# Notes on deformations and index theorems I

Ryszard Nest

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## 1 Continuous fields of C*-algebras

**Definition 1.1** *Let $X$ be a lcally compact Hausdorff topological space and $\{A_x\}_{x\in X}$ a family of C*-algebras. A structure of continuous field over $X$*

*with fibers $A_x$ is given by a *-algebra $\mathcal{A}$ of functions*

$$X \ni x \to a(x) \in A_x$$

*satisfying the following conditions.*

(i) *For each $a \in \mathcal{A}$, the function $x \to ||a(x)||_{A_x}$ is in $C_0(X)$.*

(ii) *For each $x \in X$, $\{a(x) \mid a \in \mathcal{A}\}$ is a dense *-subalgebra of the fiber $A_x$.*

*The *-algebra structure on $\mathcal{A}$ is given by the pointwise sum, product and adjoint.*

*Let $\mathbf{A}$ be the *-algebra of sections which are locally uniformly approximable by elements of $\mathcal{A}$, i. e.*

$$f \in \mathbf{A} \Longleftrightarrow \left\{ \forall_{t \in [0,1]} \exists_{\epsilon > 0,\ \sigma_n \in \mathcal{A}} \lim_{n \to \infty} \sup_{|x-t| \le \epsilon} ||\sigma_n(x) - f(x)|| = 0 \right\}$$

*The completion of $\mathcal{A}$ in the norm $||a|| = \sup_x ||a(x)||_{A_x}$ is called the C*-algebra of sections of the continious field.*

Note that the *-representation of $\mathcal{A}$ given by the evaluation map $ev_x : a \to a(x)$ extends to a *-homomorphism $ev_x : A \to A_x$ and the continuity condition for the norms above implies that

$$X \ni x \to ||ev_x(a)||$$

is continous for any $a \in A$. Our main application of the notion of continuous field is based on the following

**Observation 1**

In the notation above, assume that $p_0$ is a projection in $A_0$. There exists a continuous section $q \in A$ and an open neighbourhood $U$ of $x_0$ such that, for all $x \in U$, $q(x)$ is a projection and $q(x_0) = p_0$. Given any other $q_1 \in A$ satisfying the same conditions and a positive $\epsilon$, there exists an open neighbourhood $U_1$ of $x_0$ such that

$$\sup_{x \in U_1} ||q(x) - q_1(x)|| \le \epsilon$$

Similar statement holds for a unitary in $A_0$.

## 2 Heisenberg field

Let $t \in [0,1]$ and $\mathcal{S}(\mathbb{R})$ denote the space of Schwartz functions on $\mathbb{R}^2$. We define a linear map

$$\pi_t : \mathcal{S}(\mathbb{R}^2) \to BL^2(\mathbb{R}^2)$$

by

$$(\pi_t(f)g)(x,w) = \int dv f(x,v)g(x-tv, w-v).$$

We will prove below, that this produces a continuous field of C*-algebras.

### 2.1 The fibers

First let us look at the C*-algebras generated by $\{\pi_t(f) \mid f \in \mathcal{S}(\mathbb{R}^2)\}$.

**Case of t=0.**

Let $\mathcal{F}_2$ denote the Fourier transform in the second variable. Then

$$\mathcal{F}_2\pi_0(f)\mathcal{F}_2^{-1}g = \mathcal{F}_2(f)g$$

and since the Fourier transform is unitary, the C*-algebra generated by $\{\pi_0(f) \mid f \in \mathcal{S}(\mathbb{R}^2)\}$ is isomorphic to the C* algebra generated by the multiplication operators $g \to fg$ with $f \in \mathcal{S}(\mathbb{R}^2)$ (acting on $L^2(\mathbb{R}^2)$). As a consequence,

$$\big\{\text{norm closure of } \{\pi_0(f) \mid f \in \mathcal{S}(\mathbb{R}^2)\}\big\} \simeq C_0(\mathbb{R}^2).$$

**Case of t $\neq$ 0.**

Let $U_t$ be the unitary operator on $L^2(\mathbb{R}^2)$ given by

$$(U_t g)(x,w) = g(x, w - \frac{x}{t}).$$

A straightforward computation gives

$$U_t\pi_t(f)U_t^{-1} = \rho_t(f) \otimes 1,$$

where we write $L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, and

$$(\rho_t(f)g)(x) = \int dv f(x,v)g(x-tv)$$

As a consequence, the C*-algebra generated by $\{\pi_t(f) \mid f \in \mathcal{S}(\mathbb{R}^2)\}$ is isomorphic to the norm closure in $B(L^2(\mathbb{R}))$ of

$$\{\rho_t(f) \mid f \in \mathcal{S}(\mathbb{R}^2)\}.$$

**Lemma 2.1** *For any $f \in \mathcal{S}(\mathbb{R}^2)$and $t \neq 0$ the operator $\rho_t(f)$ is Hilbert Schmidt and*

$$||\rho_t(f)|| \leq ||\rho_t(f)||_{HS} \leq t^{-\frac{1}{2}}||f||_2.$$

*The norm closure of the set $\{\rho_t(f) \mid f \in \mathcal{S}(\mathbb{R}^2)\}$ coincides with $\mathcal{K}$, the algebra of compact operators on $L^2(\mathbb{R})$.*

Proof. First note that, by change of variables,

$$(\rho_t(f)g)(x) = -t^{-1} \int dv f(x, \frac{v-x}{t})g(v),$$

hence $\rho_t(f)$ is an integral operator with kernel

$$K_{f,t}(x,y) = -t^{-1}f(x, \frac{y-x}{t}).$$

Hence

$$||\rho_t(f)||_{HS} = ||K_{f,t}||_2 \leq t^{-\frac{1}{2}}||f||_2.$$

Given two Schwartz functions $\psi$ and $\phi$ of a single variable with $L^2$-norm one, set

$$f(x,v) = -t\psi(x)\phi(t(x+v)).$$

Then $\rho_t(f)$ is a rank one partial isometry $g \rightarrow < g, \overline{\phi} > \psi$. Hence $\{\rho_t(f) \mid f \in \mathcal{S}(\mathbb{R}^2)\}$ is dense in the set of finite rank operators and its closure coincides with $\mathcal{K}$.

## 2.2 The product

Given $f, g \in \mathcal{S}(\mathbb{R}^2)$, we set

$$f^{*_t}(x,v) = -\overline{f(x-tv,-v)}$$

and

$$(f *_t g)(x,w) = \int dv f(x,v)g(x-tv, w-v)$$

**Theorem 2.2** *Suppose that $f$ and $g$ are Schwartz funtions on $\mathbb{R}^2$. Then*

*(i)* $||\rho_t(f)\rho_t(g) - \rho_t(f *_0 g)||_{HS} = O(t^{\frac{1}{2}})$, *and*

*(ii)* $||\rho_t(f)^* - \rho_t(f^{*_0})||_{HS} = O(t^{\frac{1}{2}})$

Proof. Both proofs are more or less the same, so we will just deal with the second satement. It is straightforward to see that

$$\rho_t(f)^* = \rho_t(f^{*t}),$$

hence it is enough to estimate

$$t^{-\frac{1}{2}}||f^{*t} - f^{*0}||_2.$$

But

$$f(x-tv,-v)-f(x,-v) = \int_0^t \frac{d}{d\lambda} f(x-\lambda v,-v)d\lambda = -v\int_0^t (\partial_x f)(x-\lambda v,-v)d\lambda.$$

and since $vf(x,v)$ is rapidly decreasing in both $v$ and $x$,

$$\int dxdv \left| v\int_0^t (\partial_x f)(x-\lambda v,-v)d\lambda \right|^2$$

is of the order $t^2$. The claimed result follws.

### 2.3 The continuity of the product at $t = 0$

**Proposition 2.3** *Suppose that $\pi_0(f) \geq 0$ and $\epsilon > 0$. Then, for $t$ close enough to zero,*

$$\rho_t(f) \geq -\epsilon 1$$

Proof. By the theorem above,

$$||\rho_t(f) - \{\rho_t(\mathcal{F}_2^{-1}((\mathcal{F}_2 f)^{\frac{1}{2}}))\}^*\{\rho_t(\mathcal{F}_2^{-1}(\mathcal{F}_2 f)^{\frac{1}{2}}))\}|| \to 0$$

as $t \to 0$.

**Theorem 2.4** *Suppose that $f \in \mathcal{S}(\mathbb{R}^2)$. Then*

$$\lim_{t\to 0^+} ||\rho_t(f)|| = ||\pi_0(f)||.$$

Proof.

*Lower semicontinuity*

Given $\pi_0(f) \geq 0$ and $\epsilon > 0$, we can find a function $g \in L^2(\mathbb{R})$ such that $||g||_2 = 1$ and

$$||\pi_0(f)|| - \epsilon \leq < \pi_0 g|g > .$$

But the Lebesque dominated convergence theorem shows immediately that

$$< \pi_0(f)g|g >= \lim_{t\to 0} < \pi_t(f)g|g >$$

and hence

$$||\pi_0(f)||b \leq \lim_{t\to 0} < \pi_t(f)g|g > +\epsilon \leq \liminf ||\pi_t(f)|| + \epsilon.$$

Since $||\pi_t(f)|| = ||\rho_t(f)||$, we get

$$||\pi_0(f)|| \leq \liminf ||\rho_t(f)||.$$

*Upper semicontinuity*

We can assume that $0 \leq \pi_0(f) \leq 1$, hence

$$0 \leq \pi_0(f)^2 \leq \pi_0(f) \leq 1.$$

But then, given positive $\epsilon$, for $t$ small enough

$$||\rho_t(f)^* - \rho_t(f)|| \leq \epsilon$$

and

$$-2\epsilon \leq \rho_t(f)^2 - \epsilon \leq \rho_t(f) \leq 1 + \epsilon.$$

These imply that

$$\limsup ||\rho_t(f)|| \leq (1 + \epsilon)||\pi_0(f)||$$

for all positive $\epsilon$ and hence

$$\limsup ||\rho_t(f)|| \leq ||\pi_0(f)||$$

## 2.4 The "Heisenberg" field

We will prove a following statement which will be usefull later

**Proposition 2.5** *Let $\mathcal{F}_1$ denote the Fourier transform in the first variable. For any $f$ and $t$*

$$||\pi_t(f)|| \leq ||\mathcal{F}_1^{-1}(f)||_1$$

Proof.

$$\pi_t(f)g = \int d\omega dv \mathcal{F}_1^{-1}(f)(\omega, v) U_{\omega,v,t} g$$

where $U_{\omega,v,t}$ is the *unitary* operator (hence of norm one) given by

$$(U_{\omega,v,t}g)(x, w) = e^{i\omega x} g(x - tv, w - v)$$

Hence

$$||\rho_t(f)|| \leq \int d\omega dv ||\mathcal{F}_1^{-1}(f)(\omega, v) U_{\omega,v,t}|| = \int d\omega dv |\mathcal{F}_1^{-1}(f)(\omega, v)|$$

**Corollary 2.6** *Suppose that $\phi \in C_0(\mathbb{R}^2)$ satisfies*

$$\partial_x^i \partial_\xi^j \phi \in C_0(\mathbb{R}^2),\ i + j \leq 2.$$

*Then*

$$t \to \pi_t(\mathcal{F}_2^{-1}\phi)$$

*is norm continuous.*

Proof. We just note that

$$\phi \in C_0^2(\mathbb{R}^2) \Longrightarrow \mathcal{F}_1^{-1}(\mathcal{F}_2^{-1}\phi) \in L^1(\mathbb{R}^2)$$

with a norm estimate

$$||\mathcal{F}_1^{-1}(\mathcal{F}_2^{-1}\phi)||_1 \leq C(\sum_{i+j\leq 2} ||\partial_x^i \partial_\xi^j \phi||_\infty).$$

From here it follows that, if we approximate $\phi$ by Schwartz functions $\phi_n$ so that

$$||\partial_x^i \partial_\xi^j (\phi - \phi_n)||_\infty \to 0$$

for $n \to \infty$ and $i + j \leq 2$, then

$$\sup_t ||\pi_t(\mathcal{F}_2^{-1}\phi) - \pi_t(\mathcal{F}_2^{-1}\phi_n)|| \to 0$$

and hence $t \to \pi_t(\mathcal{F}_2^{-1}\phi)$ is norm continuous "section".

**Corollary 2.7** *Let $\mathcal{S}$ be the space of functions on $\mathbb{R}^2 \times [0, 1]$ which satisfie the following estimates.*

$$\forall_{i,j,\alpha} \sup_{x,\xi,t} |(1 + x^2 + \xi^2)^\alpha \partial_x^i \partial_\xi^j \phi(x, \xi, t)| < \infty.$$

*Then $\{t \to \pi_t(\mathcal{F}_2^{-1}\phi) \mid \phi \in \mathcal{S}\}$ is a dense algebra of norm continuous maps.*

Proof. Since $\pi_t(f)\pi_t(g) = \pi_t(f *_t g))$ and $\mathcal{S}$ is closed under the $*_t$-product, it is enough to show the continuity of $\pi(\mathcal{F}_2^{-1}\phi)$ for $\phi \in \mathcal{S}$. But any element in $\mathcal{S}$ can be approximated uniformly with its derivatives by finite sums of the form

$$\sum \chi(t)\phi(x,\xi)$$

and the result follows from the above corollary.

**Theorem 2.8** *The *-algebra generated by* $\{\pi_t(f) \mid f \in \mathcal{S}(\mathbb{R}^2)\}$ *defines a continuous field of* $C^*$*-algebras over the closed interval* $[0,1]$ *with the fiber* $C_0(\mathbb{R}^2)$ *at zero and a constant fiber* $\mathcal{K}$ *away from zero.*

Proof. It is sufficient to show that, for any finite family of Schwartz functions $f_{i_1}, \ldots, f_{i_k}$, the functions

$$t \to || \sum_{i_1 \ldots i_k} \pi_t(f_{i_1}) \ldots \pi_t(f_{i_k})||$$

are continuous. By the above results,

$$\lim_{t \to 0} || \sum_{i_1 \ldots i_k} \pi_t(f_{i_1}) \ldots \pi_t(f_{i_k})|| = || \sum_{i_1 \ldots i_k} \pi_0(f_{i_1} *_0 \ldots *_0 f_{i_k})||$$

and

$$\sum_{i_1 \ldots i_k} \pi_0(f_{i_1} *_0 \ldots *_0 f_{i_k}) = \sum_{i_1 \ldots i_k} \pi_0(f_{i_1}) \ldots \pi_0(f_{i_k}).$$

This proves continuity at zero.

For $t \neq 0$ and $f \in \mathcal{S}(\mathbb{R}^2)$, $f(x, \frac{v-x}{t})$ is continuous in $t$ as an element of $L^2(\mathbb{R}^2)$, hence

$$t \to \rho_t(f)$$

are continuous in the Hilbert Schmidt and hence operator norm. The required continuity follows immediately.

**Definition 2.9** *The continuous field of* $C^*$*-algebras constructed above will be called the "Heisenberg field".*

In particular, this gives a map:

$$\psi : K_0(C_0(\mathbb{R}^2)) \to K_0(\mathcal{K}) = K_0(\mathbb{C}).$$

constructed as follows. given a projection $p_0$ in matrices over unitalisation of $C_0(\mathbb{R}^2)$, let $p_t$ be its continuous projection-valued extension to a section in a neighbourhood of $t=0$. Let $\epsilon$ be the evaluation at infinity. Then

$$\psi([(p_0)] - [\epsilon(p_0)]) = [p_t] - [\epsilon(p_t)]$$

This is independent of the choice of $t$ small and well-defined on K-theory by the observation 1.

# 3 Some continuous sections

**Example 3.1** *Set $z = x + i\xi$. Then*

$$\begin{pmatrix} \frac{1}{1+|z|^2} & \frac{\overline{z}}{1+|z|^2} \\ \frac{z}{1+|z|^2} & \frac{|z|^2}{1+|z|^2} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

*is a matrix of functions whose Fourier transform in the second variable satisfy the conditions of the above corollary and hence*

$$b_t - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_t \circ \mathcal{F}_2(\frac{1}{1+|z|^2}) & \pi_t \circ \mathcal{F}_2(\frac{\overline{z}}{1+|z|^2}) \\ \pi_t \circ \mathcal{F}_2(\frac{z}{1+|z|^2}) & \pi_t \circ \mathcal{F}_2(\frac{|z|^2}{1+|z|^2}) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

*is a continuous section.*

### 3.0.1 Some estimates

*The following is a standard kind of result:*

$$||\rho_t \circ \mathcal{F}_2(\frac{1}{1+|z|^2}) - (1 + x^2 - t^2\partial_x^2 - 2t)^{-1}|| = 0(t)$$

*and*

$$\lim_{t\to 0} ||\rho_t \circ \mathcal{F}_2(\frac{z}{1+|z|^2}) - (x + t\partial_x)(1 + x^2 - t^2\partial_x^2)^{-1}|| = 0.$$

*We will prove the first of them, the other can be obtained from the first by approximating by terms of the form*

$$\frac{z}{(1+|z|^2)^{1+\epsilon}}.$$

*Let*

$$\Delta_t = (x - t\partial_x)(x + t\partial_x).$$

*A direct computation shows that, for a Schwartz function $\phi$ in $L^2(\mathbb{R})$, the following holds.*

$$(1+\Delta_t)\rho_t \circ \mathcal{F}_2(\frac{1}{1+|z|^2})\phi = \phi + t\rho_t(f_1)\phi + t^2\rho_t(f_2)\phi,$$

*where the functions $f_1$ and $f_2$ satisfy the conditions of Corollary 2.6. Since $1+\Delta_t \geq 1-2t$,*

$$\rho_t \circ \mathcal{F}_2(\frac{1}{1+|z|^2}) - (1+\Delta_t)^{-1} = (1+\Delta_t)^{-1}(t\rho_t(f_1) + t^2\rho_t(f_2))$$

*has the uniform norm bounded above by const$\frac{t}{1-2t}$ and the claim follows.*

*By above estimates the family of projections*

$$t \rightarrow p_t = \textit{projection onto the graph of } x + t\partial_x.$$

*satisfies*

$$||p_t - b_t|| = 0(t)$$

*Since $t \rightarrow b_t$ is a continuous section, $p_t$ is continuous at zero and, since it is obviously continuous away from zero, it is a continuous section.*

Since $A_t = x + t\partial_x$ is a Fredholm operator with index 1, we conclude the following

**Theorem 3.2**

$$\psi([\begin{pmatrix} \frac{1}{1+|z|^2} & \frac{\overline{z}}{1+|z|^2} \\ \frac{z}{1+|z|^2} & \frac{|z|^2}{1+|z|^2} \end{pmatrix}] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}])$$

*is the generator of $K_0(\mathcal{K})$.*

# 4 Atiyah-Singer index theorem for $\mathbb{R}^n$

## 4.1 A continuous field of C*-algebras

Let us recall the construction of the Heisenberg field, except that we replace everywhere $\mathbb{R}$ by $\mathbb{R}^n$. Given a Schwartz function function $f \in \mathcal{S}(\mathbb{R}^{2n})$, we write it as $f(x,v)$, where we use the splitting of variables according to

$$\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$$

To $f$ we associate a family of operators on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ given by

$$\pi_t(f)g(x,w) = \int_{\mathbb{R}^n} dv f(x,v) g(x-tv, w-v).$$

**Theorem 4.1** *The family*

$$\{[0,1] \ni t \to \pi_t(f) \mid f \in \mathcal{S}(\mathbb{R}^{2n})\}$$

*defines a continuous field of $C^*$-algebras with fiber $C_0(\mathbb{R}^{2n})$ at $t = 0$ and $\mathcal{K}$ at $t \neq 0$. We will denote by $\Gamma$ the $C^*$-algebra of all continuous sections of this continuous field.*

*Let $\mathcal{S}^\infty$ be the space of functions on $\mathbb{R}^{2n} \times [0,1]$ which satisfiy the following estimates:*

$$\forall_{i,j,\alpha} \sup_{x,\xi,t} |(1 + x^2 + \xi^2)^\alpha \partial_x^i \partial_\xi^j \phi(x, \xi, t)| < \infty,$$

*and let $\mathcal{F}_2$ denote the Fourier transform in the $\xi$-variables. Then*

$$\Gamma^\infty = \{t \to \pi_t(\mathcal{F}_2^{-1}\phi(\cdot, \cdot, t)) \mid \phi \in \mathcal{S}\}$$

*is a dense algebra of continuous sections.*

Proof. Word for word the same as for $n = 1$.

## 4.2 "Traces"

Let $\mathcal{K}^\infty$ denote the algebra of compact operators on $L^2(\mathbb{R}^n)$ with Schwartz kernels. Then

$$\delta_i(A) = \partial_{x_i} A - A \partial_{x_i} \text{ and } \delta_{i+n}(A) = \mathcal{M}_{x_i} A - A \mathcal{M}_{x_i}$$

are 2n derivations of $\mathcal{K}^\infty$ ($\mathcal{M}_x$ is as usuall the operator of multiplication with the function x).

We define a (2n+1)-linear functional $\tau$ on $\mathcal{K}^\infty$ by

$$\tau(A_0, A_1, \ldots, A_{2n}) = \frac{1}{n!} \sum_{\sigma \in S_{2n}} sgn(\sigma) \mathrm{Tr}(A_0 \delta_{\sigma(1)}(A_1) \ldots \delta_{\sigma(2n)}(A_{2n}) \tag{1}$$

We will also use the following notation. Suppose that $A^i = \{A^i\}_{s,t} \in M_k(\mathcal{K}^\infty)$. Then

$$\tau \# \mathrm{Tr}_{M_k(\mathbb{C})}(A^0, A^1, \ldots, A^{2n}) = \sum_{i_0, \ldots i_{2n}} \tau({A^0}_{i_0,i_1}, {A^1}_{i_1,i_2}, \ldots, {A^{2n}}_{i_{2n},i_0})$$

We will leave it to the reader to work out the decay properties needed for the second part of the following result.'

**Theorem 4.2** *Suppose that e is a projection in $\mathcal{K}^\infty$. Then*

$$\tau(e,\dots,e) = Tr(e)$$

*Moreover, let $\tau_t$ denote the (2n+1)-linear functional on $\mathcal{S}$ given by*

$$\tau_t(f_0,\dots,f_{2n}) = \tau(\pi_t(\mathcal{F}_2^{-1}f_0(\cdot,\cdot,t)),\dots,\pi_t(\mathcal{F}_2^{-1}f_{2n}(\cdot,\cdot,t))).$$

*Then*

$$\lim_{t\to 0}\tau_t(f_0,\dots,f_{2n}) = \frac{1}{(2\pi i)^n n!}\int_{\mathbb{R}^{2n}} f_0 df_1\dots df_{2n}|_{t=0}. \qquad (2)$$

*The meaning of the right hand side is as follows. We set $t=0$ in all $f_i$'s with $i\neq 0$, form the Jacobi determinant of all partial derivatives of $f$'s with respect to all variables $x_i,\xi_j$'s multiply by $f_0$ and integrate over $\mathbb{R}^{2n}$.*

Proof. The first statement follows by a direct computation from the fact that $[\partial_{x_i},\mathcal{M}_{x_j}] = \delta_{i,j}$.

The second statement can be proved as follows. Since

$$\pi_t(f)\pi_t(g) = \pi_t(f *_t g)$$

the operator

$$\pi_t(\mathcal{F}_2^{-1}f_0(\cdot,\cdot,t)),\dots,\pi_t(\mathcal{F}_2^{-1}f_{2n}(\cdot,\cdot,t))$$

has again the form $\pi_t(...)$, hence is given by an integral kernel and its trace is just the integration of this kernel over the diagonal. If we let $t\to 0$, the corresponding integral reduces immediately to the expression in (2).

## 4.3 Pseudodifferential operators a'la Shubin

We will collect here a few facts on pseudodifferential operators on $\mathbb{R}^n$. Let $d$ be a smooth function on $\mathbb{R}^{2n}$ satisfying estimates of the form

$$|\partial_x^\alpha\partial_\xi^\beta d| \leq C_{\alpha,\beta}(1+x^2+\xi^2)^{\frac{N-|\alpha|-|\beta|}{2}}.$$

Here $\alpha$ is a multiindex, i.e. $\alpha\in\mathbb{N}^n$ and $|\alpha| = \alpha_1+\dots\alpha_n$. Then

$$Op(d) = d(x,\partial_x)g(x) = \mathcal{S}(\mathbb{R}^n)\ni g\to\int d\xi d(x,\xi)e^{ix\xi}(\mathcal{F}g)(\xi)$$

defines a "pseudodifferential operator" on $\mathbb{R}^n$ of degree ≤N. $d$ itself is called a symbol of degree ≤N. If $d$ is a polynomial in $\xi$, the corresponding operator $d(x,\partial_x)$ is a differential operator.

For future reference, let us introduce the following notation.

**Definition 4.3** *Let $f$ be a function on $T^*\mathbb{R}^n$. Then $f_t$ denites the scaled function*

$$f_t(x, \xi) = f(x, t\xi)$$

*and*

$$Op_t(f) := Op(f_t).$$

The basic result we need is the following.

**Theorem 4.4** *A psudodifferential operator is*

- *bounded, if its degree is $\leq 0$;*
- *compact, if its degree is $< 0$;*
- *trace class, if its degree is $< -n$.*

*$d(x, \partial_x)$ is called elliptic, if it is of degree $N$ and satisfies an inequality of the form*

$$|d(x, \xi)| \geq C(||x||^2 + ||\xi||^2)^N$$

*for $||x||^2 + ||\xi||^2$ large. Suppose moreover that $N > 0$ In this case $D^*D = d(x, \partial_x)^* d(x, \partial_x)$ is essentially self adjoint (has unique self-adjoint extension) and has compact resolvent. In particular $D$ is Fredholm and has index.*

Since this goes into different subject, we will not give the proof.

### 4.4 Index theorem

**Theorem 4.5** *Suppose that $d$ is an elliptic symbol of positive order. Let $p_d$ denote the graph projection of the multiplication operator $\mathcal{M}_d$. Then*

$$index(d(x, \partial_x)) = \frac{1}{(2\pi i)^n n!} Tr_{M_2(\mathbb{C})} \int_{\mathbb{R}^{2n}} p_d dp_d \ldots dp_d$$

Proof. We just sketch the proof, the detals being more or less identical to those in the section on "continuous sections". It is easy to see that we can assume that the degree of $d$ is higher then n. Let $p_{d_t}$ denote the graph projection of $d(x, t\partial_t)$. Then

$$||Op_t(p_d) - p_{d_t}|| \to 0 \text{ as } t \to 0.$$

in the operator norm (this is the sketchy part). But then

$$\text{index}(d(x, \partial_x)) = \text{index}(d(x, t\partial_x)) = \tau \# \text{Tr}_{M_2(\mathbb{C})}(p_{d_t}, \ldots p_{d_t}) \to \lim_{t \to 0} \tau_t(p_d, \ldots p_d),$$

but

$$\lim_{t \to 0} \tau_t \# \text{Tr}_{M_2(\mathbb{C})}(p_d, \ldots p_d) = \frac{1}{(2\pi i)^n n!} Tr_{M_2(\mathbb{C})} \int_{\mathbb{R}^{2n}} p_d dp_d \ldots dp_d.$$

### 4.5 The estimates, basics

To begin with, note that a pseudodifferential operator of order N maps

$$C_c^N(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

This is proved by noticing that

$$\exp(ix\xi) = (1 + x^2 + \xi^2)^{-N/2}(1 + x^2 - \partial_x^2)^{N/2} \exp(ix\xi)$$

inserting this into the formula for the pseudodifferential operator and doing partial integration.

Another important observation is that

$$Op(S^m)/Op(S^{m-1}) \simeq C^\infty(\mathbb{R}^n \times \{\xi \mid ||\xi|| = 1\}),$$

where the right hand side is considered to be given by positively m-homogeneous functions on $T^*(\mathbb{R}^n) \setminus \mathbb{R}^n$.

We'll deal below with the expressions of type

$$||Op_t(p_d) - p_{d_t}||.$$

Since this is the standard bag of tricks for pseudodifferential operators, it is worthwhile to spend some time on them. What we need is the behaviour of, f. ex.

$$Op_t(\frac{d}{1 + |d|^2}) - Op_t(d)(1 + Op_t(d)^* Op_t(d))^{-1}.$$

The basic observation is that, given two pseudodifferential symbols f and g, so is the pointwise product, fg, the degree is additive under multiplication and, if d is an elliptic symbol, so is $(1-\phi)d^{-1}$, where $\phi$ is a smooth, compactly supported function which is equal to 1 in a neighbourhood of the (compact) set of zero's of $d$. Moreover,

$$Op_t(f)Op_t(g) - Op_t(fg) = tOp_t(r_t)$$

where $r_t$ is a pseudodifferential symbol of degree at most

degree(f)+degree(g)-1.

Moreover the estimates involved in the definition of the symbol are uniform in t when applied to $r_t$. This is all fairly straightforward to check when applied to the explicit formula

$$Op_t(f)Op_t(g) = Op_t(f *_t g).$$

We can add to it the easily seen fact that for a symbol f of degree -2(n+1) the operator $Op(f)$ is given by the square integrable kernel

$$(\mathcal{F}_2^{-1} f)(x, y - x).$$

Now the arguments of the type of 3.0.1 show easily that $||Op_t(p_d) - p_{d_t}||$ converges to zero in norm (even $\mathcal{L}^1$ norm) as soon as the order of d is high enough.

# 5 Pseudodifferential operators and deformations on compact manifolds

Let now $X$ be a smooth compact manifold, with a chosen Riemannian metric, and $T^*X$ the cotangent bundle of $X$.

## 5.1 Pseudodifferential operators

Let $\chi$ be a smooth, non-negative function on $X \times X$ satisfying the following conditions:

(1) $\chi(x, y) = \chi(y, x)$;
(2) $\chi \equiv 1$ on an open set containing the diagonal in $X \times X$.
(3) For each $x \in X$, the set $D_x = \{y \in X \mid (x, y) \in supp(\chi)$ is geodesically convex.
We denote by $\mathrm{Exp}_x^{-1}$ the unique smooth inverse to the exponential map:

$$\mathrm{Exp}_x \,:\, T_x X \to X$$

defined on $D_x$ and such that $\mathrm{Exp}_x^{-1}(x) = 0$.

Given $x \in X$, $y \in D_x$, let $z$ denote the midpoint of the unique geodesic joining $x$ and $y$ within $D_x$, and let $v \in T_z X$ be given by

$$v/2 \;=\; \mathrm{Exp}_x^{-1}(y) \;=\; -\mathrm{Exp}_z^{-1}(x) \tag{3}$$

Now, denote by $S^m(T^*X)$ the space of classical symbols of order $m$ on $X$, i. e.. smooth functions $\theta$ on $T^*X$ satisfying estimates of the form:

$$\sup_{(x,\xi)} |\partial_x^\alpha \, \partial_\xi^\beta \, \theta\, (x, \xi)| \;\leq\; C_{\alpha,\beta}\, (1 + |\xi|^2)^{\frac{m-|\beta|}{2}}$$

$S^m(T^*X)$ is given the topology of (Frechet) topological vector space by the "best" $C_{\alpha,\beta}$.

We will denote by $S^\infty(T^*X) = \cup_{m\in\mathbb{R}} S^m(T^*X)$ the set of all classical symbols on $T^*X$, and we set $S^{-\infty}(T^*X) = \cap_{m\in\mathbb{R}} S^m(T^*X)$
With the above notation (3), the expression:

$$\mathrm{Op}(\theta)\,(u)\,(x) \;=\; \int_{T_z^*X} d\xi \int dy\, \chi(x,y)\, \mathrm{e}^{i\,\xi.v}\, \theta(z,\xi)\, u(y)$$

defines a pseudo-differential operator with symbol $\theta$.

In fact,

$$C^\infty(X\times X) \ni f\otimes g \to \int_X Op(\theta) f\overline{g}$$

defines a distribution (continuous linear functional) and hence gives a linear map

$$Op(\theta): C^\infty(X) \to C^\infty(X)$$

called a pseudodifferential operator with (complete) symbol $\theta$.

In the case when $\theta \in S^{-\infty}$, $Op(\theta)$ is given by a Schwartz kernel and is smoothing (extends to a continuous operator from $C^\infty(X)^*$ to $C^\infty(X)$).

Conversely, if $P$ is a pseudo-differential operator on $X$ we define its complete symbol to be:

$$\sigma(P)(z,\xi) \;=\; P_y(\, \chi(x,y)\, \mathrm{e}^{i\,\xi.v}\,)|_{x=y=z} \tag{4}$$

where $z$ is the midpoint of the geodesic joining $x$ and $y$ and $v$ satisfies (3). We observe that $P - \mathrm{Op}(\sigma(P))$ is a smoothing operator whose Schwartz kernel vanishes to infinite order on the diagonal.

The basic property of pseudodifferential operators is the order filtration.

**Proposition 5.1** *For $f \in S^m$ and $g \in S^n$, $\sigma(Op(f)Op(g)) \in S^{n+m}$ and, moreover, $\sigma(Op(f)Op(g) - Op(fg)) \in S^{n+m-1}$. In particular, if we denote by $Op^m$ the linear space of pseudodifferential operators of order $m$ and by $\mathcal{A}$ the algebra $\cup_m Op^m$, then*

$$Op^m Op^n \subset Op^{m+n}.$$

*For $f \in S^m$*

$$Op(f) mod(Op^{m-1})$$

*depends only on $f mod(S^{m-1})$ and is a well defined function $\sigma_p(Op(f))$ on the cosphere bundle $S^*(X)$ of $X$, called the principal symbol of $Op(f)$. This induces a $C^*$-algebra extension*

$$0 \to \mathcal{K}(L^2(X)) \to \overline{Op^0} \to C(S^*(X)) \to 0.$$

*The induced boundary map in K-theory*

$$\partial : K_1(C(S^*(X))) \to K_0(\mathcal{K})$$

*is given by*

$$\sigma_p(f) \to \textit{Index } Op(f).$$

*In particular, Index $(D)$ for an elliptic pseudodifferential operator $D$ depends only on the homotopy class of $\sigma_p(D)$.*

The proofs are fairly straightforward.

$K^\infty$, the algebra of smoothing operators on $X$, is an ideal in $\mathcal{A}$ and the filtered algebra

$$\mathcal{A}/K^\infty$$

has the associated graded algebra

$$Gr(\mathcal{A}/K^\infty) := \oplus_m Op^m/Op^{m-1} \simeq C^\infty(S^*X)[[||\xi||^{-1}, ||\xi||]].$$

**All the statements from previous sections about pseudodifferential operators on $\mathbb{R}^n$ extend to this case (with obvious modifications).**

## 5.2 Formal deformation associated to Pseudodifferential operators

Let $\mathcal{P}^i$ denote the space of pseudodifferential operators of order $\leq i$ modulo the smoothing operators. The space $\mathcal{S}P$ of doubly infinite sequences

$$\{P_i\}_{i\in\mathbb{Z}},\ P_i \in \mathcal{P}^i,\ \text{and there exists } i_0 \text{ such that } P_i \in \mathcal{P}^{i_0} \text{ for } i \text{ large}$$

is a flat module over $\mathbb{C}[[t]]$, where the multiplication by $t$ acts as the right translation. We endow $\mathcal{S}P$ with the product

$$\{P_i\}_{i\in\mathbb{Z}}\{Q_i\}_{i\in\mathbb{Z}} = \{\sum_{i+j=n} P_iQ_j\}_{n\in\mathbb{Z}}$$

Note that, given the particular sequences $\{P_i\}_i$ and $\{Q_i\}_i$, there exists N such that all the pseudodifferential operators involved have order bounded by N, hence the doubly infinite sums $\sum_{i+j=n} P_iQ_j$ make sense. In fact, given n, the top order of operators appearing as products $P_iQ_{n-i}$ are at most 2N, and given any integer $k \leq 2N$, there are at most finitely many terms in the sum of the order $k$.

The space $\mathcal{S}P$ is immediately seen to be isomorphic to

$$C^\infty(S^*X)[[||\xi||^{-1}, ||\xi||][t^{-1}, t]].$$

Any trace $\tau$ on it is given by a sequence of $\mathbb{C}$-linear, $\mathbb{C}[[\hbar]]$-valued functionals $\tau_n$ on $\mathcal{P}^n$ such that

$$\tau(\{P_i\}) = \sum \tau_i(P_i).$$

The $\hbar$-linearity of $\tau$ implies that $\tau_{i+1} = \hbar\tau_i$ and the trace condition on $\tau$ implies that each $\tau_i$ is a trace on the algebra of pseudodifferential operators modulo the smoothing operators.

The following holds

**Theorem 5.2 (Wodzicki)** *$\mathcal{A}$ has a unique trace, called the residue.*

Hence $\tau$ is, up to multiplicative constant, uniquely determined by $\tau_{-n} = res$, and hence the space of traces on $\mathcal{S}P$ is one-dimensional.

## 5.3 Formal deformation of $C^\infty(T^*X)$

The $*_t$-product from the flat case has the following generalisation.

We set, for a given $\theta \in C^\infty(T^*X)$,

$$\theta_t(x, \xi) \;=\; \theta(x, t\xi)$$

.

Suppose that f and g are smoothing symbols, i.e in $S^{-\infty}$. The expression

$$t \to \sigma(Op(f_t)Op(g_t))_{t^{-1}}$$

does not converge as t goes to zero, but it does have an asymptotic expansion, of the form

$$f * g = fg + it/2\{f, g\} + \sum_{n \geq 2} t^n P_n(f, g).$$

It is easy to see that this extends to an associative product $(*)$ on

$$\mathring{A}_{T*X} = C^\infty(T^*X)[[t]].$$

Moreover recall again that for $f \in S^{-\infty}$, $Op_t(f)$ is smoothing and hence of trace-class. As easily seen, the asymptotic expansion at $t = 0$ of

$$t \to Tr(Op(f_t))$$

is well defined and defines a trace $\tau$ on $\mathring{A}_{T*X}$ with values in $\mathbb{C}[t^{-1}, t]]$ of the form

$$\tau(f) = \frac{1}{n!t^n} \int_{T^*X} f\omega^n.$$

Here n=dim X and $\omega$ is the standard symplectic two-form on $T^*X$.

This trace is, up to normalisation, the unique trace on $\mathring{A}_{T*X}$.