\mathbf{t}) = m$ for all $\mathbf{t} \in U$, and given any point $\mathbf{t}_0 \in U$, there exists an open set $U_0 \subset U$ with $\mathbf{t}_0 \in U_0$ such that $\mathbf{g}(U_0)$ is an m-manifold.

3. Differential forms

1. The **dual vector space** to $\mathbb{R}^n$ is
$$(\mathbb{R}^n)^* = \{\alpha : \mathbb{R}^n \to \mathbb{R} : \alpha \text{ is linear}\}.$$
The elements of $(\mathbb{R}^n)^*$ are called **covectors**.
2. The covectors $dx^1, \ldots, dx^n \in (\mathbb{R}^n)^*$ defined through
$$dx^i(\mathbf{e}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
form a basis for $(\mathbb{R}^n)^*$, whose dimension is then n.
3. A (covariant) k-**tensor** T is a multilinear map $T : (\mathbb{R}^n)^k \to \mathbb{R}$, i.e.
   (i) $T(\mathbf{v}_1, \ldots, \mathbf{v}_i + \mathbf{w}_i, \ldots, \mathbf{v}_k) = T(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k) + T(\mathbf{v}_1, \ldots, \mathbf{w}_i, \ldots, \mathbf{v}_k)$;
   (ii) $T(\mathbf{v}_1, \ldots, \lambda\mathbf{v}_i, \ldots, \mathbf{v}_k) = \lambda T(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k)$.
4. A k-tensor α is said to be **alternanting**, or a k-**covector**, if
$$\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_j \ldots, \mathbf{v}_k) = -\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_i \ldots, \mathbf{v}_k).$$
We denote by $\Lambda^k(\mathbb{R}^n)$ the vector space of all k-covectors.

5. Given $i_1, \dots, i_k \in \{1, \dots, n\}$, we define $dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Lambda^k(\mathbb{R}^n)$ as

$$dx^{i_1} \wedge \dots \wedge dx^{i_k}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \det \begin{bmatrix} dx^{i_1}(\mathbf{v}_1) & \dots & dx^{i_1}(\mathbf{v}_k) \\ \dots & \dots & \dots \\ dx^{i_k}(\mathbf{v}_1) & \dots & dx^{i_k}(\mathbf{v}_k) \end{bmatrix}.$$

The set $\left\{dx^{i_1} \wedge \dots \wedge dx^{i_k}\right\}_{1 \le i_1 < \dots < i_k \le n}$ is a basis for $\Lambda^k(\mathbb{R}^n)$, whose dimension is then $\binom{n}{k}$. Since $\binom{n}{0} = 1$, we define $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$.

6. If $\alpha \in \Lambda^k(\mathbb{R}^n)$ and $\beta \in \Lambda^l(\mathbb{R}^n)$,

$$\alpha = \sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k}, \qquad \beta = \sum_{j_1 < \dots < j_l} \beta_{j_1 \dots j_l} \, dx^{j_1} \wedge \dots \wedge dx^{j_l},$$

we define their **wedge product** $\alpha \wedge \beta \in \Lambda^{k+l}(\mathbb{R}^n)$ as

$$\alpha \wedge \beta = \sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_l}} \alpha_{i_1 \dots i_k} \beta_{j_1 \dots j_l} \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}.$$

If α is a 0-covetor (real number), its wedge product by α is simply the product by a scalar.

7. **Properties of the wedge product:**
   (i) $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$;
   (ii) $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ if $\alpha \in \Lambda^k(\mathbb{R}^n), \beta \in \Lambda^l(\mathbb{R}^n)$;
   (iii) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.
8. A **differential form of degree** k on $U \subset \mathbb{R}^n$ is a smooth function $\omega : U \to \Lambda^k(\mathbb{R}^n)$. We denote by $\Omega^k(U)$ the set of k-forms on U.
9. If $\mathbf{f} : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ is smooth and $\omega \in \Omega^k(V)$ then the **pull-back** of ω by $\mathbf{f}$ is the k-form $\mathbf{f}^*\omega \in \Omega^k(U)$ defined by

$$(\mathbf{f}^*\omega)(\mathbf{x})(\mathbf{v}_1, \dots, \mathbf{v}_k) = \omega(\mathbf{f}(\mathbf{x}))(D\mathbf{f}(\mathbf{x})\mathbf{v}_1, \dots, D\mathbf{f}(\mathbf{x})\mathbf{v}_k).$$

10. **Properties of the pull-back:**
    (i) $\mathbf{f}^*(\omega + \eta) = \mathbf{f}^*\omega + \mathbf{f}^*\eta$;
    (ii) $\mathbf{f}^*(\omega \wedge \eta) = \mathbf{f}^*\omega \wedge \mathbf{f}^*\eta$;
    (iii) $(\mathbf{g} \circ \mathbf{f})^*(\omega) = \mathbf{f}^*(\mathbf{g}^*\omega)$.
11. If $\omega \in \Omega^k(U)$ with $U \subset \mathbb{R}^n$,

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(\mathbf{x}) \, dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

then its **exterior derivative** is the $(k+1)$-form $d\omega \in \Omega^{k+1}(U)$ defined by

$$d\omega = \sum_{i_1 < \dots < i_k} \sum_{i=1}^{n} \frac{\partial \omega_{i_1 \dots i_k}}{\partial x^i} \, dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

12. **Properties of the exterior derivative:**
   (i) $d(\omega + \eta) = d\omega + d\eta$;
   (ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ if ω has degree k;
   (iii) $d(d\omega) = 0$;
   (iv) $\mathbf{f}^*(d\omega) = d(\mathbf{f}^*\omega)$.
13. We say that $\omega \in \Omega^k(U)$ is:
   (i) **closed** if $d\omega = 0$;
   (ii) **exact** if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$ (called a **potential** for ω).
14. If $\omega \in \Omega^k(U)$ is exact then ω is closed.
15. **Poincaré Lemma:** If $\omega \in \Omega^k(U)$ is closed and the open set U is star-shaped then ω is exact.
16. If $\mathbf{g} : U \subset \mathbb{R}^m \to M$ and $\mathbf{h} : V \subset \mathbb{R}^m \to M$ are parameterizations of the m-manifold $M \subset \mathbb{R}^n$ then $\mathbf{h}^{-1} \circ \mathbf{g}$ is a **diffeomorphism** (smooth bijection with smooth inverse).
17. We say that two parameterizations $\mathbf{g} : U \subset \mathbb{R}^m \to M$ and $\mathbf{h} : V \subset \mathbb{R}^m \to M$ of the m-manifold $M \subset \mathbb{R}^n$ induce the **same orientation** if $\det D(\mathbf{h}^{-1} \circ \mathbf{g}) > 0$, and **opposite orientations** if $\det D(\mathbf{h}^{-1} \circ \mathbf{g}) < 0$. The manifold M is called **orientable** if it is possible to choose parameterizations whose images cover M and induce the same orientation. An **orientation** on an orientable manifold is a choice of a maximal family of parameterizations under these conditions, which are said to be **positive**. An orientable manifold with a choice of orientation is said to be **oriented**.
18. If $\mathbf{g} : U \subset \mathbb{R}^m \to M$ is a positive parameterization of the oriented m-manifold $M \subset \mathbb{R}^n$ and $\omega \in \Omega^m(\mathbb{R}^n)$, we define the **integral** of ω along $\mathbf{g}(U)$ as

$$\int_{\mathbf{g}(U)} \omega = \int_U \omega(\mathbf{g}(\mathbf{t})) \left(\frac{\partial \mathbf{g}}{\partial t^1}, \dots, \frac{\partial \mathbf{g}}{\partial t^m} \right) dt^1 \dots dt^m$$
$$= \int_U \mathbf{g}^* \omega \left(\mathbf{e}_1, \dots, \mathbf{e}_m \right) dt^1 \dots dt^m.$$

19. If we think of an open set $U \subset \mathbb{R}^n$ as an n-manifold parameterized by the identity map (which we take to be positive), then

$$\int_U f(\mathbf{x})\, dx^1 \wedge \dots \wedge dx^n = \int_U f(\mathbf{x})\, dx^1 \dots dx^n,$$

   and so

$$\int_{\mathbf{g}(U)} \omega = \int_U \mathbf{g}^* \omega.$$

20. The integral of a m-form on the image of a positive parameterization of an m-manifold is well defined, that is, it is independent of the choice of parameterization.
21. If $M \subset \mathbb{R}^n$ is an oriented m-manifold and $\omega \in \Omega^m(\mathbb{R}^n)$, we define

$$\int_M \omega = \sum_{i=1}^N \int_{\mathbf{g}_i(U_i)} \omega,$$

   where $\mathbf{g}_i : U_i \to M$ are positive parameterizations whose images are disjoint and cover M except for a finite number of manifolds of dimension smaller than m. It can be shown that it is always possible to obtain a finite number of parameterizations of this kind, and that the definition above does not depend on the choice of these parameterizations.

22. Informally, an m-**manifold with boundary** is a subset $M \subset N$ of an m-manifold $N \subset \mathbb{R}^n$ delimited by an $(m-1)$-manifold $\partial M \subset M$, called the **boundary** of M, such that $M \setminus \partial M$ is again an m-manifold. We say that M is **orientable** if N is orientable. If M is oriented, the **induced orientation** on ∂M is defined as follows: if $\mathbf{g} : U \cap \{t^1 \leq 0\} \to M$ is a positive parameterization of M such that $\mathbf{h}(t^2, \ldots, t^m) = \mathbf{g}(0, t^2, \ldots, t^m)$ is a parameterization of ∂M, then $\mathbf{h}$ is positive. Moreover, if $\omega \in \Omega^m(\mathbb{R}^n)$, we define

$$\int_M \omega = \int_{M \setminus \partial M} \omega.$$

23. **Stokes Theorem:** If $M \subset \mathbb{R}^n$ is a compact, oriented m-manifold with boundary and $\omega \in \Omega^{m-1}(\mathbb{R}^n)$ then

$$\int_M d\omega = \int_{\partial M} \omega,$$

where ∂M has the induced orientation.

24. If M is an oriented compact m-manifold (without boundary) and $\omega \in \Omega^{m-1}(\mathbb{R}^n)$ then

$$\oint_M d\omega = 0.$$

4. Surfaces

1. A **surface** is a 2-dimensional differentiable manifold $S \subset \mathbb{R}^3$.
2. The **first fundamental form** of a surface S parameterized by $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is the quadratic form

$$\mathbf{I} = d\mathbf{g} \cdot d\mathbf{g} = E du^2 + 2F du\, dv + G dv^2,$$

where

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{g}}{\partial u} \cdot \frac{\partial \mathbf{g}}{\partial u} & \frac{\partial \mathbf{g}}{\partial u} \cdot \frac{\partial \mathbf{g}}{\partial v} \\ \frac{\partial \mathbf{g}}{\partial v} \cdot \frac{\partial \mathbf{g}}{\partial u} & \frac{\partial \mathbf{g}}{\partial v} \cdot \frac{\partial \mathbf{g}}{\partial v} \end{bmatrix}$$

is a positive definite matrix of functions, called the **matrix of the metric**.

3. The squared length of a vector tangent to a surface S parameterized by $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is

$$\left\| v^1 \frac{\partial \mathbf{g}}{\partial u} + v^2 \frac{\partial \mathbf{g}}{\partial v} \right\|^2 = \mathbf{I}(v^1, v^2) = E(v^1)^2 + 2F v^1 v^2 + G(v^2)^2.$$

In particular, the length of a curve $\mathbf{c} : [a, b] \to S$ given by $\mathbf{c}(t) = \mathbf{g}(u(t), v(t))$ is

$$\int_a^b \sqrt{\mathbf{I}(\dot{u}(t), \dot{v}(t))}\, dt = \int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$$

4. The **second fundamental form** of a surface S parameterized by $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is the quadratic form

$$\mathbf{II} = -d\mathbf{g} \cdot d\mathbf{n} = L du^2 + 2M du\, dv + N dv^2,$$

where

$$\mathbf{n} = \frac{\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}}{\left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\|}$$

is a unit normal vector to S and

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} = - \begin{bmatrix} \frac{\partial \mathbf{g}}{\partial u} \cdot \frac{\partial \mathbf{n}}{\partial u} & \frac{\partial \mathbf{g}}{\partial u} \cdot \frac{\partial \mathbf{n}}{\partial v} \\ \frac{\partial \mathbf{g}}{\partial v} \cdot \frac{\partial \mathbf{n}}{\partial u} & \frac{\partial \mathbf{g}}{\partial v} \cdot \frac{\partial \mathbf{n}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathbf{g}}{\partial u^2} \cdot \mathbf{n} & \frac{\partial^2 \mathbf{g}}{\partial u \partial v} \cdot \mathbf{n} \\ \frac{\partial^2 \mathbf{g}}{\partial v \partial u} \cdot \mathbf{n} & \frac{\partial^2 \mathbf{g}}{\partial v^2} \cdot \mathbf{n} \end{bmatrix}.$$

5. At a point where the second fundamental form is definite ($LN - M^2 > 0$) the surface is convex (i.e. it lies on the same side of the tangent plane); at a point where the second fundamental form is indefinite ($LN - M^2 < 0$) the surface is not convex (i.e. it lies on both sides of the tangent plane).
6. **Gauss's equations:**

$$\begin{aligned} \frac{\partial^2 \mathbf{g}}{\partial u^2} &= \Gamma^u_{uu} \frac{\partial \mathbf{g}}{\partial u} + \Gamma^v_{uu} \frac{\partial \mathbf{g}}{\partial v} + L\mathbf{n}; \\ \frac{\partial^2 \mathbf{g}}{\partial u \partial v} &= \Gamma^u_{uv} \frac{\partial \mathbf{g}}{\partial u} + \Gamma^v_{uv} \frac{\partial \mathbf{g}}{\partial v} + M\mathbf{n}; \\ \frac{\partial^2 \mathbf{g}}{\partial v \partial u} &= \Gamma^u_{vu} \frac{\partial \mathbf{g}}{\partial u} + \Gamma^v_{vu} \frac{\partial \mathbf{g}}{\partial v} + M\mathbf{n}; \\ \frac{\partial^2 \mathbf{g}}{\partial v^2} &= \Gamma^u_{vv} \frac{\partial \mathbf{g}}{\partial u} + \Gamma^v_{vv} \frac{\partial \mathbf{g}}{\partial v} + N\mathbf{n}, \end{aligned}$$

where the functions $\Gamma^u_{uu}, \Gamma^u_{uv} = \Gamma^u_{vu}, \Gamma^u_{vv}, \Gamma^v_{uu}, \Gamma^v_{uv} = \Gamma^v_{vu}, \Gamma^v_{vv}$ are called the **Christoffel symbols**.
7. **Weingarten's equations:**

$$\begin{aligned} \frac{\partial \mathbf{n}}{\partial u} &= \frac{FM - GL}{EG - F^2} \frac{\partial \mathbf{g}}{\partial u} + \frac{FL - EM}{EG - F^2} \frac{\partial \mathbf{g}}{\partial v}; \\ \frac{\partial \mathbf{n}}{\partial v} &= \frac{FN - GM}{EG - F^2} \frac{\partial \mathbf{g}}{\partial u} + \frac{FM - EN}{EG - F^2} \frac{\partial \mathbf{g}}{\partial v}. \end{aligned}$$

8. The **normal curvature** of a curve $\mathbf{c} : I \to S$ on a surface S, parameterized by arclength, is $k_n(s) = \mathbf{c}''(s) \cdot \mathbf{n}$, where $\mathbf{n}$ is a unit normal vector to S at $\mathbf{c}(s)$. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization and $\mathbf{c}(s) = \mathbf{g}(u(s), v(s))$ then $k_n(s) = \mathbf{II}(u'(s), v'(s))$.
9. The maximum and the minimum of $\mathbf{II}(v^1, v^2)$ subject to the constraint $\mathbf{I}(v^1, v^2) = 1$ are called the **principal curvatures** of S at the point under consideration. The directions of the corresponding unit tangent vectors are called the **principal directions** of S at that point. If the principal curvatures are different then the principal directions are orthogonal.
10. The **mean curvature** of a surface S at a given point is

$$H = \frac{1}{2}(k_1 + k_2) = \frac{EN + GL - 2FM}{2(EG - F^2)},$$

where k_1 and k_2 are the principal curvatures at that point. The **Gauss curvature** of S at the same point is

$$K = k_1 k_2 = \frac{LN - M^2}{EG - F^2}.$$

S is said to be **minimal** if $H \equiv 0$, and **flat** if $K \equiv 0$.

11. If $k_1 = k_2$ at some point then that point is called **umbillic**. Moreover, we call the point **elliptic** if $K > 0$, **hyperbolic** if $K < 0$, and **parabolic** if $K = 0$. The surface is convex at elliptic points, and is not convex at hyperbolic points.
12. The principal direction corresponding to the principal curvature k_1 is given by tangent vectors of the form
$$v^1 \frac{\partial \mathbf{g}}{\partial u} + v^2 \frac{\partial \mathbf{g}}{\partial v}$$
such that
$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = k_1 \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}.$$
13. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization then the **area** of $\mathbf{g}(U) \subset S$ is
$$A = \iint_U \left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\| du\, dv = \iint_U \sqrt{EG - F^2}\, du\, dv.$$
14. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization then
$$\frac{\partial \mathbf{n}}{\partial u} \times \frac{\partial \mathbf{n}}{\partial v} = K \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}.$$
In particular, if $K(u_0, v_0) \neq 0$ then
$$|K(u_0, v_0)| = \lim_{\varepsilon \to 0} \frac{A'(\varepsilon)}{A(\varepsilon)},$$
where $A(\varepsilon)$ is the area of $\mathbf{g}(B_\varepsilon(u_0, v_0)) \subset S$ and $A'(\varepsilon)$ is the area of $\mathbf{n}(B_\varepsilon(u_0, v_0)) \subset S^2$.
15. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization,
$$\mathbf{g}_\varepsilon(u, v) = \mathbf{g}(u, v) + \varepsilon f(u, v) \mathbf{n}(u, v)$$
is a small deformation of $\mathbf{g}$ and $A(\varepsilon)$ is the area of $\mathbf{g}_\varepsilon(U)$ then
$$\frac{dA}{d\varepsilon}(0) = -2 \iint_U f H \sqrt{EG - F^2}\, du\, dv.$$
In particular, if S has minimal area (for a fixed boundary) then $H \equiv 0$, and if S has minimal area while bounding a fixed volume then H is constant.
16. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{n}\}$ is an orthonormal frame and $\theta^1, \theta^2 \in \Omega^1(U)$ are such that
$$d\mathbf{g} = \theta^1 \mathbf{e}_1 + \theta^2 \mathbf{e}_2$$
then the first fundamental form is
$$\mathbf{I} = (\theta^1)^2 + (\theta^2)^2.$$
Moreover, if ${\omega_i}^j \in \Omega^1(U)$ are such that
$$d\mathbf{e}_i = \sum_{j=1}^{3} {\omega_i}^j \mathbf{e}_j,$$

we have

$$\omega_i{}^j = -\omega_j{}^i.$$

Defining the symmetric 2×2 matrix B through

$$\begin{cases} \omega_1{}^3 = b_{11}\theta^1 + b_{12}\theta^2 \\ \omega_2{}^3 = b_{21}\theta^1 + b_{22}\theta^2 \end{cases},$$

we have

$$\mathbf{II} = \sum_{i,j=1}^{2} b_{ij}\theta^i\theta^j.$$

In particular,

$$H = \frac{1}{2}\operatorname{tr} B \qquad \text{and} \qquad K = \det B$$

(that is, the eigenvalues of B are k_1 and k_2).