Preprintreihe

SFB 478 – Geometrische Strukturen in der Mathematik

Heft 141

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ISSN 1435-1188

Preprintreihe

des SFB 478 – Geometrische Strukturen in der Mathematik

des Mathematischen Instituts
der Westfälischen Wilhelms-Universität Münster

Heft 141

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Dezember 2000

ISSN 1435-1188

Diese Arbeit ist im Sonderforschungsbereich 478, Geometrische Strukturen in der Mathematik, Münster, entstanden und wurde auf seine Veranlassung unter Verwendung der ihm von der Deutschen Forschungsgemeinschaft zur Verfügung gestellten Mittel gedruckt.

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Cyclic Theory and the Bivariant Chern-Connes Character

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November 20, 2000

Abstract

We give a survey of cyclic homology/cohomology theory including a detailed discussion of cyclic theories for various classes of topological algebras. We show how to associate cyclic classes with Fredholm modules and K-theory classes and construct a completely general bivariant Chern-Connes character from bivariant K-theory to bivariant cyclic theory.

1 Introduction

The two fundamental "machines" of non-commutative geometry are cyclic homology and (bivariant) topological K-theory. In the present notes we describe these two theories and their connections. Cyclic theory can be viewed as a far reaching generalization of the classical de Rham cohomology, while bivariant K-theory includes the topological K-theory of Atiyah-Hirzebruch as a very special case.

The classical commutative theories can be extended to a striking amount of generality. It is important to note however that the new theories are by no means based simply on generalizations of the existing classical constructions. In fact, the constructions are quite different and give, in the commutative case, a new approach and an unexpected interpretation of the well-known classical theories. One aspect is that some of the properties of the two theories become visible only in the non-commutative category. For instance both theories have certain universality properties in this setting.

*Research supported by the Deutsche Forschungsgemeinschaft

Bivariant K-theory has first been defined and developed by Kasparov on the category of C^*-algebras (possibly with the action of a locally compact group) thereby unifying and decisively extending previous work by Atiyah-Hirzebruch, Brown-Douglas-Fillmore and others. Kasparov also applied his bivariant theory to obtain striking positive results on the Novikov conjecture. Very recently, it was discovered that in fact, bivariant topological K-theories can be defined on a wide variety of topological algebras ranging from discrete algebras and very general locally convex algebras to e.g. Banach algebras or C^*-algebras (possibly equipped with a group action). If E is the covariant functor from such a category of algebras given by topological K-theory or also by periodic cyclic homology, then it satisfies the following three fundamental properties:

(E1) E is diffeotopy invariant, i.e., the evaluation map ev_t in any point $t \in [0,1]$ induces an isomorphism $E(ev_t) : E(\mathfrak{A}[0,1]) \to E(\mathfrak{A})$ for any $\mathfrak{A}$ in C. Here $\mathfrak{A}[0,1]$ denotes the algebra of $\mathfrak{A}$-valued C^∞-functions on $[0,1]$.

(E2) E is stable, i.e., the canonical inclusion $\iota : \mathfrak{A} \to \mathfrak{K}\hat{\otimes}\mathfrak{A}$, where $\mathfrak{K}$ denotes the algebra of infinite $\mathbb{N} \times \mathbb{N}$-matrices with rapidly decreasing coefficients, induces an isomorphism $E(\iota)$ for any $\mathfrak{A}$ in C.

(E3) E is half-exact, i.e., each extension $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{B} \to 0$ in C admitting a continuous linear splitting induces a short exact sequence $E(\mathfrak{I}) \to E(\mathfrak{A}) \to E(\mathfrak{B})$

"Diffeotopy", i.e., differentiable homotopy is used in (E1) for technical reasons in connection with the homotopy invariance properties of cyclic homology. For K-theory, it could also be replaced by ordinary continuous homotopy.

It turns out that the bivariant K-functor is the universal functor from the given category C of algebras into an additive category D (i.e., the morphism sets $D(\mathfrak{A}, \mathfrak{B})$ are abelian groups) satisfying these three properties 21.2.

Cyclic theory is a homology theory that has been developed, starting from K-theory, independently by Connes and Tsygan. Connes' construction was in fact directly motivated by Kasparov's formalism for bivariant K-theory and in particular for K-homology. A crucial role is played by so-called Fredholm modules or spectral triples. Also Tsygan's work is closely related to K-theory, [43]. In fact, in his approach, the new theory was originally called "additive K-theory" and he pointed out that it is an additive version of Quillen's definition of algebraic K-theory. It was immediately realized that cyclic homology has close connections with de Rham theory, Lie algebra homology, group cohomology and index theorems.

The theory with the really good properties is the periodic theory introduced by Connes. Periodic cyclic homology HP_* satisfies the three properties (E1), (E2), (E3), [4], [18], [46], [47], [15]. Combining this fact with the universality property of bivariant K-theory leads to a multiplicative transformation (the bivariant Chern-Connes character) from bivariant K-theory to bivariant periodic cyclic theory. This transformation is a vast generalization of the classical Chern character in differential geometry.

The principal aim of this volume is to give an account of cyclic theory. Here, everything works in parallel algebraically as well as for locally convex algebras (to name just two examples think of the algebra of C^∞-functions on a smooth manifold or of algebras of pseudodifferential operators). Cyclic theory can be introduced using rather different complexes, each one of them having its own special virtues. Specifically, we will use the following complexes or bicomplexes

- The cyclic bicomplex with various realizations:

$$\begin{array}{rll} CC^n(A) & \text{with boundary operators} & b, b', Q, 1-\lambda \\ \overline{CC}^n(\widetilde{A}) & \text{with boundary operator} & B-b \\ \Omega(A) & \text{with boundary operator} & B-b \end{array}$$

  The cyclic bicomplex is well suited for the periodic theory as well as for the $\mathbb{Z}$-graded ordinary theory and for the connections between both.

- The Connes complex C_λ^n
  It has the advantage, that concrete *finite-dimensional* cocycles often arise naturally as elements of C_λ^n. The connection with Hochschild cohomology also fits naturally into this picture.

- The X-complex of any complete quasi-free extension of the given algebra A. This complex is very useful for a conceptual explanation of the properties of periodic theory, in particular for proving excision, for the connections with topological K-theory and the bivariant Chern-Connes character. It also is the natural framework for all infinite-dimensional versions of cyclic homology (analytic and entire as well as asymptotic and local theory).

In the following sections we discuss the basic properties of cyclic theory. We note however, that it is quite difficult to be exhaustive and we don't even try to give a complete account of all aspects of cyclic theory. For instance, an important notion which we don't treat is the one of a cyclic object. But also other important aspects have to be omitted. We focus on those notions and results which, we think, are most relevant for non-commutative geometry, including homotopy invariance, Morita invariance, excision but also explicit formulas for the Chern character associated to idempotents, invertibles, Fredholm modules etc..

After this we turn to a description of bivariant K-theory and to the construction of the bivariant Chern-Connes character, which generalizes the Chern character for idempotents, invertibles or Fredholm modules mentioned before.

As we pointed out already above, cyclic homology and bivariant K-theory can be defined on different categories of algebras - purely algebraically for algebras (over $\mathbb{R}$

or $\mathbb{C}$) or on categories of topological algebras like locally convex algebras, Banach algebras or C*-algebras. There are different variants of the two theories which are adapted to the different categories. In this text we treat cyclic theory in the purely algebraic case (however restricting to algebras over a field of characteristic 0) on the one hand. Concerning cyclic theory for topological algebras on the other hand, we have to make a choice. For the classical ("finite-dimensional") cyclic theories we concentrate on the category of what we call m-algebras. These are particularly nice locally convex algebras (projective limits of Banach algebras). Their advantage is that, both, ordinary cylic homology and bivariant topological K-theory make perfect sense and that the bivariant Chern-Connes-character can be constructed nicely on this category. It is important to note however that the restriction to the category of m-algebras is for convenience mainly.
Cyclic theory as well as bivariant K-theory can also be treated on many other categories of algebras. In particular, for the categories of Banach algebras or C^*-algebras there are special variants of cyclic theory, namely the entire and the local cyclic theory, which are especially designed for these categories. An interesting new feature here is the existence of *infinite-dimensional* cohomology classes. These theories, as well as a bivariant character from Kasparov's KK-theory to the local cyclic theory, are discussed in sections 22 and 23 which have been written by Ralf Meyer and Michael Puschnigg.

The text starts with a collection of examples of algebras, locally convex algebras and certain extensions of algebras. These serve as a reference for later and may be omitted at a first reading.

2 Some examples of algebras

We list in this section some examples of algebras and constructions on algebras that will be used later. For simplicity we always assume that the ground field is $\mathbb{C}$. Since it is important to do all constructions in the category of non-unital algebras we will frequently use the procedure of adjoining a unit to a given algebra A.

Definition 2.1 *The unitization $\tilde{A}$ of an algebra A is the vector space $\mathbb{C} \oplus A$ with multiplication given by*

$$(\lambda, x)(\mu, y) = (\lambda\mu, \lambda y + \mu x + xy)$$

$\tilde{A}$ is a unital algebra and contains A as an ideal of codimension 1.

2.1 Algebras of polynomial functions

For any algebra A, we can form the algebra $A[t]$ of polynomials with coefficients in A.

Two homomorphisms $\alpha, \beta : A \to B$ of algebras are called polynomially homotopic, if there is a homomorphism $\varphi : A \to B[t]$ such that $\varphi(x)(0) = \alpha(x), \varphi(x)(1) = \beta(x)$ for $x \in A$. Polynomial homotopy is not an equivalence relation, in general.
We denote by $A[z, z^{-1}]$ or by $A(z)$ the algebra of Laurent polynomials with coefficients in A.

2.2 The tensor algebra

Given any vector space V, we define the (non-unital) tensor algebra over V as

$$TV = V \oplus V{\otimes}V \oplus V^{\otimes^3} \oplus \dots$$

TV is an algebra with the product given by concatenation of tensors. We denote by $\sigma : V \to TV$ the linear map, that maps V to the first summand in TV. The map σ has the following universal property: Let $s : V \to \mathfrak{A}$ be an arbitrary linear map from V into an algebra A. Then there is a unique algebra homomorphism $\tau_s : TV \to A$ such that $\tau_s \circ \sigma = s$.

The tensor algebra is polynomially contractible, i.e., the identity map of TV is polynomially homotopic to 0. A polynomial homotopy $\varphi_t : TV \to TV$, such that $\varphi_0 = 0, \varphi_1 = \mathrm{id}$, is given by $\varphi_t = \tau_{t\sigma}, t \in [0, 1]$.

2.3 The free product of two algebras

Let A and B be algebras. The algebraic free product (in the non-unital category) of A and B then is the following algebra

$$A * B \;=\; A \oplus B \oplus (A{\otimes}B) \oplus (B{\otimes}A) \oplus (A{\otimes}B{\otimes}A) \oplus \dots$$

This direct sum is taken over all tensor products where the factors A and B alternate. The multiplication is given by concatenation of tensors but where the multiplication $A \otimes A \to A$ and $B \otimes B \to B$ is used to simplify all terms where two elements in A or two elements in B meet.
The canonical inclusions $\iota_1 : A \to A * B$ and $\iota_2 : B \to A * B$ have the following universal property: Let $\alpha : A \to E$ and $\beta : B \to E$ be two homomorphisms into an algebra E. Then there is a unique homomorphism $\alpha * \beta : A * B \to E$, such that $(\alpha * \beta) \circ \iota_1 = \alpha$ and $(\alpha * \beta) \circ \iota_2 = \beta$.

2.4 The algebra of finite matrices of arbitrary size

For any algebra A, we can form the algebra $M_\infty(A)$ consisting of matrices $(a_{ij})_{i,j \in \mathbb{N}}$ with $a_{ij} \in A$, $i, j = 0, 1, 2 \dots$ and only finitely many a_{ij} non-zero. Equivalently

$$M_\infty(A) = \varinjlim_n M_n(A) \tag{1}$$

where, as usual, $M_n(A)$ is embedded into $M_{n+1}(A)$ in the upper left corner.

2.5 The algebraic Toeplitz algebra

The algebraic Toeplitz algebra Ψ is the algebra of (row- and column-finite) $\mathbb{N}\times\mathbb{N}$-matrices generated by the matrices

$$v = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and

$$u = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

As a vector space, the algebraic Toeplitz algebra $\mathfrak{T}$ is isomorphic to the direct sum $\Psi = M_\infty(\mathbb{C}) \oplus \mathbb{C}[z, z^{-1}]$.

Under this isomorphism v^k and u^k correspond to the elements $(0, z^k)$ and $(0, z^{-1})$, respectively, $k \in \mathbb{N}$. Moreover, $v^i(1-vu)u^j$corresponds to $(e^{ij}, 0)$, where e^{ij} is given by the matrix (a_{kl}) with $a_{kl} = 1$, for $k = i, l = j$, and $a_{kl} = 0$ otherwise (with the convention that $e^{ij} = 0$ for $i < 0$ or $j < 0$).

3 Locally convex algebras

Cyclic homology and cohomology can be defined readily for topological algebras and, in particular, locally convex algebras. Since there is a certain freedom in setting up cyclic homology in this case, we will now fix the conventions. A locally convex algebra A is, in general, an algebra with a locally convex topology for which the multiplication $A \times A \to A$ is (jointly) continuous. In the present survey we will restrict our attention however to locally convex algebras that can be represented as projective limits of Banach algebras. Most of what will be said below works for general locally convex algebras but some of the constructions (in particular the free extension) and the proofs (in particular excision for the periodic theory) become much more involved.
A locally convex algebra $\mathfrak{A}$ that can be represented as a projective limit of Banach algebras can equivalently be defined as a complete locally convex space whose topology is determined by a family $\{p_\alpha\}$ of submultiplicative seminorms, [33]. Thus for each α we have $p_\alpha(xy) \le p_\alpha(x)p_\alpha(y)$. The algebra $\mathfrak{A}$ is then automatically a topological algebra, i.e., multiplication is (jointly) continuous. **We call such algebras m-algebras.**
The unitization $\tilde{\mathfrak{A}}$ of an m-algebra is again an m-algebra with the seminorms $\tilde{p}$ given

by $\tilde{p}(\lambda 1 + x) = |\lambda| + p(x)$ for all submultiplicative seminorms p on $\mathfrak{A}$.
The direct sum $\mathfrak{A} \oplus \mathfrak{B}$ of two m-algebras is again an m-algebra with the topology defined by all seminorms $p \oplus q$ with $(p \oplus q)(x, y) = p(x) + q(y)$, where p is a continuous seminorm on $\mathfrak{A}$ and q a continuous seminorm on $\mathfrak{B}$.
Recall the definition of the projective tensor product by Grothendieck, [19], [42]. For two locally convex vector spaces V and W the projective topology on the tensor product $V \otimes W$ is given by the family of seminorms $p \otimes q$, where p is a continuous seminorm on V and q a continuous seminorm on W. Moreover $p \otimes q$ is defined by

$$p \otimes q\,(z) = \inf \Big\{ \sum_{i=1}^{n} p(a_i) q(b_i) \,|\, z = \sum_{i=1}^{n} a_i \otimes b_i, a_i \in V, b_i \in W \Big\}$$

for $z \in V \otimes W$. We denote by $V \hat{\otimes} W$ the completion of $V \otimes W$ with respect to this family of seminorms. For m-algebras $\mathfrak{A}$ and $\mathfrak{B}$, the projective tensor product $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ is again an m-algebra (if p and q are submultiplicative, so is $p \otimes q$).
Basically all definitions and constructions in the subsequent sections on cyclic theory will carry over to the category of m-algebras. All the definitions and results on cyclic theory of the previous sections hold not only in the algebraic setting, but also for m-algebras.
To obtain the appropriate definitions in the category of m-algebras, in general, all morphisms should be continuous, duals should be topological duals, tensor products (also in the definition of differential forms) should be completed projective tensor products (see below), extensions should admit a continuous linear splitting, there will be a natural notion of differentiable homotopy in that category and - very importantly - the tensor algebra construction gives a natural free extension for any m-algebra. We restrict to the class of m-algebras (rather than general locally convex algebras) mainly because of this natural construction of a free extension. Note however that other choices are conceivable. Note also that in the definition of the homology of a complex we divide by the image of the boundary operator and not by its closure (in many cases the image is automatically closed). We will comment on these matters along the way.

We now list some important examples of m-algebras and of constructions with m-algebras that will be used later on. This list is parallel to the list in Section 2 which describes the algebraic versions of the m-algebras in the present section.

3.1 Algebras of differentiable functions

Let $[a, b]$ be an interval in $\mathbb{R}$. We denote by $\mathbb{C}[a, b]$ the algebra of complex-valued $\mathcal{C}^\infty$-functions f on $[a, b]$, all of whose derivatives vanish in a and in b (while f itself may take arbitrary values in a and b).

Also the subalgebras $\mathbb{C}(a,b], \mathbb{C}[a,b)$ and $\mathbb{C}(a,b)$ of $\mathbb{C}[a,b]$, which, by definition consist of functions f, that vanish in a, in b, or in a and b, respectively, will play an important role.

The topology on these algebras is the usual Fréchet topology, which is defined by the following family of submultiplicative norms p_n:

$$p_n(f) = \|f\| + \|f'\| + \tfrac{1}{2}\|f''\| + \cdots + \tfrac{1}{n!}\|f^{(n)}\|$$

Here of course $\|g\| = \sup\{|g(t)| \,\big|\, t \in [a,b]\}$.

We note that $\mathbb{C}[a,b]$ is nuclear in the sense of Grothendieck [19] and that, for any complete locally convex space V, the space $\mathbb{C}[a,b]\hat{\otimes}V$ is isomorphic to the space of $\mathcal{C}^\infty$-functions on $[a,b]$ with values in V, whose derivatives vanish in both endpoints, [42], § 51. Exactly the same comments apply to the algebra $\mathcal{C}^\infty(M)$ of smooth functions on a compact smooth manifold where an m-algebra topology is defined locally by seminorms like the p_n above.

Given an m-algebra $\mathfrak{A}$, we write $\mathfrak{A}[a,b]$, $\mathfrak{A}[a,b)$ and $\mathfrak{A}(a,b)$ for the m-algebras $\mathfrak{A}\hat{\otimes}\mathbb{C}[a,b]$, $\mathfrak{A}\hat{\otimes}\mathbb{C}[a,b)$ and $\mathfrak{A}\hat{\otimes}\mathbb{C}(a,b)$.

Two continuous homomorphisms $\alpha, \beta : A \to B$ between complete locally convex algebras are called differentiably homotopic, or *diffeotopic*, if there is a family $\varphi_t : A \to B$, $t \in [0,1]$, of continuous homomorphisms, such that $\varphi_0 = \alpha, \varphi_1 = \beta$ and such that the map $t \mapsto \varphi_t(x)$ is infinitely often differentiable for each $x \in A$. An equivalent condition is that there should be a continuous homomorphism $\varphi : A \to \mathcal{C}^\infty([0,1])\hat{\otimes}B$ such that $\varphi(x)(0) = \alpha(x), \varphi(x)(1) = \beta(x)$ for each $x \in A$.

Let $h : [0,1] \to [0,1]$ be a monotone and bijective $\mathcal{C}^\infty$-map, whose restriction to $(0,1)$ gives a diffeomorphism $(0,1) \to (0,1)$ and whose derivatives in 0 and 1 all vanish. Replacing φ_t by $\psi_t = \varphi_{h(t)}$ one sees that α and β are diffeotopic if and only if there is a continuous homomorphism $\psi : A \to \mathbb{C}[0,1]\hat{\otimes}B$ such that $\psi(x)(0) = \alpha(x), \psi(x)(1) = \beta(x)$, $x \in A$. This shows in particular that diffeotopy is an equivalence relation.

Note that this definition of differentiable homotopy applies not only to m-algebras but to arbitrary complete locally convex algebras. In particular, it applies to any algebra A over $\mathbb{C}$ if we regard it as a locally convex space with the topology given by all possible semi-norms on A. In this case $\mathbb{C}[0,1]\hat{\otimes}A$ is simply the algebraic tensor product $\mathbb{C}[0,1] \otimes A$.

3.2 The smooth tensor algebra

Let V be a complete locally convex space. We define the smooth tensor algebra T^sV as the completion of the algebraic tensor algebra (see Section 2.2)

$$TV = V \,\oplus\, V\otimes V \,\oplus\, V^{\otimes^3} \oplus \ldots$$

with respect to the family $\{\widehat{p}\}$ of seminorms, which are given on this direct sum as

$$\widehat{p} = p \oplus p{\otimes}p \oplus p^{\otimes^3} \oplus \ldots$$

where p runs through all continuous seminorms on V. The seminorms $\widehat{p}$ are submultiplicative for the multiplication on TV. The completion T^sV therefore is an m-algebra.

In the simplest case $V = \mathbb{C}$, $T^s\mathbb{C}$ is naturally isomorphic to the algebra of holomorphic functions on the complex plane that vanish at 0 (under the isomorphism which maps a sequence (λ_n) in $T\mathbb{C}$ to the function f with $f(z) = \sum_{n=1}^{\infty} \lambda_n z^n$). The topology is the topology of uniform convergence on compact subsets.

We denote by $\sigma : V \to TV$ the map, which maps V to the first summand in TV. The map σ has the following universal property: Let $s : V \to \mathfrak{A}$ be any continuous linear map from V to an m-algebra $\mathfrak{A}$. Then there is a unique momorphism $\tau_s : TV \to \mathfrak{A}$ of m-algebras such that $\tau_s \circ \sigma = s$.

The smooth tensor algebra is differentiably contractible, i.e., the identity map on TV is diffeotopic to 0. A differentiable family $\varphi_t : TV \to TV$ of homomorphisms for which $\varphi_0 = 0, \varphi_1 = \mathrm{id}$ is given by $\varphi_t = \tau_{t\sigma}, t \in [0,1]$.

If $\mathfrak{A}$ is an m-algebra, by abuse of notation, we write $T\mathfrak{A}$ (rather than $T^s\mathfrak{A}$) for the smooth tensor algebra over $\mathfrak{A}$. Thus $T\mathfrak{A}$ is again an m-algebra.

3.3 The free product of two m-algebras

Let $\mathfrak{A}$ and $\mathfrak{B}$ be m-algebras. The (non-unital) algebraic free product $\mathfrak{A} * \mathfrak{B}$ is the direct sum over all tensor products

$$\mathfrak{A} * \mathfrak{B} \;=\; \mathfrak{A} \oplus \mathfrak{B} \oplus (\mathfrak{A}{\otimes}\mathfrak{B}) \oplus (\mathfrak{B}{\otimes}\mathfrak{A}) \oplus (\mathfrak{A}{\otimes}\mathfrak{B}{\otimes}\mathfrak{A}) \oplus \ldots$$

where the factors $\mathfrak{A}$ and $\mathfrak{B}$ alternate (see 2.3). Multiplication is, as in 2.3 concatenation of tensors combined with simplification using the multiplication in $\mathfrak{A}$ and $\mathfrak{B}$.

We denote by $\mathfrak{A} \hat{*} \mathfrak{B}$ the completion of $\mathfrak{A} * \mathfrak{B}$ with respect to the family of all seminorms of the form $p * q$, given by:

$$p * q = p \oplus q \oplus (p \otimes q) \oplus (q \otimes p) \oplus (p \otimes q \otimes p) \oplus \ldots$$

where p and q are continuous seminorms on $\mathfrak{A}$ and $\mathfrak{B}$, respectively. If p and q are submultiplivative, then so is the seminorm $p * q$ and $\mathfrak{A} \hat{*} \mathfrak{B}$ therefore is an m-algebra.

$\mathfrak{A} \hat{*} \mathfrak{B}$ is in fact the free product of $\mathfrak{A}$ and $\mathfrak{B}$ in the category of m-algebras. The canonical inclusions $\iota_1 : \mathfrak{A} \to \mathfrak{A} \hat{*} \mathfrak{B}$ and $\iota_2 : \mathfrak{B} \to \mathfrak{A} \hat{*} \mathfrak{B}$ have the following universal property: Let $\alpha : \mathfrak{A} \to \mathfrak{C}$ and $\beta : \mathfrak{B} \to \mathfrak{C}$ be two continuous homomorphisms into an

m-algebra $\mathfrak{C}$. Then there is a unique continuous homomorphism $\alpha * \beta : \mathfrak{A}\hat{*}\mathfrak{B} \to \mathfrak{C}$, such that $(\alpha * \beta)\circ\iota_1 = \alpha$ and $(\alpha * \beta)\circ\iota_2 = \beta$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are m-algebras, by abuse of notation, we write $\mathfrak{A} * \mathfrak{B}$ (rather than $\mathfrak{A}\hat{*}\mathfrak{B}$) for the m-algebra free product of $\mathfrak{A}$ and $\mathfrak{B}$.

3.4 The algebra of smooth compact operators

The m-algebra $\mathfrak{K}$ of "smooth compact operators" consists of all $\mathbb{N}\times\mathbb{N}$-matrices (a_{ij}) with rapidly decreasing matrix elements $a_{ij} \in \mathbb{C}$, $i, j = 0, 1, 2 \ldots$. The topology on $\mathfrak{K}$ is given by the family of norms $p_n, n = 0, 1, 2 \ldots$, which are defined by

$$p_n\big((a_{ij})\big) = \sum_{i,j} |1 + i + j|^n \, |a_{ij}|$$

It is easily checked that the p_n are submultiplicative and that $\mathfrak{K}$ is complete. Thus $\mathfrak{K}$ is an m-algebra. As a locally convex vector space, $\mathfrak{K}$ is isomorphic to the sequence space s and therefore is nuclear.

The map that sends $(a_{ij}) \otimes (b_{kl})$ to the $\mathbb{N}^2 \times \mathbb{N}^2$-matrix $(a_{ij}b_{kl})_{(i,k)(j,l)\in\mathbb{N}^2\times\mathbb{N}^2}$ obviously gives an isomorphism Θ between $\mathfrak{K}\hat{\otimes}\mathfrak{K}$ and $\mathfrak{K}$ (see also [36], 2.7).

Lemma 3.4.1 *Let $\Theta : \mathfrak{K} \to \mathfrak{K}\hat{\otimes}\mathfrak{K}$ be as above and let $\iota : \mathfrak{K} \to \mathfrak{K}\hat{\otimes}\mathfrak{K}$ be the inclusion that maps x to $e^{00} \otimes x$ (where e^{00} is the matrix with elements a_{ij}, for which $a_{ij} = 1$, whenever $i = j = 0$, and $a_{ij} = 0$ otherwise). Then Θ is diffeotopic to ι. The same holds for the corresponding maps $\Theta' : \mathfrak{K} \to M_2(\mathfrak{K})$ and $\iota' : \mathfrak{K} \to M_2(\mathfrak{K})$.*

Remark 3.4.2 *Let V be a Banach space. Then $\mathfrak{K}\hat{\otimes}V$ consists exactly of those matrices (or the sequences indexed by $\mathbb{N} \times \mathbb{N}$) $(v_{ij})_{i,j\in\mathbb{N}}$, for which the expression*

$$\bar{p}_n((v_{ij})) \underset{def}{=} \sum_{i,j} (1 + i + j)^n \|v_{ij}\|$$

is finite for all n. The topology on $\mathfrak{K}\hat{\otimes}V$ is of course given exactly by the norms $\bar{p}_n$. To see this, consider the tensor product α_n of the norm p_n on $\mathfrak{K}$ with the norm $\|\cdot\|$ given on V. Then, if x^{ij} denotes the matrix, which has $x \in V$ as element on the place i, j and is 0 otherwise, we have

$$\alpha_n(x^{ij}) = (1 + i + j)\|x\|$$

by [42], Prop. 43.1. This immediately shows that

$$\alpha_n((v_{ij})) \leq \bar{p}_n((v_{ij}))$$

for all matrices (v_{ij}) in the algebraic tensor product $\mathfrak{K} \otimes V$. The converse inequality follows from the definition of the projective tensor product. Therefore, for each fixed n, the completion $(\mathfrak{K}\otimes V)_{\bar{p}_n}$ is isometrically isomorphic to $(\mathfrak{K})_{p_n}\hat{\otimes}V$ and consists exactly of those matrices (v_{ij}), for which $\bar{p}_n((v_{ij}))$ is finite.

3.5 The Schatten ideals $\ell^p(H)$

Let H be an infinite-dimensional separable Hilbert space (i.e., $H \cong \ell^2(\mathbb{N})$), and $(\xi_n)_{n\in\mathbb{N}}$ an orthonormal basis. Let also $\mathcal{L}(H)$ denote the algebra of bounded operators on H. For each positive x in $\mathcal{L}(H)$, we can define the trace $Tr(x)$ by

$$Tr\,(x) = \sum_n (x\xi_n \mid \xi_n)$$

This expression is ≥ 0 and may be infinite. It does not depend on the choice of the orthonormal basis. Moreover $Tr(x^*x) = Tr(xx^*)$ holds for all x.

Definition 3.5.1 *Let $p \geq 1$. We define $\ell^p(H)$ to be the set of all operators $x \in \mathcal{L}(H)$ for which*

$$Tr\,|x|^p < \infty$$

where $|x| = \sqrt{x^*x}$.

Proposition 3.5.2 *For each $p \geq 1$, $\ell^p(H)$ is an ideal in $\mathcal{L}(H)$ which is contained in the ideal $\mathcal{K}(H)$ of compact operators on H. It is a Banach algebra with respect to the norm*

$$\|x\|_p = (\,Tr\,|x|^p\,)^{1/p}$$

Moreover $\ell^p(H) \subset \ell^q(H)$ for $p \leq q$

3.6 The smooth Toeplitz algebra

The elements of the algebra $\mathcal{C}^\infty S^1$ can be written as power series in the generator z (defined by $z(t) = t$, $t \in S^1 \subset \mathbb{C}$). The coefficients of these power series are rapidly decreasing. Thus

$$\mathcal{C}^\infty(S^1) = \Big\{ \sum_{k\in\mathbb{Z}} a_k z^k \;\;\Big|\; \sum_{k\in\mathbb{Z}} |a_k|\,|k|^n \; < \infty \text{ for all } n \in \mathbb{N} \Big\}$$

Submultiplicative norms that determine the topology are given by

$$q_n\left(\sum a_k z^k\right) = \sum |1+k|^n\,|a_k|$$

As a topological vector space, the smooth Toeplitz algebra $\mathfrak{T}$ may be defined as the direct sum $\mathfrak{T} = \mathfrak{K} \oplus \mathcal{C}^\infty(S^1)$.

To describe the multiplication in $\mathfrak{T}$, we write v_k for the element $(0, z^k)$ in $\mathfrak{T}$ and simply x for the element $(x, 0)$ with $x \in \mathfrak{K}$. Moreover, e^{ij} denotes the element of $\mathfrak{T}$, which is determined by the matrix (a_{kl}), where $a_{kl} = 1$, whenever $k = i, l = j$, and $a_{kl} = 0$ otherwise (with the convention that $e^{ij} = 0$, if $i < 0$ oder $j < 0$). Multiplication in $\mathfrak{T}$ then is determined by the following rules:

$$e^{ij}e^{kl} = \delta_{jk}e^{il}$$

$$v_k e^{ij} = e^{(i+k),j} \qquad e^{ij} v_k = e^{i,(j-k)}$$

$(i,j,k \in \mathbb{Z})$; and

$$v_k v_{-l} = \begin{cases} v_{k-l}(1 - E_{l-1}) & l > 0 \\ v_{k-l} & l \leq 0 \end{cases}$$

where $E_l = e^{00} + e^{11} + \cdots + e^{ll}$. If p_n are the norms on $\mathfrak{K}$ defined in 3.4 and q_n the norms on $\mathcal{C}^\infty(S^1)$ defined above, it is easy to see that each norm $p_n \oplus q_m$ is submultiplicative on $\mathfrak{T} = \mathfrak{K} \oplus \mathcal{C}^\infty(S^1)$ with the multiplication introduced above. Obviously, $\mathfrak{K}$ is a closed ideal in $\mathfrak{T}$ and the quotient $\mathfrak{T}/\mathfrak{K}$ is $\mathcal{C}^\infty(S^1)$.

4 Standard extensions of a given algebra.

Extensions play a fundamental role in all aspects of non-commutative geometry. In this section we describe standard extensions that can be associated to any given algebra. All of these extensions exist in a purely algebraic setting, but also in the category of m-algebras. We treat both cases in parallel.

4.1 The suspension extension

This is the fundamental extension of algebraic topology. It has the form

$$0 \to \mathbb{C}(0,1) \to \mathbb{C}[0,1) \to \mathbb{C} \to 0$$

or more generally

$$0 \to \mathfrak{A}(0,1) \to \mathfrak{A}[0,1) \to \mathfrak{A} \to 0$$

with an arbitrary m-algebra $\mathfrak{A}$.

Since cyclic homology is homotopy invariant only for differentiable homotopies (not for continuous ones) we have to work with algebras of differentiable functions. Recall that $\mathbb{C}(0,1)$ and $\mathbb{C}[0,1)$ denote the algebras of $\mathcal{C}^\infty$- functions on the interval $[0,1]$, whose derivatives to arbitrary order vanish in 0 and 1, or or only in 1, respectively, and that the algebra $\mathbb{C}[0,1)$ is differentiably contractible.

The algebraic suspension extension for an algebra A is

$$0 \to t(1-t)A[t] \to tA[t] \to A \to 0$$

4.2 The free extension

For any algebra there exists a natural extension by the free algebra TA—the nonunital tensor algebra over A.

$$0 \to JA \to TA \to A \to 0. \tag{2}$$

This extension is universal in the sense that, given any extension $0 \to I \to E \to A \to 0$ of A, there is a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \to & JA & \to & TA & \to & A & \to & 0 \\ & & \downarrow \gamma_A & & \downarrow & & \downarrow id & & \\ 0 & \to & I & \to & E & \to & A & \to & 0 \end{array}$$

(the map $TA \to E$ is obtained by choosing a linear splitting $s : A \to E$ in the given extension). The map $JA \to I$ in this commutative diagram is called the *classifying map* of the extension $0 \to I \to E \to A \to 0$.
More generally, any n-step extension of the form

$$0 \to I \to E_1 \to E_2 \to \ldots \to E_n \to A \to 0$$

can be compared with the free n-step extension

$$0 \to J^n A \to T(J^{n-1}A) \to T(J^{n-2}A) \to \ldots \to TA \to A \to 0$$

and thus has a classifying map $J^n A \to I$ (which is unique up to homotopy) (here $J^n A$ is defined recursively by $J^n A = J(J^{n-1}A)$).
For any m-algebra $\mathfrak{A}$, the (smooth) tensor algebra $T\mathfrak{A}$ has been defined in 3.2.

Recall that the linear map $\sigma : V \to TV$ that sends V to the first summand in $T\mathfrak{A}$ has the following universal property:
Let $s : \mathfrak{A} \to \mathfrak{B}$ be an arbitrary continuous linear map from $\mathfrak{A}$ to an m-Algebra $\mathfrak{B}$. Then there exists a unique homomorphism $\tau_s : T\mathfrak{A} \to \mathfrak{B}$ of m-algebras with the property that $\tau_s \circ \sigma = s$. Thus exactly as in the algebraic case we get an extension

$$0 \to J\mathfrak{A} \to T\mathfrak{A} \to \mathfrak{A} \to 0. \tag{3}$$

by defining $J\mathfrak{A}$ as the kernel of the natural map $T\mathfrak{A} \to \mathfrak{A}$. This extension has the property that, given any extension $0 \to \mathfrak{I} \to \mathfrak{E} \to \mathfrak{A} \to 0$ of $\mathfrak{A}$ **with a continuous linear splitting** $s : \mathfrak{A} \to \mathfrak{E}$, there is a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \to & J\mathfrak{A} & \to & T\mathfrak{A} & \to & \mathfrak{A} & \to & 0 \\ & & \downarrow \gamma_A & & \downarrow & & \downarrow id & & \\ 0 & \to & \mathfrak{I} & \to & \mathfrak{E} & \to & \mathfrak{A} & \to & 0 \end{array}$$

The map $J\mathfrak{A} \to \mathfrak{I}$ in this commutative diagram is called the *classifying map* of the extension $0 \to \mathfrak{I} \to \mathfrak{E} \to \mathfrak{A} \to 0$.
More generally, any n-step extension of the form

$$0 \to \mathfrak{I} \to \mathfrak{E}_1 \to \mathfrak{E}_2 \to \ldots \to \mathfrak{E}_n \to \mathfrak{A} \to 0$$

has a classifying map $J^n\mathfrak{A} \to \mathfrak{I}$ (which is unique up to homotopy) (here $J^n\mathfrak{A}$ is defined recursively by $J^n\mathfrak{A} = J(J^{n-1}\mathfrak{A})$).

4.3 The universal two-fold trivial extension.

With any m-algebra $\mathfrak{A}$ we associate as in [9] the m-algebra $Q\mathfrak{A} = \mathfrak{A} * \mathfrak{A}$. We denote by ι and $\bar{\iota}$ the two canonical inclusions of $\mathfrak{A}$ into $Q\mathfrak{A}$. The algebra $Q\mathfrak{A}$ is, in a natural way, $\mathbb{Z}/2$-graded by the involutive automorphism τ which exchanges $\iota(\mathfrak{A})$ and $\bar{\iota}(\mathfrak{A})$.

The ideal $q\mathfrak{A}$ in $Q\mathfrak{A}$ is by definition the kernel of the homomorphism $\pi = \mathrm{id} * \mathrm{id} : \mathfrak{A} * \mathfrak{A} \to \mathfrak{A}$. The extension

$$0 \to q\mathfrak{A} \to Q\mathfrak{A} \xrightarrow{\pi} \mathfrak{A} \to 0 \tag{4}$$

admits two natural splittings via the algebra homomorphisms ι and $\bar{\iota}$. It has the following universal property: Let

$$0 \to \mathfrak{E}_0 \to \mathfrak{E}_1 \to \mathfrak{A} \to 0 \tag{5}$$

be an extension with two splittings $\alpha, \bar{\alpha} : \mathfrak{A} \to \mathfrak{E}_1$, which are continuous algebra homomorphisms. Then there is a morphism of extensions (i.e., a commutative diagram) as follows:

$$\begin{array}{ccccccc}
0 \to & q\mathfrak{A} & \to & Q\mathfrak{A} & \to & \mathfrak{A} & \to 0 \\
 & \downarrow {\scriptstyle \alpha * \bar{\alpha}} & & \downarrow {\scriptstyle \alpha * \bar{\alpha}} & & \downarrow {\scriptstyle \mathrm{id}} & \\
0 \to & \mathfrak{E}_0 & \to & \mathfrak{E}_1 & \to & \mathfrak{A} & \to 0
\end{array}$$

By construction, this morphism sends ι and $\bar{\iota}$ to α and $\bar{\alpha}$.

The same construction can of course be performed purely algebraically and we obtain, for any algebra A, an extension

$$0 \to qA \to QA \to A \to 0$$

where $QA = A * A$. This extension has the same universal property in the category of algebras without topology.

4.4 The Toeplitz extension

The smooth Toeplitz algebra $\mathfrak{T}$ was introduced in 3.6. By definition, $\mathfrak{T}$ contains the algebra $\mathfrak{K}$ of smooth compact operators as a closed ideal and we obtain the following extension (Toeplitz extension)

$$0 \to \mathfrak{K} \to \mathfrak{T} \xrightarrow{\pi} \mathcal{C}^\infty(S^1) \to 0$$

which, of course, admits a continuous linear splitting.

Let $\kappa : \mathfrak{T} \to \mathbb{C}$ be the canonical homomorphism, which maps v_1 and v_{-1} to 1 (with the notation of 3.6) and put $\mathfrak{T}_0 = \mathrm{Ker}\,\kappa$. By restriction we obtain the following extension

$$0 \to \mathfrak{K} \to \mathfrak{T}_0 \to \mathcal{C}_0^\infty(S^1 \backslash 1) \to 0$$

The algebraic Toeplitz extension has the following form

$$0 \to M_\infty(\mathbb{C}) \to \Psi \to \mathbb{C}[\,t, t^{-1}] \to 0$$

where Ψ is the algebraic Toeplitz algebra, 2.5

5 Preliminaries on homological algebra

A complex is a $\mathbb{Z}$-graded vector space C with an endomorphism ∂ of degree 1 such that $\partial^2 = 0$. The map δ is called the boundary operator of C. We will also use $\mathbb{Z}/2$-graded complexes ("supercomplexes"). They can be viewed as periodic ordinary complexes.

Definition 5.1 *The homology $H(C)$ of a complex C is the quotient space*

$$H(C) = \operatorname{Ker}\partial / \operatorname{Im}\partial$$

$H(C)$ is a graded vector space with $H_n(C) = (\operatorname{Ker}\partial \cap C_n)/\partial(C_{n+1})$.

A linear map between two complexes C and D is said to have degree k if it maps C_n to D_{n+k} for all n. The Hom-complex $\operatorname{Hom}(C, D)$ is the (graded) vector space consisting of linear maps φ from the vector space C to the vector space D which are linear combinations of such homogeneous maps. The boundary map on $\operatorname{Hom}(C, D)$ is given by the graded commutator

$$\partial(\varphi) = \partial_Y \circ \varphi - (-1)^{\deg}\varphi\varphi \circ \partial_X .$$

wit the boundary maps. A map $\varphi : C \to D$ of degree 0 is called a morphism of complexes if $\partial\varphi = 0$, i.e., if it commutes with the boundary operators. Any such morphism induces a morphism

$$\varphi_* : H(C) \longrightarrow H(D)$$

Two morphisms in $\operatorname{Hom}(C, D)$ are said to be (chain) homotopic if they differ by a boundary in $\operatorname{Hom}(X, Y)$. Homotopic morphisms induce the same maps on homology. Two complexes are called homotopy equivalent if there exists

$$\varphi \in \operatorname{Hom}(C, D)\,,\ \partial\varphi = 0$$

and

$$\psi \in \operatorname{Hom}(C, D)\,,\ \partial\psi = 0$$

$\deg\varphi = \deg\psi = 0$ such that $\varphi\psi$ and $\psi\varphi$ are homotopic to id. Finally, the complex C will be said to be contractible if it is homotopy equivalent to 0 (this is the case iff there exists a map $s : C \to C$ of degree 1, such that $\mathrm{id}_C = \partial s + s\partial$). The homology of a contractible complex is 0.

A sequence

$$\ldots \xrightarrow{\alpha_{n+2}} G_{n+1} \xrightarrow{\alpha_{n+1}} G_n \xrightarrow{\alpha_n} G_{n-1} \xrightarrow{\alpha_{n-1}} \ldots$$

of morphisms between vector spaces or abelian groups is called exact, if, for each n, $\operatorname{Ker}\alpha_n = \operatorname{Im}\alpha_{n+1}$.

Proposition 5.2 *Let*

$$0 \longrightarrow C \xrightarrow{\iota} D \xrightarrow{\pi} E \longrightarrow 0$$

be an extension of complexes, i.e., an exact sequence of (graded) vector spaces, where the maps ι and π are morphisms of complexes. Then there is a map $\delta : H(E) \to H(C)$ of degree 1 and an exact sequence of the form

$$\begin{tikzcd} & H(D) \arrow[dr, "\pi_*"] & \\ H(C) \arrow[ur, "\iota_*"] & & H(E) \arrow[ll, "\delta"] \end{tikzcd}$$

This can be written as a long exact sequence of the form

$$\xrightarrow{\pi_*} H_{n+1}E \xrightarrow{\delta} H_nC \xrightarrow{\iota_*} H_nD \xrightarrow{\pi_*} H_nE \xrightarrow{\delta} H_{n-1}C \xrightarrow{\iota_*} H_{n-1}A \xrightarrow{\pi_*}$$

Proof. Given $[x]$ in $H(E)$ with $\partial(x) = 0$, put $\delta([x]) = \partial(z)$, where $z \in D$ is such that $\pi(z) = x$. □

Definition 5.3 *A bicomplex is a $\mathbb{Z}\times\mathbb{Z}$-graded vector space or, equivalently, a family $(C_{pq})_{p,q\in\mathbb{Z}}$ of vector spaces with two endomorphisms ∂_h and ∂_v of degree $(-1,0)$ and $(0,-1)$, respectively, such that*

$$\partial_h^2 = \partial_v^2 = \partial_h\partial_v + \partial_v\partial_h = 0$$

∂_h and ∂_v are called the horizontal and vertical boundary maps of C.
The total complex for C is the infinite product

$$D = \prod_{p,q\in\mathbb{Z}} C_{pq}$$

It is a $\mathbb{Z}$-graded vector space with the grading given by $D_n = \prod_{p+q=n} C_{pq}$ (The letter D stands for diagonals). With the boundary operator $\partial = \partial_h + \partial_v$ it becomes a complex.

Remark 5.4 *Instead of the direct product, one can also take the direct sum in the definition of the total complex. Both variants will be used in the sequel.*

For each p, the family $(C_{pq})_{q\in\mathbb{Z}}$ defines a complex with boundary operator ∂_v. We call its homology the vertical homology. The horizontal homology is defined similarly for each q. We will often consider bicomplexes where $C_{pq} = 0$ for $p < 0$ or $q < 0$. The following proposition is then often useful.

Proposition 5.5 *Let $C = C_{pq}$ and $E = E_{pq}$ be bicomplexes with $C_{pq} = 0$ and $E_{pq} = 0$ for $p < 0$ or $q < 0$, and let D and F denote the total complexes of C and E, respectively. Assume that $\alpha, \beta : C \to E$ are two morphisms of bicomplexes which, for each p, induce the same map on the vertical homology of C and E, respectively. Then the induced maps $\alpha_*, \beta_* : H(D) \to H(F)$ coincide.*

6 Definition of cyclic homology/cohomology using the cyclic bicomplex and the Connes complex

In this section, we introduce cyclic homology using, what are probably the most fundamental operators in cyclic theory (b, b', λ, Q), and we explain the long exact sequence connecting cyclic homology with Hochschild homology. The important properties of cyclic theory will be discussed only much later.

The cyclic bicomplex $CC_{pq}(A)$, $p, q \in \mathbb{N}$, for an algebra A is the following

$$
\begin{array}{lllllll}
\downarrow & & \downarrow & & \downarrow & & \\
A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & A^{\otimes 3} & \xleftarrow{1-\lambda} & \dots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & A^{\otimes 2} & \xleftarrow{1-\lambda} & \dots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & A & \xleftarrow{1-\lambda} & \dots
\end{array}
$$

i.e.,

$$CC_{pq}(A) = A^{\otimes(q+1)} \qquad p, q \geq 0$$

In this section and later, bicomplexes are understood to continue with 0 outside the represented region (thus here $C_{pq} = 0$ for $p < 0$ or $q < 0$). The map b' is given by

$$b'(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = \sum_{j=0}^{n-1} (-1)^j x_0 \otimes \ldots \otimes x_j x_{j+1} \otimes \ldots \otimes x_n$$

while b is the usual Hochschild operator

$$b(x_0 \otimes \ldots \otimes x_n) = b'(x_0 \otimes \ldots \otimes x_n) + (-1)^n x_n x_1 \otimes x_0 \otimes \ldots \otimes x_{n-1}$$

Finally, λ is Connes' signed cyclic permutation operator

$$\lambda(x_0 \otimes \ldots \otimes x_n) = (-1)^n \, x_n \otimes x_0 \otimes \ldots \otimes x_{n-1}$$

and $Q = \sum_{j=0}^{n} \lambda^j$ is averaging with respect to signed cyclic permutations, thus

$$Q(x_0 \otimes \ldots \otimes x_n) = \sum_{j=0}^{n} (-1)^{jn} x_j \otimes \ldots \otimes x_n \otimes x_0 \otimes \ldots \otimes x_{j-1}$$

One has $b^2 = 0, b'^2 = 0$ and $(1-\lambda)Q = Q(1-\lambda) = 0$. Moreover, $CC_{pq}(A)$ is in fact a bicomplex, due to the fundamental identities

$$Qb = b'Q$$

$$(1-\lambda)b' = b(1-\lambda)$$

The rows of the cyclic bicomplex $CC_{pq}(A)$ are obviously exact as long as $p > 0$.

Definition 6.1 *HC_nA is the homology of the total complex of the cyclic bicomplex $CC_{pq}(A)$, i.e., HC_nA is the homology of the complex*

$$\ldots \longrightarrow D_n \xrightarrow{\partial} D_{n-1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} D_1 \xrightarrow{\partial} D_0 \longrightarrow 0$$

where

$$\begin{aligned} D_n &= CC_{0,n} \oplus CC_{1,(n-1)} \oplus \ldots \oplus CC_{n,0} \\ &= A^{\otimes(n+1)} \oplus A^{\otimes n} \oplus \ldots \oplus A \end{aligned}$$

are the diagonals in $CC_{pq}(A)$ and ∂ is the sum of the respective boundary operators $b, b', 1-\lambda, Q$.

Consider the bicomplex $CC_*^{Hochschild}$ consisting of the first two columns of CC_*:

$$\begin{array}{ccc} \downarrow & & \downarrow \\ A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} \\ \downarrow b & & \downarrow -b' \\ A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} \\ \downarrow b & & \downarrow -b' \\ A & \xleftarrow{1-\lambda} & A \end{array} \qquad (CC_*^{Hochschild})$$

and its total complex

$$\ldots \xrightarrow{\partial'} D_n^{Hochschild} \xrightarrow{\partial'} D_{n-1}^{Hochschild} \xrightarrow{\partial'} \ldots \xrightarrow{\partial'} D_0^{Hochschild} \longrightarrow 0$$

where

$$D_n^{Hochschild} = A^{\otimes(n+1)} \oplus A^{\otimes n}, \quad n \geq 1, \qquad D_0^{Hochschild} = A$$

$$\text{and} \quad \partial' = \begin{pmatrix} b & (1-\lambda) \\ 0 & -b' \end{pmatrix}$$

Definition 6.2 *For any (possibly nonunital) algebra A, the Hochschild homology $HH_*(A)$ is defined as the homology of the complex*

$$\ldots \xrightarrow{\partial'} D_n^{Hochschild} \xrightarrow{\partial'} D_{n-1}^{Hochschild} \xrightarrow{\partial'} \ldots \xrightarrow{\partial'} D_0^{Hochschild} \longrightarrow 0$$

If A has a unit, then the second column in $CC_*^{Hochschild}$ is obviously exact and can therefore be disregarded for the computation of $HH_*(A)$. We therefore find in this case the usual definition of the Hochschild homology of A (with coefficients in A), namely as the homology of the complex

$$\ldots \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A \longrightarrow 0$$

More generally, if this holds, we call A H-unital, see [46].

Definition 6.3 *The algebra A is called H-unital, if the second column in $CC_*^{Hochschild}$, i.e., the complex*

$$\ldots \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{b'} A \longrightarrow 0$$

is exact.

For any H-unital algebra A, the Hochschild homology $HH(A)$ is the homology of the complex

$$\ldots \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A \longrightarrow 0$$

If A is unital, then the Hochschild homology of A has a well known interpretation as a derived functor. In particular it can be computed from any projective A-bimodule resolution of the A-bimodule A. This method applies in many cases.

Theorem 6.4 *There is a long exact sequence (Connes's SBI-sequence) of the form*

$$\begin{aligned} \longrightarrow \quad & HC_nA \xrightarrow{S} HC_{n-2}A \xrightarrow{B} HH_{n-1}A \xrightarrow{I} HC_{n-1}A \xrightarrow{S} HC_{n-3}A \xrightarrow{B} \\ & \ldots \xrightarrow{I} HC_1A \xrightarrow{S} HC_{-1}A \longrightarrow HH_0A \xrightarrow{I} HC_0A \longrightarrow 0 \end{aligned}$$

Proof. This is the long exact sequence associated to the obvious exact sequence of complexes

$$0 \longrightarrow D_n^{Hochschild} \longrightarrow D_n \longrightarrow D_{n-2} \longrightarrow 0$$

□

The operators I and S in the above sequence are simply induced by the inclusion $D_n^{Hochschild} \to D_n$ and by the projection $D_n \to D_{n-2}$, while the operator B is given by the boundary map. Note that $HC_{-1}A = 0$ and that $I : HH_0A \xrightarrow{\cong} HC_0A$ is an isomorphism. If the Hochschild homology of A is known, then the SBI-sequence allows in many cases to determine the cyclic homology groups for A recursively.

Consider the following sub-bicomplex $CC_*^{trivial}(A)$ of $CC_{pq}(A)$

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
(1-\lambda)A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & A^{\otimes 3} & \xleftarrow{1-\lambda} & \ldots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
(1-\lambda)A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & A^{\otimes 2} & \xleftarrow{1-\lambda} & \ldots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
(1-\lambda)A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & A & \xleftarrow{1-\lambda} & \ldots
\end{array} \qquad (CC_*^{trivial}(A))$$

Since the rows in $CC_*^{trivial}(A)$ are exact, its total complex $D^{trivial}$ has no homology. We denote by $C^\lambda(A)$ the quotient complex $D/D^{trivial}$. Explicitly

$$C_n^\lambda(A) = D_n/D_n^{trivial} = A^{\otimes(n+1)}/(1-\lambda)A^{\otimes(n+1)}$$

Since $H_n(D^{trivial}) = 0$, we immediately get

Proposition 6.5 *For any algebra A, the cyclic homology $HC_n(A)$ can be computed as the homology of the complex*

$$\ldots \longrightarrow C_n^{\lambda}(A) \xrightarrow{b} C_{n-1}^{\lambda}(A) \xrightarrow{b} \ldots \longrightarrow C_0^{\lambda}(A) \longrightarrow 0$$

This is Connes' original definition of cyclic homology.

The two-sided (or periodic) cyclic complex $CC_*^{twosided}$ continues CC_* to the left. It is defined by $CC_{pq}^{twosided}(A) = A^{\otimes(q+1)} \quad p, q \in \mathbb{Z}, q \geq 0$ with boundary operators as follows

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \xleftarrow{Q} & A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & A^{\otimes 3} & \xleftarrow{1-\lambda} & \ldots \\
 & & \downarrow b & & \downarrow -b' & & \downarrow b & & \\
\ldots & \xleftarrow{Q} & A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & A^{\otimes 2} & \xleftarrow{1-\lambda} & \ldots \\
 & & \downarrow b & & \downarrow -b' & & \downarrow b & & \\
\ldots & \xleftarrow{Q} & A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & A & \xleftarrow{1-\lambda} & \ldots
\end{array} \qquad (CC_{pq}^{twosided}(A))$$

The left cyclic complex CC_*^{left} is the left-hand side of $CC_*^{twosided}$

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} \\
 & & \downarrow -b' & & \downarrow b & & \downarrow -b' \\
\ldots & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} \\
 & & \downarrow -b' & & \downarrow b & & \downarrow -b' \\
\ldots & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & A & \xleftarrow{1-\lambda} & A
\end{array} \qquad (CC_*^{left})$$

Consider the total complexes D^{left}, $D^{twosided}$, D of the bicomplexes $CC_{pq}^{left}(A)$, $CC_{pq}^{twosided}(A)$ and $CC_{pq}(A)$, respectively, where here we use direct sums rather than direct products for the definition of the diagonals in the total complex. We get an exact sequence of complexes

$$0 \longrightarrow D^{left} \longrightarrow D^{twosided} \longrightarrow D \longrightarrow 0$$

Since the rows of $CC_{pq}^{twosided}(A)$ are exact, a standard staircase argument shows that $D^{twosided}$ has no homology (this argument would not work for the direct product total complex). Therefore

$$HC_n(A) = H_n(D) \cong H_{n-1}(D^{left})$$

whence the following alternative description of cyclic homology

Proposition 6.6 *For any algebra A, the cyclic homology $HC_n(A)$ can be computed as*

$$HC_n(A) = H_{n-1}(D^{left})$$

where D^{left} is the total complex of $CC^{left}_{pq}(A)$.

By using the dual complexes, the analogue of any statement in this section is also valid for cohomology in place of homology.

In particular, cyclic cohomology can be defined using the dual $C^n_\lambda A = (C^\lambda_n A)'$ of $C^\lambda_n A$, i.e.,

$$C^n_\lambda A = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is an } (n+1)\text{-linear functional on } A^{n+1} \text{ such that} \\ f(x_n, x_0, x_1, \ldots, x_{n-1}) = (-1)^n f(x_0, x_1, \ldots, x_n) \end{array} \right\}$$

The boundary operator on $C^n_\lambda A$ is the restriction of the Hochschild operator b.

A **cyclic n-cocycle** then is an $(n+1)$-linear functional such that

$$f(x_n, x_0, x_1, \ldots, x_{n-1}) = (-1)^n f(x_0, x_1, \ldots, x_n)$$

and

$$\begin{aligned} & f(x_0x_1, x_2, \ldots, x_{n+1}) - f(x_0, x_1x_2, \ldots, x_{n+1}) + \ldots \\ & \quad + (-1)^n f(x_0, \ldots, x_nx_{n+1}) + (-1)^{n+1} f(x_{n+1}x_0, x_1, \ldots, x_n) = 0 \end{aligned}$$

This is the original definition of a cyclic cocycle due to A. Connes. It also is the form in which cyclic cocycles appear often in practice.

7 The algebra ΩA of abstract differential forms over A and its operators

The boundary operators in the cyclic bicomplex take a simpler form and can be "explained" if interpreted as operators on the algebra ΩA of abstract differential forms over A. Another deeper and even more conceptual explanation will be presented in Section 10.1.

Given an algebra A, we denote by ΩA the universal algebra generated by $x \in A$ with relations of A and symbols $dx, x \in A$, where dx is linear in x and satisfies $d(xy) = xdy + d(x)y$. We do not impose $d1 = 0$, i.e., if A has a unit, $d1 \neq 0$. ΩA is a direct sum of subspaces $\Omega^n A$ generated by linear combinations of $x_0dx_1 \ldots dx_n$, and $dx_1 \ldots dx_n$, $x_j \in A$. This decomposition makes ΩA into a graded algebra. As usual, we write $\deg(\omega) = n$ if $\omega \in \Omega^n A$.

As a vector space, for $n \geq 1$,

$$\Omega^n A \cong \widetilde{A} \otimes A^{\otimes n} \cong A^{\otimes(n+1)} \oplus A^{\otimes n} \tag{6}$$

(where $\widetilde{A}$ is A with a unit adjoined and $1 \otimes x_1 \otimes \cdots \otimes x_n$ corresponds to $dx_1 \ldots dx_n$). The operator d is defined on ΩA by

$$\begin{aligned} d(x_0 dx_1 \ldots dx_n) &= dx_0 dx_1 \ldots dx_n \\ d(dx_1 \ldots dx_n) &= 0 \end{aligned} \tag{7}$$

The operator b is defined by

$$\begin{aligned} b(\omega dx) = (-1)^{\deg \omega}[\omega, x] \\ b(dx) = 0\,,\, b(x) \;=\; 0\,, \qquad x \in A\,,\, \omega \in \Omega A \end{aligned} \tag{8}$$

Then clearly $d^2 = 0$ and one easily computes that also $b^2 = 0$.
Under the isomorphism in equation (6) d becomes

$$d(x_0 \otimes \ldots \otimes x_n) = 1 \otimes x_0 \otimes \ldots \otimes x_n \quad x_0 \in A$$
$$d(1 \otimes x_1 \otimes \ldots \otimes x_n) = 0$$

while b corresponds to the usual Hochschild operator

$$\begin{aligned} &b(\widetilde{x_0} \otimes x_1 \otimes \ldots \otimes x_n) = \\ &\quad \widetilde{x_0} x_1 \otimes \ldots \otimes x_n + \textstyle\sum_{j=2}^n (-1)^{j-1} \widetilde{x_0} \otimes \ldots \otimes x_{j-1} x_j \otimes \ldots \otimes x_n \\ &\quad + (-1)^n x_n \widetilde{x_0} \otimes x_1 \otimes \ldots \otimes x_{n-1}\,, \qquad \widetilde{x_0} \in \widetilde{A},\; x_1, \ldots, x_n \in A \end{aligned}$$

Another important natural operator is the degree (or number) operator:

$$N(\omega) \;=\; \deg(\omega)\omega \tag{9}$$

We also define the Karoubi operator κ on ΩA by

$$\kappa = 1 - (db + bd) \tag{10}$$

Explicitly, κ is given by

$$\kappa(\omega dx) = (-1)^{deg \omega} dx \omega$$

The operator κ satisfies $(\kappa^n - 1)(\kappa^{n+1} - 1) = 0$ on Ω^n. Therefore, by linear algebra, there is a projection operator P on ΩA corresponding to the generalized eigenspace for 1 of the operator $\kappa = 1 - (db + bd)$.

Lemma 7.1 *Let $L = (Nd)b + b(Nd)$. Then $\Omega A = Ker L \oplus Im L$ and P is exactly the projection onto $Ker L$ in this splittting.*

Proof. This follows from the identity

$$L = (\kappa - 1)^2 (\kappa^{n-1} + 2\kappa^{n-2} + 3\kappa^{n-3} + \ldots + (n-1)\kappa + n)$$

$\square$

The operator L thus behaves like a "selfadjoint" operator. It can be viewed as an abstract Laplace operator on the algebra of abstract differential forms ΩA. The elements in the image of P are then "abstract harmonic forms".

By construction, P commutes with b, d, N. Thus setting $B = NPd$ one finds $Bb + bB = PL = 0$ and $B^2 = 0$.
Explicitly, B is given on $\omega \in \Omega$ by the formula

$$B(\omega) = \sum_{j=0}^{n} \kappa^j d\omega$$

Under the isomorphism in equation (6), this corresponds to:

$$\begin{aligned} B(x_0 dx_1 \dots dx_n) &= dx_0 dx_1 \dots dx_n + (-1)^n dx_n dx_0 \dots dx_{n-1} \\ &\quad + \dots + (-1)^{nn} dx_1 \dots dx_n dx_0 \end{aligned}$$

The preceding identities show that we obtain a bicomplex - the (B, b)-bicomplex - in the following way

$$\begin{array}{ccccccc} \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\ \Omega^3 A & \xleftarrow{B} & \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A \\ \downarrow b & & \downarrow b & & \downarrow b & & \\ \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & \\ \downarrow b & & \downarrow b & & & & \\ \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & & & \\ \downarrow b & & & & & & \\ \Omega^0 A & & & & & & \end{array} \tag{11}$$

One can rewrite the (B, b)-bicomplex (11) using the isomorphism $\Omega^n A \cong A^{\otimes(n+1)} \oplus A^{\otimes n}$ in equation (6). An easy computation shows that the operator $b : \Omega^n A \to \Omega^{n-1} A$ corresponds under this isomorphism to the operator $A^{\otimes(n+1)} \oplus A^{\otimes n} \to A^{\otimes n} \oplus A^{\otimes(n-1)}$ which is given by the matrix

$$\begin{pmatrix} b & (1-\lambda) \\ 0 & -b' \end{pmatrix}$$

where b', b and λ are the operators discussed in Section 6.
Similarly, the operator $B : \Omega^n A \to \Omega^{n+1} A$ corresponds to the operator $A^{\otimes(n+1)} \oplus A^{\otimes n} \to A^{\oplus(n+2)} \oplus A^{\otimes(n+1)}$ given by the 2×2-matrix

$$\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$$

where Q is as in Section 6.
This shows that the (B,b)-bicomplex is just another way of writing the cyclic bicomplex, i.e., the total complex D^Ω for the (B,b)-bicomplex is exactly isomorphic to the total complex D for the cyclic bicomplex. As a consequence we have

Proposition 7.2 *HC_nA is the homology of the complex*

$$\longrightarrow D_n^\Omega \stackrel{B'-b}{\longrightarrow} D_{n-1}^\Omega \stackrel{B'-b}{\longrightarrow} \dots \stackrel{B'-b}{\longrightarrow} D_1^\Omega \stackrel{B'-b}{\longrightarrow} D_0^\Omega \longrightarrow 0$$

where

$$\begin{aligned} D_{2n}^\Omega &= \Omega^0 A \oplus \Omega^2 A \oplus \dots \oplus \Omega^{2n} A \\ D_{2n+1}^\Omega &= \Omega^1 A \oplus \Omega^3 A \oplus \dots \oplus \Omega^{2n+1} A \end{aligned}$$

and B' is the truncated B-operator, i.e., $B' = B$ on the components $\Omega^k A$ of D_n^Ω, except on the highest component $\Omega^n A$, where it is 0.
$HH_n(A)$ is the homology of the complex

$$\longrightarrow \Omega^n A \stackrel{b}{\longrightarrow} \Omega^{n-1} A \stackrel{b}{\longrightarrow} \dots \stackrel{b}{\longrightarrow} \Omega^1 A \stackrel{b}{\longrightarrow} \Omega^0 A \longrightarrow 0$$

Remark 7.3 *Assume that A has a unit 1. We may introduce in ΩA the additional relation $d(1) = 0$, i.e., divide by the ideal M generated by $d(1)$ (this is equivalent to introducing the relation $1 \cdot \omega = \omega$ for all ω in ΩA). We denote the quotient by $\overline{\Omega} A$. Now M is a graded subspace, invariant under b and B, and its homology with respect to b is trivial. The preceding proposition is thus still valid if we use $\overline{\Omega} A$ in place of ΩA. The convention $d(1) = 0$ is often used (implicitly) in the literature. In some cases it simplifies computations considerably. There are however situations where one cannot reduce the computations to the unital situation (in particular this is true for the excision problem).*

If $\mathfrak{A}$ is an m-algebra, then $\Omega\mathfrak{A}$ is defined using the completed projective tensor product

$$\Omega\mathfrak{A} = \widetilde{\mathfrak{A}} \hat{\otimes} \mathfrak{A}^{\hat{\otimes} n} \cong \mathfrak{A}^{\hat{\otimes}(n+1)} \oplus \mathfrak{A}^{\hat{\otimes} n}$$

The operators on $\Omega\mathfrak{A}$ considered above are of course continuous and extend to the completed tensor products.

8 Periodic cyclic homology and the bivariant theory.

Let A be an algebra. We denote by $\widehat{\Omega} A$ the infinite product

$$\widehat{\Omega} A = \prod_n \Omega^n A$$

and by $\widehat{\Omega}^{ev}A$, $\widehat{\Omega}^{odd}A$ its even and odd part, respectively. $\widehat{\Omega}A$ may be viewed as the (periodic) total complex for the bicomplex

$$
\begin{array}{ccccccccc}
 & \downarrow_{-b} & & \downarrow_{-b} & & \downarrow_{-b} & & \downarrow_{-b} \\
\longleftarrow & \Omega^3 A & \stackrel{B}{\longleftarrow} & \Omega^2 A & \stackrel{B}{\longleftarrow} & \Omega^1 A & \stackrel{B}{\longleftarrow} & \Omega^0 A \\
 & \downarrow_{-b} & & \downarrow_{-b} & & \downarrow_{-b} & & \\
\longleftarrow & \Omega^2 A & \stackrel{B}{\longleftarrow} & \Omega^1 A & \stackrel{B}{\longleftarrow} & \Omega^0 A & & \\
 & \downarrow_{-b} & & \downarrow_{-b} & & & & \\
\longleftarrow & \Omega^1 A & \stackrel{B}{\longleftarrow} & \Omega^0 A & & & & \\
 & \downarrow_{-b} & & & & & & \\
\longleftarrow & \Omega^0 A & & & & & &
\end{array}
$$

Similarly, the (continuous) dual $(\widehat{\Omega}A)'$ of $\widehat{\Omega}A$ is

$$(\widehat{\Omega}A)' = \bigoplus_n (\Omega^n A)'$$

Definition 8.1 *The periodic cyclic homology $HP_*(A)$, $* = 0,1$, is defined as the homology of the $\mathbb{Z}/2$-graded complex*

$$\widehat{\Omega}^{ev}A \quad \underset{B-b}{\overset{B-b}{\rightleftarrows}} \quad \widehat{\Omega}^{odd}A$$

and the periodic cyclic cohomology $HP^*(A)$, $* = 0,1$, is defined as the homology of the $\mathbb{Z}/2$-graded complex

$$(\widehat{\Omega}^{ev}A)' \quad \underset{B-b}{\overset{B-b}{\leftrightarrows}} \quad (\widehat{\Omega}^{odd}A)'$$

Now by definition, S is the projection

$$D_n^\Omega = \Omega^n \oplus \Omega^{n-2} \oplus \ldots \quad \longrightarrow \quad D_{n-2}^\Omega = \Omega^{n-2} \oplus \Omega^{n-4} \oplus \ldots$$

where D_n^Ω is as in 7.2. Therefore we get

$$\widehat{\Omega}A = \varprojlim_S (D_{2n}^\Omega \oplus D_{2n+1}^\Omega)$$

and

$$(\widehat{\Omega}A)' = \varinjlim_{S'} (D_{2n}^\Omega \oplus D_{2n+1}^\Omega)'$$

We deduce

Proposition 8.2 *For any algebra A and $* = 0,1$ one has*

$$HP^*(A) = \varinjlim_S HC^{2n+*}A$$

and an exact sequence

$$0 \to \varprojlim_S{}^1 HC_{2n+*+1}A \to HP_*A \to \varprojlim_S HC_{2n+*}A \to 0$$

(where as usual $\varprojlim_S{}^1 HC_{2n+*+1}A$ is defined as $(\prod_n HC_{2n+*+1}A)/((1-s)(\prod_n HC_{2n+*+1}A))$, s being the shift on the infinite product).

Now $\widehat{\Omega}A$ is in a natural way a complete metric space (with the metric induced by the filtration on ΩA—the distance of families (x_n) and (y_n) in $\prod \Omega^n A$ to 0 is 2^{-k} if the k first x_i and y_i agree). We call this the filtration topology and denote by $\mathrm{Hom}\,(\widehat{\Omega}A,\, \widehat{\Omega}B)$ the set of continous linear maps $\widehat{\Omega}A \to \widehat{\Omega}B$. It can also be described as

$$\mathrm{Hom}\,(\widehat{\Omega}A,\, \widehat{\Omega}B) = \varprojlim_m (\varinjlim_n \mathrm{Hom}\,(\bigoplus_{i\le n}\Omega^i A,\, \bigoplus_{j\le m}\Omega^j B).$$

It is a $\mathbb{Z}/2$-graded complex with boundary map

$$\partial\varphi = \varphi\circ\partial - (-1)^{deg(\varphi)}\partial\circ\varphi$$

where $\partial = B - b$. If A and B are themselves topological algebra (m-algebras) of course one takes, in the definition of Hom above, continuous (for the locally convex vector space topology) linear maps from $\Omega^i A$ to $\Omega^j B$.

Definition 8.3 *Let A and B be algebras. Then the bivariant periodic cyclic homology $HP_*(A,B)$ is defined as the homology of the Hom-complex*

$$HP_*(A,B) = H_*(\mathrm{Hom}\,(\widehat{\Omega}A,\, \widehat{\Omega}B)) \quad * = 0,1$$

It is not difficult to see that the $\mathbb{Z}/2$-graded complex $\widehat{\Omega}\mathbb{C}$ is (continuously with respect to the filtration topology) homotopy equivalent to the trivial complex

$$\mathbb{C} \quad \overrightarrow{\longleftarrow} \quad 0$$

Therefore

$$HP_*(\mathbb{C},B) = HP_*(B) \quad \text{and} \quad HP_*(A,\mathbb{C}) = HP^*(A)$$

There is an obvious product

$$HP_i(A_1,A_2)\times HP_j(A_2,A_3) \to HP_{i+j}(A_1,A_3)$$

induced by the composition of elements in $\mathrm{Hom}\,(\widehat{\Omega}A_1, \widehat{\Omega}A_2)$ and $\mathrm{Hom}\,(\widehat{\Omega}A_2, \widehat{\Omega}A_3)$, which we denote by $(x, y) \mapsto x \cdot y$. In particular, $HP_0(A, A)$ is a unital ring with unit 1_A given by the identity map on $\widehat{\Omega}A$.
An element $\alpha \in HP_*(A, B)$ is called invertible if there exists $\beta \in HP_*(B, A)$ such that $\alpha \cdot \beta = 1_A \in HP_0(A, A)$ and $\beta \cdot \alpha = 1_B \in HP_0(B, B)$. An invertible element of degree 0, i.e., in $HP_0(A, B)$ will also be called an HP-equivalence. Such an HP-equivalence exists in $HP_0(A, B)$ if and only if the supercomplexes $\widehat{\Omega}A$ and $\widehat{\Omega}B$ are continuously homotopy equivalent. Multiplication by an invertible element α on the left or on the right induces natural isomorphisms $HP_*(B, D) \cong HP_*(A, D)$ and $HP_*(D, A) \cong HP_*(D, B)$ for any algebra D.

One can also define a $\mathbb{Z}$-graded version of the bivariant cyclic theory, [], as follows. Say that a linear map $\alpha : \widehat{\Omega}A \to \widehat{\Omega}B$, continuous for the filtration topology, is of order $\leq k$ if $\alpha(F^n\widehat{\Omega}A) \subset F^{n-k}\widehat{\Omega}B$ for $n \geq k$, where $F^n\widehat{\Omega}A$ is the infinite product of $b(\Omega^{n+1})$, Ω^{n+1}, Ω^{n+2}, ... We denote by $\mathrm{Hom}^k(\widehat{\Omega}A, \widehat{\Omega}B)$ the set of all maps of order $\leq k$. This is, for each k, a subcomplex of the $\mathbb{Z}/2$-graded complex $\mathrm{Hom}\,(\widehat{\Omega}A, \widehat{\Omega}B)$. We can define

$$HC_n(A, B) \;=\; H_i(\mathrm{Hom}^k(\widehat{\Omega}A, \widehat{\Omega}B)) \text{ where } i \in \{0, 1\},\ i \equiv n \bmod 2$$

The bivariant theory $HC_n(A, B)$ has a product $HC_n(A, B) \times HC_m(B, C) \longrightarrow HC_{n+m}(A, C)$ and satisfies

$$HC_n(\mathbb{C}, B) = HC_n(B) \quad \text{and} \quad HC_n(A, \mathbb{C}) = HC^n(A)$$

In general, there exist elements in $HP_*(A, B)$ which are not in the range of the natural map $HC_{2n+*}(A, B) \to HP_*(A, B)$ for any n, [].

9 Mixed Complexes.

In this section we present still another construction of cyclic theory which is sometimes convenient for explicit computations of cyclic homology in terms of Hochschild homology.
Following [26], a *mixed complex* is a $\mathbb{Z}$-graded complex $\bigoplus_n M_n$, bounded below, whose differential (usually denoted b) has degree -1, which is equipped with an operator B of degree $+1$ satisfying $[b, B] = B^2 = 0$. The homology of M as a complex, i.e., with respect to b, is by definition the *Hochschild homology* of M:

$$HH_nM = H_nM$$

Associated to M is the $\mathbb{Z}/2$-graded complex given by

$$\widehat{M} = \prod_n M_n$$

equipped with the usual even-odd grading and the differential $B - b$. *The periodic cyclic homology* is then the homology of this supercomplex:

$$HP_\nu M = H_\nu \widehat{M}, \qquad \nu \in \mathbb{Z}/2$$

The *cyclic homology* of M is defined as

$$HC_n M = H_n(D)$$

where D is the complex given by

$$D_n = \bigoplus_{p \geq 0} M_{n-2p}$$

with boundary operator $B' - b$, where $B' : M_n \to M_{n-1}$ is, as in 7.2, the truncated B-operator.
We have seen that, for any algebra A, there is a natural mixed complex, given by $(\Omega(A), b, B)$ and the Hochschild, cyclic and periodic cyclic homology of A is by definition the Hochschild, cyclic and periodic cyclic homology of this mixed complex.
The following simple observation is useful for computations of cyclic homology invariants.

Proposition 9.1 *Let $\varphi : (M, b, B) \to (M', b, B)$ be a morphism of mixed complexes (i.e., φ maps M_n to M'_n for all n and φ commutes with the b- and B-operators).*
If φ is a quasi-isomorphism from (M, b) to (M', b), then φ induces isomorphisms on the cyclic and periodic cyclic homology of M and M'.

Proof. This follows from the analogue of 6.4 and 8.2 for mixed complexes. □

For instance, the cyclic homology of smooth (in the sense of Grothendieck) algebras, of the locally convex algebra of smooth functions on a compact differentiable manifold or of the irrational rotation algebra (noncommutative torus) can be computed that way. In these cases, the Hochschild homology $HH(A)$ can be computed and there is an operator $\dot{B}$ of degree 1 on $HH(A)$ as well as a surjective map of mixed complexes

$$(\Omega(A), b, B) \;\to\; (HH(A), 0, \dot{B})$$

see [4], [31]. In the case of a smooth algebra or of the smooth functions on a compact manifold, $HH(A)$ is simply the algebra of ordinary classical differential forms and the operator $\dot{B}$ is the the classical de Rham operator d.

10 The X-complex description of cyclic homology.

The simple X-complex provides a useful alternative description of cyclic theory. It has been introduced first by Quillen [40] in connection with cyclic homology for coalgebras and then been used systematically in [13], [14], [15]. This different approach to cyclic theory is particularly useful for the study of periodic theories and cyclic theories for topological algebras, for proving the excision theorem, and for understanding the connection with topological K-theory.

Definition 10.1 *Let A be an algebra. The X-complex $X(A)$ is the $\mathbb{Z}/2$-graded complex*

$$A \underset{b}{\overset{\natural d}{\rightleftarrows}} \Omega^1 A_\natural$$

where $\Omega^1 A_\natural = \Omega^1 A / b(\Omega^2 A) = \Omega^1 A / [A, \Omega^1 A]$ and $\natural : \Omega^1 A \to \Omega^1 A_\natural$ is the canonical quotient map.

Recall that, from the definition of b, its image exactly consists of all commutators, thus $b(\Omega^2 A) = [A, \Omega^1 A]$.
The X-complex is constructed from the lowest level of the (B, b)-bicomplex

$$\begin{array}{ccccccc}
 & \downarrow b & & \downarrow b & & \downarrow b \\
\longleftarrow & \Omega^2 A & \overset{B}{\longleftarrow} & \boxed{\Omega^1 A} & \overset{B=d}{\longleftarrow} & \boxed{\Omega^0 A} \\
 & \downarrow b & & \downarrow b & & \\
\longleftarrow & \boxed{\Omega^1 A} & \overset{B=d}{\longleftarrow} & \boxed{\Omega^0 A} & & \\
 & \downarrow b & & & & \\
\longleftarrow & \boxed{\Omega^0 A} & & & &
\end{array}$$

To be precise, the X-complex is the total complex of the quotient of the bicomplex

$$\begin{array}{ccccccccc}
 & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
\longleftarrow & \Omega^3 A & \overset{B}{\longleftarrow} & \Omega^2 A & \overset{B}{\longleftarrow} & \Omega^1 A & \overset{B}{\longleftarrow} & \Omega^0 A \\
 & \downarrow b & & \downarrow b & & \downarrow b & & \\
\longleftarrow & \Omega^2 A & \overset{B}{\longleftarrow} & \Omega^1 A & \overset{B}{\longleftarrow} & \Omega^0 A & & \\
 & \downarrow b & & \downarrow b & & & & \\
\longleftarrow & \Omega^1 A & \overset{B}{\longleftarrow} & \Omega^0 A & & & & \\
 & \downarrow b & & & & & & \\
\longleftarrow & \Omega^0 A & & & & & &
\end{array}$$

by the sub-bicomplex

$$
\begin{array}{ccccccc}
 & & \downarrow b & & \downarrow b & & \downarrow b \\
\longleftarrow & & \Omega^3 A & \xleftarrow{B} & \Omega^2 A & \xleftarrow{B} & b(\Omega^2 A) \\
 & & \downarrow b & & \downarrow b & & \\
\longleftarrow & & \Omega^2 A & \xleftarrow{B} & b(\Omega^2 A) & & \\
 & & \downarrow b & & & & \\
\longleftarrow & & (\Omega^2 A) & & & &
\end{array}
\tag{12}
$$

where $\Omega^0 A$ is replaced by 0 and $\Omega^1 A$ is replaced by $b(\Omega^2 A)$. It is periodic of period two and specifically designed to compute the periodic cyclic theory for a special class of "smooth" algebras - the quasi-free algebras, because for quasi-free algebras the subcomplex by which we divide is contractible.

Proposition-Definition 10.2 *Let A be an algebra. The following conditions are equivalent:*

(1) Let $0 \to N \to S \xrightarrow{q} B \to 0$ be an extension of algebras where the ideal N is nilpotent (i.e., $N^k = \{0\}$ for some $k \geq 1$) and $A \xrightarrow{\alpha} B$ a homomorphism. Then there exists a homomorphism $A \xrightarrow{\alpha'} S$ such that $q \circ \alpha' = \alpha$.

(2) A has cohomological dimension ≤ 1 with respect to Hochschild cohomology.

(3) $\Omega^1 A$ is a projective bimodule over A.

The algebra A is called quasi-free if these equivalent conditions are satisfied.

The list of equivalent conditions characterizing quasi-freeness can be extended. In particular, (2) is equivalent to

(2)' There exists a linear map $\varphi : A \to \Omega^2 A$ such that

$$\varphi(xy) = x\varphi(y) + \varphi(x)y - dxdy \tag{13}$$

(i.e., the universal Hochschild 2-cocycle $c : A \times A \to \Omega^2 A$ defined by $c(x, y) = dxdy$ is the boundary of φ: $c = b\varphi$).

and (3) is equivalent to

(3)' There exists a linear map $\nabla : \Omega^1 A \to \Omega^2 A$ (a "right connection") satisfying $\nabla(a\omega) = a\nabla\omega$

$$\nabla(\omega a) = (\nabla\omega)a + \omega da \qquad (a \in A, \omega \in \Omega^2 A)$$

The equivalence between (2)' and (3)' is based on the correspondence between φ and ∇ given by the equality $\nabla(xdy) = x\varphi(y)$. For explicit computations the property (13) of quasi-free algebras often is particularly useful.

Proposition 10.3 *Let A be quasi-free and $HX_*(A), * = 0, 1$ be the homology of the complex $X(A)$. Then $HX_*(A) = HP_*A$.*

Proof. The proof is based on a contraction of the infinite product $(b\Omega^2 A, \Omega^2 A, \Omega^3 A, \ldots)$ inside $\hat{\Omega} A$ obtained from the connection ∇. Or, equivalently on showing that the the total complex of the sub-bicomplex (12) is contractible. $\square$

The preceding characterizations of quasi-free algebras carry over directly to the category of m-algebras. In condition (1) of 10.2 N^k has to be interpreted as the image of the n-fold projective tensor power of N under the multiplication map and the nilpotent extension has to be assumed to admit a continuous linear splitting. The most important examples of quasi-free algebras are free algebras. In particular, for any vector space V, the (non-unital) tensor algebra

$$TV \ = \ V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \ldots$$

is quasi-free. It obviously satisfies condition (1) in 10.2. In fact, it admits liftings in any (not necessarily nilpotent) extension. The map φ of (13) is determined by the condition that $\varphi(x) = 0$ for all generators $x \in V$. If $\mathfrak{A}$ is an m-algebra, then the tensor algebra $T\mathfrak{A}$ in the category of m-algebras is quasi-free. Other examples of quasifree algebras include $\mathbb{C}$, $M_n(\mathbb{C})$ or $\mathbb{C}[z, z^{-1}]$. One obtains from this

Corollary 10.4

$$\begin{aligned} HP_0(\mathbb{C}) &= \mathbb{C} \qquad & HP_1(\mathbb{C}) &= 0 \\ HP_0(\mathbb{C}[z, z^{-1}]) &= \mathbb{C} & HP_1(\mathbb{C}[z, z^{-1}]) &= \mathbb{C} \end{aligned}$$

Definition 10.5 *Let T be an algebra and J an ideal in T. The J-adic filtration on $X(T)$ is defined by*

$$\begin{aligned} F^{2n+1}X(T) &= J^{n+1} \oplus \natural(J^{n+1}dT + J^n dJ) \\ F^{2n}X(T) &= (J^{n+1} + [J^n, T]) \oplus \natural(J^n dT) \end{aligned}$$

Remark 10.6 *This defines a decreasing filtration which is closely related to the inverse system $X(T/J^n)$. In fact. let F^n be the J-adic filtration of $X(T)$ as above and G^n the kernel of the quotient map $X(T) \to X(T/J^n)$. Then*

$$G^{n+1} \subset F^{2n+1} \subset G^{2n} \subset G^n \subset \ldots$$

Therefore $\widehat{X}(T) = X(\widehat{T})$ where

$$\widehat{T} = \varprojlim_n T/J^n \qquad \text{and} \qquad \widehat{X}(T) = \varprojlim_n X(T)/F^n$$

The connection between the X-complex for free-algebras and the cyclic bicomplex is based on the following results.

Proposition 10.7 *Let A be an algebra.*

(1) The following defines an associative product (Fedosov product) on ΩA:

$$\omega_1 \circ \omega_2 \;=\; \omega_1\omega_2 + (-1)^{deg(\omega_1)} d\omega_1 d\omega_2$$

where $\omega_1\omega_2$ is the ordinary product of ω_1, ω_2 in ΩA. Clearly, the even forms $\Omega^{ev}A$ form a subalgebra of $(\Omega A, \circ)$ for $\circ$.

(2) Let $p : A \to TA$ be the canonical linear inclusion of A into the tensor algebra over A and, for $x, y \in A$, $\omega(x,y) = p(xy) - pxpy$. Then the map

$$\alpha : x_0 dx_1 dx_2 \ldots dx_{2n-1} dx_{x_{2n}} \;\mapsto\; x_0\omega(x_1,x_2)\ldots\omega(x_{2n-1},x_{2n})$$

defines an <u>isomorphism</u> of algebras $(\Omega^{ev}A, \circ) \xrightarrow{\alpha} TA$.

(3) The map $\omega dx \mapsto \natural(\alpha(\omega)d(px))$ (where α and p are as in (2)) defines a linear isomorphism $\Omega^{odd}A \xrightarrow{\alpha'} \Omega^1 TA_\natural$.

(4) The isomorphism $\alpha \oplus \alpha' : \Omega A \to X(TA)$ described in (2) and (3) identifies the natural filtration $F^n\Omega A \;=\; b\Omega^{n+1}A \,\oplus\, \Omega^{n+1}A \,\oplus\, \Omega^{n+2}A \,\oplus\, \ldots$ on $\Omega(A)$ with the JA-adic filtration

$$\begin{aligned} F^{2n+1}X(TA) &= (JA)^{n+1} \oplus \natural((JA)^{n+1}d(TA) + (JA)^n d(JA)) \\ F^{2n}X(TA) &= ((JA)^{n+1} + [(JA)^n, TA]) \oplus \natural((JA)^n d(TA)) \end{aligned}$$

on $X(TA)$, see 10.5, where JA is the ideal in TA given as the kernel of the natural homomorphism $TA \to A$.

Remark 10.8 *The full algebra $(\Omega A, \circ)$ with Fedosov product is isomorphic to the algebra $QA = A * A$, 4.3. The isomorphism sends $x + dx$ to $\iota(x)$ and $x - dx$ to $\bar\iota(x)$. Thus, in particular, the algebra TA is isomorphic to the even part, for the natural $\mathbb{Z}/2$-grading of QA.*

As a consequence of the preceding proposition, the X-complex $X(TA) \;:\; TA \underset{b}{\overset{\natural d}{\rightleftarrows}} \Omega^1(TA)_\natural$ is isomorphic to a complex of the form $\Omega^{ev}A \underset{\beta}{\overset{\delta}{\rightleftarrows}} \Omega^{odd}A$ where the maps δ and β have to be determined.

This is the content of the following key theorem.

Theorem 10.9 *Let δ, β be as above and $\Omega A \overset{\kappa}{\to} \Omega A$ the Karoubi operator. Then*

(a)

$$\begin{aligned}\beta(\omega) &= b\omega - (1+\kappa)d\omega, & \omega \in \Omega^{odd}A\\ \delta(\omega) &= \sum_{j=0}^{2n+1}\kappa^j d\omega - \sum_{j=0}^{n-1}\kappa^{2j} b\omega, & \omega\in\Omega^{2n}A\end{aligned}$$

(b) Let P be the spectral projection to 1 for κ (see 7.1). P commutes with b, B and β,δ and

$$\begin{aligned}P\beta &= P(b-\tfrac{2}{N}B)\\ P\delta &= P(B-b\tfrac{N}{2})\end{aligned}$$

(b) follows immediately from (a) by using the equality $P\kappa = P$ and $B = NPd$.

The complex $(1-P)\widehat{\Omega}A$ is contractible, both with the boundary operator $B-b$ and with β and δ. The same holds for the complexes in the associated filtration. For a precise statement we refer to [14], §6.

We still denote by P the projection operator corresponding to P under the isomorphism $\Omega A \cong X(TA)$. Let c be the scaling operator on ΩA which is multiplication by c_q on Ω^q where $c_{2n} = c_{2n+1} = (-1)^n n!$. It follows from 10.9 that $c: PX(TA) \to P\Omega A$ is an isomorphism identifying the differential $\natural d$ and $\dot{b}$ on $PX(TA)$ with the differential $B-b$ on $P\Omega A$. It obviously is compatible with the filtration F^n on both sides described in 10.7(4).

This proves the following theorem

Theorem 10.10 *Let A be an algebra*

(a) For each n we have

$$HC_n(A) = H_i\Big(X(TA)\,/\,F^nX(TA)\Big)$$

where $H_i, i=0,1$ is the homology of the $\mathbb{Z}/2$-graded complex ("supercomplex") $X(TA)/F^nX(TA)$ and $i\equiv n \bmod 2$.

(b) Let $\widehat{\Omega}A = \varprojlim_n \Omega A/F^n\Omega A$ and $\widehat{X}(TA) = \varprojlim_n X(TA)/F^nX(TA)$ be the completions. Then

$$HP_iA = H_i(\widehat{\Omega}A) \cong H_i(\widehat{X}(TA))$$

where $i=0,1$ and the differentials on $\widehat{\Omega}A$ and $\widehat{X}(TA)$ are given by $B-b$ and by $(\natural d, \dot{b})$, respectively.

Corollary 10.11 *For each n we have*

$$HC_n(A) = H_{i+1}(F^nX(TA)) \qquad HC^n(A) = H_{i+1}((F^nX(TA))')$$

where $H_i, i = 0, 1$ is the homology of the $\mathbb{Z}/2$-graded complex $F^n X(TA)$ and its dual $(F^n X(TA))'$ while $i \equiv n \bmod 2$. In particular, any trace τ on $(JA)^n$ defines an element in $HC^{2n-1}A$. This element is represented by the cyclic cocycle f_{2n-1} given by

$$\begin{aligned} f_{2n-1}(x_0, x_1, \ldots x_{2n-1}) = \;& \tau(\omega(x_0, x_1)\omega(x_2, x_3) \ldots \omega(x_{2n-2}, x_{2n-1})) \\ & -\tau(\omega(x_{2n-1}, x_0)\omega(x_1, x_2) \ldots \omega(x_{2n-3}, x_{2n-2})) \end{aligned}$$

where $\omega(x, y) = p(xy) - p(x)p(y)$ for the canonical linear inclusion $p : A \to TA$.

Proof. The first assertion follows from 10.10 and the exact sequence of complexes $0 \to F^n X(TA) \to X(TA) \to X(TA)/(F^n X(TA)) \to 0$ together with the fact that the homology of $X(TA)$ is trivial.
The second assertion is an easy consequence of the fact that $(JA)^n$ is exactly the even part of $F^{2n-1}X(TA)$. □

The preceding corollary is a translation of 6.6. There is a $\mathbb{Z}/2$-graded version using supertraces. A supertrace on a $\mathbb{Z}/2$-graded algebra with grading automorphism $x \to \bar{x}$ is by definition a linear functional f such that $f(xy) = (-1)^{\deg x \deg y} f(yx)$. f is called odd if $f(\bar{x}) = -f(x)$.

Proposition 10.12 *Let qA be the canonical ideal in the free product $QA = A * A$ defined in 4.3. Any odd supertrace τ on $(qA)^{2n+1}$ defines an element of HC^{2n} represented by the cyclic cocycle*

$$f(x_0, x_1, \ldots, x_{2n}) = \tau(q(x_0)q(x_1) \ldots q(x_{2n}))$$

Theorem 10.13 *Let $0 \to J \to T \to A \to 0$ be any extension of A where T is quasi-free. Then the complex $X(T)$ is homotopy equivalent to $X(TA)$ by a chain map respecting the filtrations induced by J and JA as in 10.5. Therefore $HC_n(A)$ and $HP_{0/1}A$ can be computed from $X(T)$. Explicitly*

$$HC_n(A) = H_i\Big(X(T) \,/\, F^n X(T)\Big) \quad \text{for } i \equiv n \bmod 2$$

$$HP_i A = H_i(\widehat{X}(T)) \quad \text{for } i = 0, 1$$

Proof. This follows from the fact that the X-complex is homotopy invariant for quasi-free algebras (see 12.1) and the fact that T and TA are homotopy equivalent by a homomorphism that respects the filtrations of T and TA by the powers of the ideals J and JA. □

Remark 10.14 *(a) The operator $S : HC_{n+2} \to HC_n$ is induced by the natural map $\Omega/F^{n+2}\Omega \to \Omega/F^n\Omega$ or by the map $X/F^{n+2}X \to X/F^nX$.*
(b) Also the Hochschild homology $HH_n(A)$ can be read off from the filtered complex $X(T)$, if T is any quasi-free extension of A. In fact, $HH_n(A)$ is the homology of the $\mathbb{Z}/2$-graded complex $F^{n-1}XT/F^nXT$ in dimension $i \in \{0, 1\}$ with $i \equiv n \bmod 2$.

Corollary 10.15 *(Goodwillie's theorem). Let I be an ideal in the algebra A that is nilpotent, i.e., $I^k = 0$ for some k. Then the quotient map $A \to A/I$ induces an invertible element in $HP_*(A, A/I)$. In particular $HP_*(A) \cong HP_*(A/I)$.*

Proof. We can choose TA as a quasi-free extension for both A and A/I. Since I is nilpotent, the resulting filtrations on $X(TA)$ are continuous with respect to each other. □

All constructions in this section also work in the category of m-algebras. If $\mathfrak{I}$ is an m-algebra then, by definition, $\mathfrak{I}^2$ is the image of the multiplication map $\mathfrak{I}\hat{\otimes}\mathfrak{I} \to \mathfrak{I}$. Note the following important property of ideals in quasi-free m-algebras:

If $\mathfrak{I}$ is the ideal in an extension

$$0 \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{I} \longrightarrow 0$$

of m-algebras with a continuous splitting and $\mathfrak{A}$ is quasi-free, then also $\mathfrak{I}^n$ is an ideal which is a direct summand in $\mathfrak{A}$ as a topological vector space for each n. For instance $(J\mathfrak{A})^n$ is a direct summand in $T\mathfrak{A}$ for each n.

Remark 10.16 *The fact that cyclic and periodic cyclic homology can be reduced to the homology of the X-complex applied to a quasi-free algebra, means that the properties of the theory can be checked by computations that only involve the cyclic homology of a quasi-free algebra* in dimensions 0 and 1*! This fact is essential for establishing some of the most important properties of the theory.*

11 Cyclic homology as non-commutative de Rham theory

For smooth commutative algebras the cyclic homology/cohomology can be determined by computing first the Hochschild homology and then using Connes' SBI-sequence. This applies in essentially the same way to the coordinate ring of a smooth variety or to the Fréchet algebra of smooth functions on a compact $\mathcal{C}^\infty$-manifold.

Definition 11.1 *A commutative algebra A is smooth if any homomorphism $\alpha : A \to B/N$ where B is a commutative algebra and N an ideal in B with $N^2 = 0$, can be lifted to a homomorphism $\widehat{\alpha} : A \to B$ such that $\pi \circ \widehat{\alpha} = \alpha$ for the quotient map $\pi : B \to B/N$.*

There are various other characterizations of smoothness. Note that, in particular, the coordinate ring of a smooth algebraic variety is smooth.
For a unital commutative algebra we define $\Omega^*_{comm}A$ as the largest graded commutative quotient of ΩA, i.e., as ΩA divided by the ideal generated by all differences $xdy - d(y)x$ and $dxdy + dydx$—with the additional relation $d(1) = 0$. It also can

be defined as the exterior algebra over A on the A-bimodule $\Omega^1 A$, with additional relation $d(1) = 0$.
The Hochschild homology of a smooth algebra is determined by the following theorem.

Theorem 11.2 *(Hochschild-Kostant-Rosenberg) If A is a smooth unital commutative algebra then the natural map $\Omega^*_{comm} A \to HH_*(A)$ is an isomorphism (of graded algebras).*

The de Rham cohomology $H^{dR}A$ of A is by definition the homology of the complex

$$\dots \quad \to \quad \Omega^k_{comm} A \quad \xrightarrow{d} \quad \Omega^{k+1}_{comm} A \quad \to \quad \dots$$

For $* = 0, 1$ we write $H^{dR}_0 A$ for the direct sum of all de Rham homology groups in even dimensions and $H^{dR}_1 A$ for the sum of all de Rham homology groups in odd dimensions.
We have

Theorem 11.3 *Let A be a unital smooth algebra of homological dimension n. Then*

(a) $HP_* A \cong H^{dR}_* A$

(a) If $k > n$, then $HC_k A \cong H^{dR}_* A$ for $* \equiv k$ mod 2.

Proof. The quotient map $\Omega^n A \to \Omega^n_{comm} A$ induces a natural map from the (B, b)-bicomplex for A to the bicomplex

$$\begin{array}{ccccc}
\downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
\Omega^2_{comm} A & \xleftarrow{d} & \Omega^1_{comm} A & \xleftarrow{d} & \Omega^0_{comm} A \\
\downarrow 0 & & \downarrow 0 & & \\
\Omega^1_{comm} A & \xleftarrow{d} & \Omega^0_{comm} A & & \\
\downarrow 0 & & & & \\
\Omega^0_{comm} A & & & &
\end{array}$$

By the Hochschild-Kostant-Rosenberg theorem this map is a quasi-isomorphism on the columns. Thus the theorem follows for instance from the spectral sequence comparison theorem (or also from Connes's SBI-sequence). □

Remark 11.4 *If A is the coordinate ring for a not necessarily smooth variety V then $HP_* A$ is isomorphic to the infinitesimal cohomology of V in the sense of Grothendieck which is in this case the correct replacement for de Rham theory.*

In the case of the Fréchet algebra $\mathcal{C}^\infty M$ of course we use the cyclic homology for locally convex algebras.

Theorem 11.5 *(Connes) Let $\mathcal{C}^\infty M$ be the algebra of smooth functions on a smooth compact manifold M and $\mathcal{C}^\infty(\Lambda T^*M)$ be the algebra of differential forms on M. Then the natural map $\mathcal{C}^\infty(\Lambda^k T^*M) \to HH_k(\mathcal{C}^\infty M)$ is an isomorphism for each k.*

Theorem 11.6 *Let M be a smooth compact manifold of dimension n. Then*

(a) $HP_*(\mathcal{C}^\infty M) \cong H_*^{dR}(\mathcal{C}^\infty M)$

(a) If $k > n$, then $HC_k(\mathcal{C}^\infty M) \cong H_*^{dR}(\mathcal{C}^\infty M)$ for $* \equiv k \bmod 2$.

All results in this section are of course also valid in cohomology in place of homology.

Remark 11.7 *Karoubi has defined de Rham homology for an arbitrary algebra A as the homology of the complex given by the A-bimodules $\Omega^n(A)/[\Omega^*(A), \Omega^*(A)]$, where the quotients are by graded commutators, and has shown that it is isomorphic to the image of $HC_{n+2}(A)$ in $HC_n(A)$ under the operator S, see [24].*

There is an even more significant and important analogy between cyclic homology and Grothendieck's notion of infinitesimal homology which describes the topology of non-smooth varieties in algebraic geometry. In this analogy, quasi-free algebras play the role of smooth varieties in the non-commutative category and the X-complex corresponds to the de Rham complex.
In fact, in algebraic geometry, the infinitesimal cohomology is defined by writing the coordinate ring A of the variety as a quotient of the coordinate ring of a smooth variety T and by taking the de Rham cohomology of the completion $\widehat{T}$. It can be shown that this does not depend on the choice of the embedding into a smooth variety.
The periodic cyclic homology of an algebra A is obtained by writing A as a quotient of a quasi-free algebra T and taking the X-complex-homology of the completion $\widehat{T}$. This does not depend on the choice of the quasi-free extension T, see 10.10.

12 Homotopy invariance for cyclic theory

The important basic features of cyclic homology are homotopy invariance (under differentiable homotopies), Morita invariance, and excision. These properties hold very generally for the periodic theory HP—but only in a very limited sense for the non-periodic theory HC.
In this section we discuss homotopy invariance. The first (and main) step consists in showing that derivations act trivially on HP (or on HC after stabilizing by the S-operator). This can be done by guessing the correct homotopy formula on the complex $(\Omega^n(A), B - b)$ computing the cyclic homology $HC_*(A)$, or, in the X-complex-picture, by using the fact that the X-complex is naturally homotopy invariant for quasi-free algebras.
Given an algebra B, we denote by $B[0,1]$ the algebra $B \otimes \mathcal{C}^\infty[0,1]$ where $\mathcal{C}^\infty[0,1]$

stands for the algebra of $\mathbb{C}$-valued $\mathcal{C}^\infty$-functions on $[0,1]$ (i.e., restrictions of functions in $\mathcal{C}^\infty(\mathbb{R})$ to $[0,1]$). **When we work in the category of m-algebras, we use the completed projective tensor product. In this case, $B[0,1]$ can be identified with the algebra of B-valued $\mathcal{C}^\infty$-functions on $[0,1]$.**
By a differentiable homotopy from an algebra A to an algebra B we mean a homomorphism α from A to the algebra $B[0,1]$. Given such a family α, evaluation for each $t \in [0,1]$ defines a homomorphism $\alpha_t : A \to B$. We say that two homomorphisms $\beta_0, \beta_1 : A \to B$ are differentiably homotopic if there exists a differentiable homotopy α such that $\alpha_0 = \beta_0$, $\alpha_1 = \beta_1$.
Let $u : A \to B[0,1]$ be a differentiable family of homomorphisms between algebras A and B. For each fixed $t \in [0,1]$ let $u_t : A \to B$ be the morphism given by u at the point t and $\dot{u}_t$ the derivative, with respect to t, at the point t, which is a linear map $A \to B$ and behaves like a derivation with respect to u_t, i.e.,

$$\dot{u}_t(xy) = u_t(x)\dot{u}_t(y) + \dot{u}_t(x)u_t(y)\,,$$

$x, y \in A$.
The Lie derivative

$$L_t = L(u_t, \dot{u}_t) : X(A) \to X(B)$$

is defined by

$$L_t(x) = \dot{u}_t(x)\,,\; L_t(\natural(xdy)) = \natural(\dot{u}_t(x)d(u_t y) + u_t(x)d(\dot{u}_t y))\,,$$

$x, y \in A$. Then L_t is a morphism of supercomplexes, see [14], beginning of §7.

Proposition 12.1 *Let $u : A \to B[0,1]$ be as above and assume that A is quasi-free. Then, for each $t \in [0,1]$, there is a map of supercomplexes of odd degree $h_t : X(A) \to X(B)$ such that $L_t = \partial h_t + h_t \partial$ where ∂ is the boundary operator in the X-complex.*

Proof. Let φ be the fundamental map $A \to \Omega^2 A$ for the quasi-free algebra A, see (13). Put

$$\begin{aligned} h_t(x) &= \natural i\varphi(x) \\ h_t(\natural(xdy)) &= i(xdy + b(x\varphi(y)) \end{aligned}$$

$x, y \in A$, where

$$i(x_0 dx_1 \ldots dx_k) = u_t(x_0)\dot{u}_t(x_1)du_t(x_2)\ldots du_t(x_k)\,.$$

One computes that $L = \partial h + h\partial$, see [14]. □

By integrating h_t with respect to t, see [14], 8.1, we obtain an operator $H = \int_0^1 u_t dt$ and

Proposition 12.2 *Let $u : A \to B[0,1]$ be as above and A quasi-free. There is an odd map of supercomplexes $H : X(A) \to X(B)$ such that $u_{1*} - u_{0*} = \partial H + H\partial$ for the maps $u_{t*} : X(A) \to X(B)$ induced by u_t, $t = 0, 1$.*

Assume now that A and B are written as quotients of free algebras T and R, respectively:

$$A \cong T/I \qquad B \cong R/J$$

and that $u : A \to B[0,1]$ is a homotopy as before. Since T is free, u induces a homomorphism $w : T \to R[0,1]$ which makes the following diagram commutative:

$$\begin{array}{ccc} T & \xrightarrow{w} & R[0,1] \\ \downarrow & & \downarrow \\ A & \xrightarrow{u} & B[0,1] \end{array}$$

Necessarily, w maps I to $J[0,1]$ and thus each w_t maps I to J. Therefore the map induced by w_t on the X-complexes maps the subspace F^n in the filtration of $X(T)$ induced by I, see 10.5, to the corresponding subspace F^n in the filtration of $X(R)$ induced by J. The formulas used in the definition of $L(u_t, \dot{u}_t)$ and H show that these maps send F^n to F^{n-1}. As a consequence, we get

Theorem 12.3 *Let $u : A \to B[0,1]$ be a differentiable homotopy and $u_t : A \to B$ the corresponding family of homomorphisms.*

(a) The element $HP_0(u_t) \in HP_0(A, B)$ defined by u_t does not depend on t. Thus, in particular, the maps induced by u_t on HP_* and on HP^* are constant.

(b) Let $(u_t)_*$ be the map induced by u_t on HC_n and $S : HC_{n-2} \to HC_n$ the S-operator. Then $(u_t)_* \circ S : HC_{n-2}(A) \to HC_n(B)$ is constant.

13 Morita invariance for periodic cyclic theory

Theorem 13.1 *Let E and F be linear subspaces of the algebra A and $A(EF)$, $A(FE)$ the subalgebras of A generated by the spaces of products EF and FE. Then there is an invertible element in $HP_0(A(EF), A(FE))$*

This theorem can be proved using the following quasi-free extensions of $A(EF)$ and $A(FE)$:

$$0 \to K(EF) \to T(E \otimes F) \xrightarrow{\pi_1} A(EF) \to 0$$

$$0 \to K(FE) \to T(F \otimes E) \xrightarrow{\pi_2} A(FE) \to 0$$

Here, π_1 and π_2 are the homomorphisms induced by the linear maps $E \otimes F \ni x \otimes y \mapsto xy \in A(EF)$, $F \otimes E \ni y \otimes x \mapsto yx \in A(FE)$ and $K(EF), K(FE)$ are the kernels of π_1, π_2.

The invertible element in $HP_0(A(EF), A(FE))$ is induced by the linear isomorphism

$$\lambda : X(T(E \otimes F)) \to X(T(F \otimes E))$$

which is simply cyclic permutation on tensor products

$$\begin{aligned}
&\lambda(x_1 \otimes y_1 \otimes x_2 \otimes y_2 \otimes \ldots \otimes x_k \otimes y_k) \\
&\qquad = \quad y_k \otimes x_1 \otimes y_1 \otimes x_2 \otimes \ldots \otimes x_k \\
&\lambda(\natural(x_1 \otimes y_1 \otimes \ldots \otimes x_{k-1} \otimes y_{k-1} d(x_k \otimes y_k))) \\
&\qquad = \quad \natural(y_k \otimes x_1 \otimes y_1 \otimes \ldots \otimes x_{k-1} d(y_{k-1} \otimes x_k)) \\
&\qquad\qquad x_1, \ldots, x_k \in E;\ y_1, \ldots, y_k \in F
\end{aligned}$$

It is trivially checked that λ commutes with the boundary operators $\natural d$ and $\dot{b}$ of the X-complex, using the definition of these operators.
One can also show that $\lambda : X(T(E \otimes F)) \to X(T(F \otimes E))$ is "continuous" with respect to the natural filtration, see 10.5, i.e., that for each n there is k, such that λ maps F^k to F^n, see [12].

A Morita context

$$\begin{pmatrix} A & X \\ Y & B \end{pmatrix}$$

for two algebras A and B is by definition an algebra with a splitting into four linear subspaces such that when the elements are written as 2×2 matrices the multiplication is consistent with matrix multiplication.

Corollary 13.2 *Let A and B be algebras which are related by a Morita context as above and assume that $A = XY$, $B = YX$. Then there is an invertible element in $HP_0(A, B)$.*

The results of this section and of the following one are also valid for m-agebras replacing $A(EF)$, $A(FE)$, XY and YX by the closed subalgebras generated by the respective products.

14 Morita invariance for the non-periodic theory

In the non-periodic theory, Morita invariance only holds for unital or at least H-unital algebras, [47].

Theorem 14.1 *Let E and F be linear subspaces of the algebra A and $A(EF)$, $A(FE)$ the subalgebras of A generated by the spaces of products EF and FE. Assume that $A(EF), A(FE)$ are both H-unital (see 6.3). Then there is an invertible element in $HC^0(A(EF), A(FE))$. This element induces isomorphisms*

$$HC^n(A(EF)) \cong HC^n(A(FE))$$

for all n.

Corollary 14.2 *Let A and B be H-unital algebras. Let E be a $A-B$-bimodule and F be a $B-A$-bimodule such that*

$$E \otimes_B F \cong A \qquad F \otimes_A E \cong B$$

Then there is an invertible element in $HC^0(A,B)$ inducing isomorphisms

$$HC^n(A) \cong HC^n(B)$$

for all n.

15 Excision for periodic cyclic theory

Theorem 15.1 *Let $0 \to S \to P \to Q \to 0$ be an extension of algebras and A an algebra. There are two natural six-term exact sequences*

$$\begin{array}{ccccc} HP_0(A,S) & \to & HP_0(A,P) & \to & HP_0(A,Q) \\ \uparrow & & & & \downarrow \\ HP_1(A,Q) & \leftarrow & HP_1(A,P) & \leftarrow & HP_1(A,S) \end{array}$$

and

$$\begin{array}{ccccc} HP_0(S,A) & \leftarrow & HP_0(P,A) & \leftarrow & HP_0(Q,A) \\ \downarrow & & & & \uparrow \\ HP_1(Q,A) & \to & HP_1(P,A) & \to & HP_1(S,A) \end{array}$$

where the horizontal arrows are induced by the maps in the given extension (a description of the vertical arrows will be given below).

In the category of m-algebras, this theorem is valid for extensions admitting a continuous linear splitting, see [10]. We include a brief discussion of the vertical arrows in 15.1 (see [15] and also [27], III 2.2). Given an extension of algebras $0 \to S \to P \to Q \to 0$ let δ, δ' be the connecting maps in the following two six-term exact sequences.

$$\longrightarrow HP_*(P,S) \longrightarrow HP_*(S,S) \xrightarrow{\delta} HP_{*+1}(Q,S) \longrightarrow$$

$$\longrightarrow HP_*(Q,P) \longrightarrow HP_*(Q,Q) \xrightarrow{\delta'} HP_{*+1}(Q,S) \longrightarrow$$

Let 1_S and 1_Q denote the class of id in $HP_0(S,S)$ and $HP_0(Q,Q)$ respectively.

Proposition 15.2 *One has $\delta(1_S) = -\delta'(1_Q)$.*

Theorem 15.3 *Let $E:\ 0 \to S \to P \to Q \to 0$ be an extension of algebras. We write $\mathrm{ch}(E)$ for the element $\delta'(1_Q) = -\delta(1_S) \in HP_1(Q,S)$. The connecting map in the first exact sequence in Theorem 5.3 acts on $HP_j(A,Q)$ by multiplication by $(-1)^j chE$ on the right, while the connecting map in the second exact sequence in Theorem 5.3 is given by multiplication by $-\mathrm{ch}(E)$ on the left.*

16 Excision for the non-periodic theory

Let I be a (non-unital) algebra and $(\Omega^n(I), b)$ and $(D_n^\Omega(I), B' - b)$ be the standard complexes computing Hochschild and cyclic homology of I, respectively, see 7.2. If $0 \to I \to A \to B \to 0$ with I as ideal, we let $(\Omega^n(I : A), b)$ and $(D_n^\Omega(I : A)$ denote the kernels of the maps $\Omega^n(A) \to \Omega^n(B)$ and $D_n^\Omega(A) \to D_n^\Omega(B)$, respectively, so that we have exact sequences of complexes:

$$0 \to \Omega^n(I : A) \to \Omega^n(A) \to \Omega^n(B) \to 0$$

$$0 \to D_n^\Omega(I : A) \to D_n^\Omega(A) \to D_n^\Omega(B) \to 0$$

We say that I is *excisive* in Hochschild homology (respectively in cyclic homology), if the inclusion $\Omega^n(I) \to \Omega^n(I : A)$ (respectively $D_n^\Omega(I) \to D_n^\Omega(I : A)$) is a homotopy equivalence of complexes. If this is the case then obviously the exact sequences of complexes above induce long exact sequences in Hochschild (respectively cyclic) homology.

Theorem 16.1 *([46], [47]) Let I be a (non-unital) algebra. The following conditions are equivalent:*

(a) I is H-unital, (see 6.3).

(b) I is excisive in Hochschild homology, i.e., any extension of the form $0 \to I \to A \to B \to 0$ induces a long exact sequence in Hochschild homology

$$\begin{aligned} \ldots HH_n(I) \to HH_n(A) \to HH_n(B) \to HH_{n-1}(I) \\ \to HH_{n-1}(A) \to HH_{n-1}(B) \to \ldots \end{aligned}$$

(c) I is excisive in cyclic homology, i.e., any extension of the form $0 \to I \to A \to B \to 0$ induces a long exact sequence in cyclic homology

$$\begin{aligned} \ldots HC_n(I) \to HC_n(A) \to HC_n(B) \to HC_{n-1}(I) \\ \to HC_{n-1}(A) \to HC_{n-1}(B) \to \ldots \end{aligned}$$

17 Cyclic homology for Schatten ideals

The cyclic homology of the m-algebra $\ell^p(H)$, see 3.5 can—somewhat unexpectedly—be computed using the excision property of periodic cyclic homology.

Proposition 17.1 *Let $\mathfrak{I}$ and $\mathfrak{A}$ be m-algebras. We assume that there are continuous homomorphisms $\alpha : \mathfrak{I} \to \mathfrak{A}$ and $\mu : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{I}$ with the following properties:*

(a) $\alpha \circ \mu$ is the multiplication on $\mathfrak{A}$.

(b) $\mu \circ (\alpha \otimes \alpha)$ *is the multiplication on* $\mathfrak{I}$.

(thus, in particular $\alpha(\mathfrak{I})$ *is an ideal in* $\mathfrak{A}$ *with* $\mathfrak{A}^2 \subset \alpha(\mathfrak{I})$*). Then* α *induces an invertible element* $HP_0(\alpha)$ *in* $HP_0(\mathfrak{I}, \mathfrak{A})$.

Proof. This follows by applying the excision theorem in HP for m-algebras to the following extension.

$$0 \to \mathfrak{I}(0,1) \to \mathfrak{I}(0,1) + \mathfrak{A}t \to \mathfrak{A} \to 0 \tag{14}$$

The m-algebra $\mathfrak{I}(0,1)+\mathfrak{A}t$ is defined in the following way. As a locally convex vector space it simply is the direct sum of $\mathfrak{I}(0,1)$ and $\mathfrak{A}$. The symbol t denotes the identity function on $[0,1]$. The elements of $\mathfrak{A}t$ are regarded as functions on $[0,1]$ with values in $\mathfrak{A}$, which are multiples of t by elements of $\mathfrak{A}$. The multiplication on the first summand is the one of $\mathfrak{I}(0,1)$. The product of a function f in $\mathfrak{I}(0,1)$ by an element $xt \in \mathfrak{A}t$ is $\mu(\alpha(f) \otimes xt)$ (we extend μ and α canonically to functions). The product of xt and yt in the second factor is defined as $\mu(x \otimes y)(t^2 - t) + \alpha\mu(x \otimes y)t$, where the first summand is in $\mathfrak{I}(0,1)$ and the second one in $\mathfrak{A}t$. The extension constructed that way obviously has a continuous linear splitting and defines an element u in $HP_1(\mathfrak{A}, \mathfrak{I}(0,1)) \cong HP_0(\mathfrak{A}, \mathfrak{I})$. This element is inverse to $HP_0(\alpha)$. □

We now apply this result to the Schatten ideals $\ell^p = \ell^p(H)$. More generally, consider the case where $\mathfrak{I} = \ell^p \hat{\otimes} \mathfrak{B}$ and $\mathfrak{A} = \ell^q \hat{\otimes} \mathfrak{B}$ for an arbitrary m-Algebra $\mathfrak{B}$ and $p \leq q \leq 2p$. The maps α and μ are obtained from the continuous inclusion $\ell^p \to \ell^q$ and from the multiplication map $\ell^q \hat{\otimes} \ell^q \to \ell^p$.

Corollary 17.2 *For any* m*-algebra* $\mathfrak{B}$, $\ell^p \hat{\otimes} \mathfrak{B}$ *is equivalent in* HP_* *to* $\ell^q \hat{\otimes} \mathfrak{B}$*, i.e., the map* $\ell^p \hat{\otimes} \mathfrak{B} \to \ell^q \hat{\otimes} \mathfrak{B}$ *induced by the inclusion* $\ell^p \to \ell^q$ *gives an invertible element in* $HP_0(\ell^p \hat{\otimes} \mathfrak{B}, \ell^q \hat{\otimes} \mathfrak{B})$ *for* $q \geq p$.

Proposition 17.3 *The inclusion map* $\mathbb{C} \to \ell^1(H)$ *(which maps* $1 \in \mathbb{C}$ *to a one-domensional projector in* ℓ^1*) and the canonical trace* $Tr : \ell^1(H) \to \mathbb{C}$ *define elements in* $HP_0(\mathbb{C}, \ell^1(H))$ *and* $HP_0(\ell^1(H), \mathbb{C})$*, respectively, which are inverse to each other. Thus in particular* $HP_*(\ell^1(H)) \cong HP_*(\mathbb{C})$.

Corollary 17.4 *The inclusion map* $\mathbb{C} \to \ell^p(H)$ *defines an invertible element in* $HP_0(\mathbb{C}, \ell^p(H))$ *for each* $p \geq 1$*. Thus in particular* $HP_*(\ell^p(H)) \cong HP_*(\mathbb{C})$.

Remark 17.5 *Assume (for simplicity) that* p *is an odd integer. The generator of* $HP^0(\ell^p(H)) = \varinjlim HC^{2n}(\ell^p(H)$ *is represented by the cyclic cocycle* f *in* C_λ^{p-1} *given by*

$$f(x_0, x_1, \ldots, x_{p-1}) = \mathrm{Tr}\ (x_0 x_1 \ldots x_{p-1})$$

It follows from the SBI*-sequence that the class of* f *is not in the image of the* S*-operator and thus that* $p-1$ *is the smallest integer* k *for which the map* $HC^k(\ell^p(H)) \to HP^0(\ell^p(H))$ *is non-trivial.*

18 The Chern character for K-theory classes given by idempotents and invertibles

With homotopy classes of idempotents and invertibles in a matrix algebra over A one can associate periodic cyclic homology classes as follows.
Let A be unital, e an idempotent in a matrix algebra $M_n(A)$ and $[e]$ its homotopy class. e may be thought of as a homomorphism $\mathbb{C} \to M_n(A)$. It induces a map $HP_0\mathbb{C} \to HP_0(M_n(A)) \cong HP_0 A$. We denote by $ch([e])$ the image of the generator of $HP_*\mathbb{C} \cong \mathbb{C}$ (see 10.4) under this map.
Similarly, an invertible element u in $M_n(A)$ may be thought of as a map from the algebra $\mathbb{C}[z, z^{-1}]$ to $M_n(A)$ mapping z to u. This induces a map $HP_1\mathbb{C}[z, z^{-1}] \to HP_1(M_n(A)) \cong HP_1 A$. We denote by $ch([u])$ the image in $HP_1 A$ of the generator of $HP_1\mathbb{C}[z, z^{-1}] \cong \mathbb{C}$ (see 10.4) under this map.

Remark 18.1 *It is very easily seen that the classes $ch([e])$ and $ch([u])$ are additive for the natural K-theory addition. Therefore one may view these classes as defining homomorphisms $ch : K_*^{alg} A \to HP_* A$, $* = 0, 1$ from the algebraic K-theory to HP_* (which in fact can be extended to the higher-dimensional algebraic K-theory in a rather straightforward way). This homomorphism even passes to the quotient of algebraic K-theory by homotopy. Since there is no good definition of topological K-theory for arbitrary m-algebras based only on idempotents and invertibles in matrix algebras, it is not clear, a priori, how to interpret this as a map from topological K-theory though.*

To obtain explicit formulas for the cycles $ch([e])$ and $ch([u])$ in the periodic cyclic complex it is very convenient to use the description of $HP_*(A)$ as the homology of the completed X-complex $\widehat{X}(TA)$ and its homotopy equivalence with the (B, b)-bicomplex $\widehat{\Omega} A$.
The idempotent e admits a canonical lifting to an idempotent $\hat{e}$ in each nilpotent extension, and therefore in the projective limit $M_n(\widehat{T}A)$ of nilpotent extensions of $M_n(A)$. This is easily seen putting $f = 2e - 1$ in the unitization and defining a lift $\hat{f}$ for f by the formula $\hat{f} = s(f)(s(f)^2)^{-1/2}$, given by a power series in the nilpotent variable $1 - s(f)^2$) (s denotes the canonical linear lift $M_n(A) \to M_n(TA)$). Under the isomorphism $M_n(\widehat{T}A) \cong M_n(\widehat{\Omega}^{ev} A)$ with Fedosov product, 10.7, $\hat{e}$ is given by the following formula

$$\hat{e} = e + \sum_{n \geq 1} \frac{(2n)!}{(n!)^2} \left(e - \frac{1}{2} \right) (de)^{2n}$$

where de is to be considered as the matrix (de_{ij}) if $e = (e_{ij})$. Now, $Tr(\hat{e})$ is an even cycle in the complex $\widehat{X}(TA) = X(\widehat{T}A)$, representing $ch([e])$ (since the generator of $HP_0(\mathbb{C})$ is represented by 1 in $X(\mathbb{C})$). Under the identification of $\widehat{X}(TA)$ with $\widehat{\Omega} A$, this becomes

$$ch([e]) = Tr(e) + \sum_{n\geq 1}(-1)^n \frac{(2n)!}{n!} Tr\left(\left(e-\frac{1}{2}\right)(de)^{2n}\right)$$

as an element of the complex $\widehat{\Omega} A$ with boundary operator $B-b$.

Let further u be an invertible element in the unitization $M_n(A)^\sim$ and consider the reduced X-complex $\overline{X}(M_n(A)^\sim)$ where we introduce the relation $d(1)=0$ for the adjoined unit 1. By a similar argument as before, the element $\natural(u^{-1}du)$ can be lifted to a cycle $\natural(\hat{u}^{-1}d\hat{u})$ in $\overline{X}(M_n(\widehat{T}A)^\sim)$. The trace on M_n induces a map Tr : $\overline{X}(M_n(\widehat{T}A)^\sim) \to \overline{X}(\widehat{T}A^\sim)$ and $TR(\natural(\hat{u}^{-1}d\hat{u}))$ will represent $ch([u])$ (the generator of $HP_1(\mathbb{C}[z,z^{-1}])$ is represented by $\natural(z^{-1}dz)$ in $X(\mathbb{C}[z,z^{-1}])$). Then $ch([u])$ corresponds to

$$ch([u]) = \sum_{n\geq 0} n!\, Tr((u^{-1}-1)d(u-1)\,(d(u^{-1}-1)d(u-1))^n)$$

19 Cyclic cocycles associated with Fredholm modules

The concept of a Fredholm module over an algebra A has been introduced by Atiyah and Kasparov as a framework for K-homology. The bivariant version (over two algebras A and B) is due to Kasparov and is usually called a Kasparov module. It describes an extension of B by an algebra Morita equivalent to A which admits an inverse (i.e, there exists another extension such that its sum with the given extension is equivalent to a trivial extension). The notion of an unbounded Fredholm module or a spectral triple plays a fundamental role in Alain Connes's "dictionary" for noncommutative geometry. We treat in this section only the simplest but typical case of a "bounded" Fredholm module.
Given a Hilbert space H, we denote by $\mathcal{L}(H)$ the algebra of bounded operators on H and by $\ell^p(H)$ the Schatten ideal of p-summable operators in $\mathcal{L}(H)$.

Definition 19.1 *Let A be an algebra*

(a) A p-summable odd Fredholm module over A is a representation $\pi : A \to \mathcal{L}(H)$ together with an operator $F \in \mathcal{L}(H)$ such that $F = F^*$, $F^2 = 1$ and such that the commutators $[F, \pi(x)]$, $x \in A$ are all in $\ell^p(H)$.

(b) A p-summable even Fredholm module over A is a representation $\pi : A \to \mathcal{L}(H)$ together with an operator $F \in \mathcal{L}(H)$ such that $F = F^*$, $F^2 = 1$ and $[F, \pi(x)] \in \ell^p(H) \in$, $x \in A$ and a grading operator γ on H, $\gamma = \gamma^*$, $\gamma^2 = 1$, such that $\gamma F \gamma = -F$ and $\gamma\pi(x)\gamma = \pi(x)$, $x \in A$ (i.e., F has degree 1, while $\pi(A)$ has degree 0 for the $\mathbb{Z}/2$-grading induced by γ).

An odd p-summable Fredholm module determines an extension of the following form

$$0 \longrightarrow \ell^{p/2}(H) \longrightarrow (\ell^{p/2}(H) + \sigma(A)) \longrightarrow A_0 \longrightarrow 0 \tag{15}$$

where $\sigma(x) = P\pi(x)P$, $P = 1/2(F+1)$ the spectral projection for 1 and $A_0 = A/\mathrm{Ker}(\sigma)$. Note that we can work with $p/2$ rather than p, since $\sigma(xy) - \sigma(x)\sigma(y) = (1/2)P[F,x][F,y]P \in \ell^{p/2}(H)$.
We know from 17.5 that $HP^0(\ell^{2n+1}(H))$ is one-dimensional with generator the image $\dot{g}$ in HP^0 of the cyclic cocycle $g(x_0, \ldots, x_{2n}) = \mathrm{Tr}(x_0 x_1 \ldots x_{2n})$. By excision for HP^*, the image of $\dot{g}$ under the boundary map in the long exact sequence associated to the extension (15) gives an element $ch(A,F)$ in $HP^1(A)$ which is called the character of the Fredholm module (A,F). A more refined version of this argument is the following. By 4.2, the extension (15) determines a classifying map $\gamma : JA \to \ell^p(H)$ which maps the p-th power $(JA)^p$ of JA to the p-th power of $\ell^p(H)$, i.e., to $\ell^1(H)$. Composition of γ with the trace on $\ell^1(H)$ gives a trace τ on $(JA)^p$. Now, by 10.11, any trace on $(JA)^p$ determines for each n with $2n+2 \geq p$ cyclic cocycles $f_{2n+1} \in C_\lambda^{2n+1}(A)$ which are related by the S-operator. According to 10.11 the explicit formula for f_{2n+1} is

$$\begin{aligned} f_{2n+1}(x_0, x_1, \ldots x_{2n+1}) = \quad & \mathrm{Tr}(\omega(x_0,x_1)\omega(x_2,x_3)\ldots\omega(x_{2n},x_{2n+1})) \\ & -\mathrm{Tr}(\omega(x_{2n+1},x_0)\omega(x_1,x_2)\ldots\omega(x_{2n-1},x_{2n})) \end{aligned}$$

where $\omega(x,y) = \sigma(xy) - \sigma(x)\sigma(y) = P\pi(xy)P - P\pi x P \pi y P$. A simple computation shows that this formula can be rewritten in terms of commutators $[F, \cdot]$ as follows

$$f_{2n+1}(x_0, x_1, \ldots x_{2n+1}) = \mathrm{Tr}\left(F([F,x_0][F,x_1]\ldots[F,x_{2n+1}])\right)$$

It follows from the proof of excision in HP^* that $ch(A,F)$ is the image of $\frac{(-1)^n}{n!} f_{2n+1}$ under the map $HC^{2n+1} \to HP^1$. There is a simple index theorem based on this cocycle. Recall that a Fredholm operator on a Hilbert space is a bounded operator which is invertible modulo compact operators and that the index of a Fredholm operator L is defined as $\mathrm{ind}\, L = \dim \mathrm{Ker}\, L - \dim \mathrm{Coker}\, L$.

Theorem 19.2 *(Connes) Let A be unital (and unitally represented), $u \in A$ invertible and (A,F) a p-summable Fredholm module over A. Then the $u_P = P\pi(u)P$ is a Fredholm operator on the Hilbert space PH. Its index is given by*

$$\begin{aligned} \mathrm{ind}\, u_P &= \langle ch(A,F) | ch(u) \rangle \\ &= f_{2n+1}(u^{-1}, u, u^{-1}, \ldots, u) \end{aligned}$$

With obvious modifications, the theorem is still valid for invertibles in a matrix algebra over A.

An even p-summable Fredholm module determines an extension of the form

$$0 \longrightarrow \ell^p(H) \longrightarrow (\ell^p(H) + \pi(A)) \longrightarrow A_0 \longrightarrow 0$$

with two different splittings π, $\bar{\pi}$: $A \to (\ell^p(H) + \pi(A))$ where $\bar{\pi}(x) = F\pi(x)F$. Using excision for HP we see that $HP^0(\ell^p(H) + \pi(A)) \cong HP^0(\ell^p(H)) \oplus HP^0(A)$. Denote by $ch(A, F)$ the image of 1 under the map

$$\mathbb{C} = HP^0(\ell^p(H)) \quad \overset{\pi^* - \bar{\pi}^*}{\longrightarrow} \quad HP^0(A)$$

Again there is an explicit formula for a cyclic cocycle representing $ch(A, F)$. In fact, the universality property of the extension

$$0 \longrightarrow qA \longrightarrow QA \longrightarrow A \longrightarrow 0$$

shows that there is a natural homomorphism $qA \to \ell^p(H)$. Composing again with the supertrace on $\ell^1(H)$ (recall that H is graded) we obtain an odd supertrace on $(qA)^p$ and thus by 10.12 cyclic cocycles determined for each n with $2n + 1 \geq p$ by the formula

$$f_{2n}(x_0, x_1, \ldots x_{2n}) = \mathrm{Tr}(\gamma q(x_0) q(x_1) \ldots q(x_{2n}))$$

where $q(x) = \pi(x) - \bar{\pi}(x)$ and γ is the grading automorphism on H. Again, one may rewrite this as

$$f_{2n}(x_0, x_1, \ldots x_{2n}) = \mathrm{Tr}\left(\gamma F([F, x_0][F, x_1] \ldots [F, x_{2n}])\right)$$

The class $ch(A, F)$ is represented by $\frac{(-1)^n n!}{(2n)!} f_{2n}$

Theorem 19.3 *(Connes) Let A be unital, $e \in A$ idempotent and (A, F) a p-summable Fredholm module over A. Then the $F_e = \pi(e)F\pi(e)$ is a Fredholm operator from the Hilbert space $\pi(e)H_+$ to the Hilbert space $\pi(e)H_-$, where $H = H_+ \oplus H_-$ is the eigenspace-decomposition of H with respect to the grading operator γ. Its index is given by*

$$\begin{aligned} ind\, F_e &= \langle ch(A, F) | ch(e) \rangle \\ &= f_{2n}(e, e, \ldots, e) \end{aligned}$$

20 Bivariant K-theory for locally convex algebras

A natural category on which topological K-theory can be defined with good properties, is the one of Banach algebras. It has been studied with particular intensity on the category of C*-algebras. On the category of C*-algebras Kasparov, [25], extended K-theory to a bivariant theory KK in such a way that, for a C*-algebra A, one has $K_*(A) = KK_*(\mathbb{C}, A)$, $* = 0, 1$. A closely related similar bivariant theory on the category of C*-algebras is the E-theory of Connes-Higson, [8]. It is an alternative extension of K-theory to a bivariant theory. In particular it also satisfies $K_*(A) = E_*(\mathbb{C}, A)$, $* = 0, 1$.
Now it is a well-known fact that cyclic homology is degenerate on Banach or C*-algebras. Thus, for instance, if we define cyclic and periodic cyclic homology for a

locally convex algebra as in Section 8, we obtain for the algebra $\mathcal{C}(X)$ of continuous functions on the compact space X

$$HP^0(\mathcal{C}(X)) = \{ \text{bounded complex measures on} X \}$$
$$HP^1(\mathcal{C}(X)) = 0$$

It is possible to define cyclic homology theories (even in the bivariant setting) with good properties on the category of C*-algebras (or Banach algebras) and to define a bivariant transformation from $KK(A,B)$ or from $E(A,B)$ into this theory, [37], [38], [39].
On the other hand, it is in fact also possible to construct a bivariant topological K-theory on the category of locally convex algebras and then to define a bivariant Chern-Connes character from this theory into the ordinary bivariant cyclic theory $HP_*(A,B)$, as we discussed it in Section 8. We will describe the construction in this section. As before, to simplify constructions, we work within the category of m-algebras (projective limits of Banach algebras), rather than with general locally convex algebras. Since cyclic theory is homotopy invariant only for differentiable homotopies (called diffeotopies below), we have to set up the theory in such a way that it uses only diffeotopies in place of general homotopies.

For an m-algebra $\mathfrak{A}$ we write $\mathfrak{A}[a,b], \mathfrak{A}[a,b)$ and $\mathfrak{A}(a,b)$ for the m-algebras $\mathfrak{A}\hat{\otimes}\mathbb{C}[a,b]$, $\mathfrak{A}\hat{\otimes}\mathbb{C}[a,b)$ and $\mathfrak{A}\hat{\otimes}\mathbb{C}(a,b)$, see 3.1 .

Definition 20.1 *(see 3.1) Two continuous linear maps $\alpha, \beta : V \to W$ between two complete locally convex spaces are called differentiably homotopic or* **diffeotopic**, *if there is a family $\varphi_t : V \to W, t \in [0,1]$ of continuous linear maps, such that $\varphi_0 = \alpha, \varphi_1 = \beta$ and such that the map $t \mapsto \varphi_t(x)$ is infinitely differentiable for each $x \in V$.*

Another formulation of the condition in this definition is that there exists a continuous linear map $\varphi : V \to \mathcal{C}^\infty([0,1])\hat{\otimes}W$ with the property that $\varphi(x)(0) = \alpha(x), \varphi(x)(1) = \beta(x)$ for each $x \in V$.
Let $h : [0,1] \to [0,1]$ be a monotone and bijective $\mathcal{C}^\infty$-map, whose restriction to $(0,1)$ is a diffeomorphism from $(0,1) \to (0,1)$ and whose derivatives in 0 and 1 all vanish. Replacing φ_t by $\psi_t = \varphi_{h(t)}$ one sees that α and β are diffeotopic if and only if there is a continuous linear map $\psi : V \to \mathbb{C}[0,1]\hat{\otimes}W$, for which $\psi(x)(0) = \alpha(x), \psi(x)(1) = \beta(x)$, $x \in V$. This shows that diffeotopy is an equivalence relation. Given a homomorphism α we denote by $\langle\alpha\rangle$ its diffeotopy class and given two m-algebras $\mathfrak{A}$ and $\mathfrak{B}$, we denote by $\langle\mathfrak{A}, \mathfrak{B}\rangle$ the set of diffeotopy classes of homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

Definition 20.2 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be m-algebras. For any homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ of m-algebras, we denote by $\langle\varphi\rangle$ the equivalence class of φ with respect to diffeotopy and we set*

$$\langle\mathfrak{A}, \mathfrak{B}\rangle = \{\langle\varphi\rangle \,|\, \varphi \text{ is a continuous homomorphism } \mathfrak{A} \to \mathfrak{B} \}$$

Let $\mathfrak{K}$ be the algebra of smooth compact operators defined in 3.4. For two continuous homomorphisms $\alpha, \beta : \mathfrak{A} \to \mathfrak{K}\hat{\otimes}\mathfrak{B}$ we define the direct sum $\alpha \oplus \beta$ as

$$\alpha \oplus \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \mathfrak{A} \longrightarrow M_2(\mathfrak{K}\hat{\otimes}\mathfrak{B}) \cong \mathfrak{K}\hat{\otimes}\mathfrak{B}$$

With the addition defined by $\langle\alpha\rangle + \langle\beta\rangle = \langle\alpha \oplus \beta\rangle$ the set $\langle\mathfrak{A}, \mathfrak{K}\hat{\otimes}\mathfrak{B}\rangle$ of diffeotopy classes of homomorphisms from $\mathfrak{A}$ to $\mathfrak{K}\hat{\otimes}\mathfrak{B}$ is an abelian semigroup with 0-element $\langle 0\rangle$.

Definition 20.3 *(see 4.2) Given an m-algebra $\mathfrak{A}$ we define the ideal $J\mathfrak{A}$ in the standard free extension as in (2). $J\mathfrak{A}$ is, for each m-algebra $\mathfrak{A}$, again an m-algebra. By iteration we can therefore form $J^2\mathfrak{A} = J(J\mathfrak{A}), \ldots, J^n\mathfrak{A} = J(J^{n-1}\mathfrak{A})$. For any extension of $\mathfrak{A}$ of length n there is a classifying map from $J^n\mathfrak{A}$ into the first ideal in the n-step extension.*

Consider the set $H_k = \langle J^k\mathfrak{A}, \mathfrak{K}\hat{\otimes}\mathfrak{B}\,\rangle$, where $H_0 = \langle\mathfrak{A}, \mathfrak{K}\hat{\otimes}\mathfrak{B}\,\rangle$. Each H_k is an abelian semigroup with 0-element for the K-theory addition defined in 20.2.
To define a map $S : H_k \to H_{k+2}$, we use the classifying map $\varepsilon : J^2\mathfrak{A} \to \mathfrak{K}\hat{\otimes}\mathfrak{A}$ for the two-step extension which is obtained by composing the Toeplitz with the suspension extension, see 4.4 and 4.1.

Definition 20.4 *Let $\varphi : J\mathfrak{A} \to \mathcal{C}^\infty(S^1)\hat{\otimes}\mathfrak{A}$ be the composition of the classifying map $J\mathfrak{A} \to \mathfrak{A}(0,1)$, belonging to the extension*

$$0 \longrightarrow \mathfrak{A}(0,1) \longrightarrow \mathfrak{A}[0,1) \longrightarrow \mathfrak{A} \longrightarrow 0$$

with the inclusion map $\mathfrak{A}(0,1) \longrightarrow \mathcal{C}^\infty(S^1)\hat{\otimes}\mathfrak{A}$. The map

$$\varepsilon : J^2\mathfrak{A} \longrightarrow \mathfrak{K}\hat{\otimes}\mathfrak{A}$$

is defined as the composition $\varepsilon = \gamma \circ \varphi$ with the classifying map γ for the extension

$$0 \longrightarrow \mathfrak{K}\hat{\otimes}\mathfrak{A} \longrightarrow \mathfrak{T}\hat{\otimes}\mathfrak{A} \overset{s}{\longrightarrow} \mathcal{C}^\infty(S^1)\hat{\otimes}\mathfrak{A} \longrightarrow 0$$

For $\langle\alpha\rangle \in H_k$, $\alpha : J^k\mathfrak{A} \longrightarrow \mathfrak{K}\hat{\otimes}\mathfrak{B}$, let $S\langle\alpha\rangle = \langle(\mathrm{id}_\mathfrak{K}\otimes\alpha)\circ\varepsilon\rangle$. Here $\varepsilon : J^{k+2}\mathfrak{A} \longrightarrow \mathfrak{K}\otimes J^k\mathfrak{A}$ is the ε-map for $J^k\mathfrak{A}$. Let $\varepsilon_- : J^{k+2}\mathfrak{A} \longrightarrow \mathfrak{K}\otimes J^k\mathfrak{A}$ be the map which is obtained by replacing, in the definition of ε, the Toeplitz extension by the inverse Toeplitz extension. The sum $\varepsilon \oplus \varepsilon_-$ is then diffeotopic to 0. Therefore $S\langle\alpha\rangle + S_-\langle\alpha\rangle = 0$, putting $S_-\langle\alpha\rangle = \langle(\mathrm{id}_\mathfrak{K} \otimes \alpha) \circ \varepsilon_-\rangle$.

Definition 20.5 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be m-algebras and $* = 0$ or 1. We define*

$$kk_*(\mathfrak{A},\, \mathfrak{B}\,) = \varinjlim_k\, H_{2k+*} = \varinjlim_k \langle J^{2k+*}\mathfrak{A},\, \mathfrak{K}\hat{\otimes}\mathfrak{B}\,\rangle$$

The preceding discussion shows that $kk_*(\mathfrak{A},\,\mathfrak{B})$ is not only an abelian semigroup, but even an abelian group (every element admits an inverse).

Theorem 20.6 *(a) There is an associative product*

$$kk_i(\mathfrak{A},\mathfrak{B})\times kk_j(\mathfrak{B},\mathfrak{C})\longrightarrow kk_{i+j}(\mathfrak{A},\mathfrak{C})$$

($i,j\in\mathbb{Z}/2$; $\mathfrak{A},\mathfrak{B}$ and $\mathfrak{C}$ m-algebras), which is additive in both variables.

(b) There is a bilinear, graded commutative, exterior product

$$kk_i(\mathfrak{A}_1,\mathfrak{A}_2)\times kk_j(\mathfrak{B}_1,\mathfrak{B}_2)\longrightarrow kk_{i+j}(\mathfrak{A}_1\hat{\otimes}\mathfrak{A}_2,\mathfrak{B}_1\hat{\otimes}\mathfrak{B}_2)$$

(c) Each homomorphism $\varphi:\mathfrak{A}\to\mathfrak{B}$ defines an element $kk(\varphi)$ in the group $kk_0(\mathfrak{A},\mathfrak{B})$. If $\psi:\mathfrak{B}\to\mathfrak{C}$, is another homomorphism, then

$$kk(\psi\circ\varphi)=kk(\varphi)\cdot kk(\psi)$$

$kk_*(\mathfrak{A},\mathfrak{B}\,)$ is a contravariant functor in $\mathfrak{A}$ and a covariant functor in $\mathfrak{B}$. If $\alpha:\mathfrak{A}'\to\mathfrak{A}$ and $\beta:\mathfrak{B}\to\mathfrak{B}'$ are homomorphisms, then the induced maps, in the first and second variable of kk_*, are given by left multiplication by $kk(\alpha)$ and right multiplication by $kk(\beta)$.

(d) $kk_*(\mathfrak{A},\mathfrak{A})$ is, for each m-algebra $\mathfrak{A}$, a $\mathbb{Z}/2$-graded ring with unit element $kk(\mathrm{id}_{\mathfrak{A}})$.

(e) The functor kk_* is invariant under diffeotopies in both variables.

(f) The canonical inclusion $\iota:\mathfrak{A}\to\mathfrak{K}\hat{\otimes}\mathfrak{A}$ defines an invertible element in $kk_0(\mathfrak{A},\mathfrak{K}\hat{\otimes}\mathfrak{A})$. In particular, $kk_*(\mathfrak{A},\mathfrak{B})\cong kk_*(\mathfrak{K}\hat{\otimes}\mathfrak{A},\mathfrak{B})$ and $kk_*(\mathfrak{B},\mathfrak{A})\cong kk_*(\mathfrak{B},\mathfrak{K}\hat{\otimes}\mathfrak{A})$ for each m-algebra $\mathfrak{B}$.

(g) (Bott periodicity) The suspension extension gives rise to classifying maps $J\mathfrak{A}\to\mathfrak{A}(0,1)$ and $J^2\mathfrak{A}\to\mathfrak{A}(0,1)^2$ (where $\mathfrak{A}(0,1)^2=\mathfrak{A}(0,1)(0,1)$). The elements in $kk_0(J\mathfrak{A},\mathfrak{A}(0,1))$, in $kk_1(\mathfrak{A},\mathfrak{A}(0,1))$ and in $kk_0(\mathfrak{A},\mathfrak{A}(0,1)^2)$ defined by these homomorphisms are invertible.

(h) Let $\mathfrak{D}$ be any m-algebra. Every extension admitting a continuous linear split

$$E:\;0\to\mathfrak{J}\xrightarrow{\;i\;}\mathfrak{A}\xrightarrow{\;q\;}\mathfrak{B}\to 0$$

induces exact sequences in $kk(\mathfrak{D},\cdot\,)$ and $kk(\,\cdot\,,\mathfrak{D})$ of the following form:

$$\begin{array}{ccccc} kk_0(\mathfrak{D},\mathfrak{J}) & \stackrel{\cdot kk(i)}{\longrightarrow} & kk_0(\mathfrak{D},\mathfrak{A}) & \stackrel{\cdot kk(q)}{\longrightarrow} & kk_0(\mathfrak{D},\mathfrak{B}) \\ \uparrow & & & & \downarrow \\ kk_1(\mathfrak{D},\mathfrak{B}) & \stackrel{\cdot kk(q)}{\longleftarrow} & kk_1(\mathfrak{D},\mathfrak{A}) & \stackrel{\cdot kk(i)}{\longleftarrow} & kk_1(\mathfrak{D},\mathfrak{J}) \end{array} \tag{16}$$

and

$$\begin{array}{ccccc} kk_0(\mathfrak{J},\mathfrak{D}) & \overset{kk(q)\cdot}{\longleftarrow} & kk_0(\mathfrak{A},\mathfrak{D}) & \overset{kk(i)\cdot}{\longleftarrow} & kk_0(\mathfrak{B},\mathfrak{D}) \\ \downarrow & & & & \uparrow \\ kk_1(\mathfrak{B},\mathfrak{D}) & \overset{kk(i)\cdot}{\longrightarrow} & kk_1(\mathfrak{A},\mathfrak{D}) & \overset{kk(q)\cdot}{\longrightarrow} & kk_1(\mathfrak{J},\mathfrak{D}) \end{array} \tag{17}$$

The given extension E defines a classifying map $J\mathfrak{B} \to \mathfrak{J}$ and thus an element of $kk_1(\mathfrak{J},\mathfrak{B})$, which we denote by $kk(E)$. The vertical arrows in (16) and (17) are (up to a sign) given by right and left multiplication, respectively, by this class $kk(E)$.

Finally, we have to compare kk with the ordinary topological K-theory. For a Fréchet m-algebra $\mathfrak{A}$ one can define the following abelian groups:

$$\begin{aligned} K_0(\mathfrak{A}) &= \Big\{ \langle e \rangle \,\Big|\, e \text{ is an idempotent element in} \\ &\qquad M_2(\mathfrak{K}\hat{\otimes}\mathfrak{A}^{\sim}) \text{ such that } e - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathfrak{K}\hat{\otimes}\mathfrak{A}) \Big\} \\ K_1(\mathfrak{A}) &= K_0(\mathfrak{A}(0,1)) \end{aligned} \tag{18}$$

Here, as usual $\mathfrak{K}\hat{\otimes}\mathfrak{A}^{\sim}$ denotes the algebra obtained by adjoining a unit to $\mathfrak{K}\hat{\otimes}\mathfrak{A}$.

Phillips shows in [36] that this defines a topological K-theory with the usual properties (homotopoy invariance, stability, Bott periodicity, six-term exact sequences) on the category of Fréchet m-algebras and in addition gives the usual well-known K-theory if $\mathfrak{A}$ is a Banach algebra. We have

Theorem 20.7 *For every Fréchet m-algebra $\mathfrak{A}$, the groups $kk_*(\mathbb{C},\mathfrak{A})$ and $K_*\mathfrak{A}$ are naturally isomorphic.*

Remark 20.8 *V.Lafforgue, [29] has introduced a bivariant K-theory KK^{Ban} which is defined on the category of Banach algebras. It does not have a product but it has very good properties for very general Morita equivalences of Banach algebras. Lafforgue uses his theory to obtain striking positive results on the Baum-Connes conjecture. For Banach algebras, it is possible to construct an analogue kk_*^{Ban} of kk_* on the category of Banach algebras using continuous homotopies rather than differentiable one (this is straightforward) and to show that there is a natural transformation $KK_*^{Ban} \to kk_*^{Ban}$.*

21 The bivariant Chern-Connes character

In this section we describe the construction of a bivariant multiplicative transformation from kk_* to the bivariant theory HP_* on the category of m-algebras.

Consider a covariant functor E from the category of m-algebras to the category of abelian groups which satisfies the following conditions:

(E1) E is diffeotopy invariant, i.e., the evaluation map ev_t in any point $t \in [0,1]$ induces an isomorphism $E(ev_t) : E(\mathfrak{A}[0,1]) \to E(\mathfrak{A})$

(E2) E is stable, i.e., the canonical inclusion $\iota : \mathfrak{A} \to \mathfrak{K}\hat{\otimes}\mathfrak{A}$ induces an isomorphism $E(\iota)$.

(E3) E is half-exact, i.e., each extension $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0$ admitting a continuous linear splitting induces a short exact sequence $E(\mathfrak{J}) \to E(\mathfrak{A}) \to E(\mathfrak{B})$

(The same conditions can of course be formulated analogously for a contravariant functor E.) We note that a standard construction from algebraic topology, using property (E1) and mapping cones, permits to extend the short exact sequence in (E3) to an infinite long exact sequence of the form

$$\begin{aligned} \cdots \to E(\mathfrak{B}(0,1)^2) \to E(\mathfrak{J}(0,1)) \to E(\mathfrak{A}(0,1)) \\ \to E(\mathfrak{B}(0,1)) \to E(\mathfrak{J}) \to E(\mathfrak{A}) \to E(\mathfrak{B}) \end{aligned} \tag{19}$$

see, e.g., [25] or [1].

Theorem 21.1 *Let E be a covariant functor with the properties (E1), (E2), (E3). Then we can associate in a unique way with each $h \in kk_0(\mathfrak{A}, \mathfrak{B})$ a morphism of abelian groups $E(h) : E(\mathfrak{A}) \to E(\mathfrak{B})$, such that $E(h_1 \cdot h_2) = E(h_2) \circ E(h_1)$ for the product $h_1 \cdot h_2$ of $h_1 \in kk_0(\mathfrak{A}, \mathfrak{B})$ and $h_2 \in kk_0(\mathfrak{B}, \mathfrak{C})$ and such that $E(kk(\alpha)) = E(\alpha)$ for each morphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ of m-algebras.*
An analogous statement holds for contravariant functors.

Proof. Let h be represented by $\eta : J^{2n}\mathfrak{A} \to \mathfrak{K}\hat{\otimes}\mathfrak{B}$. We set

$$E(h) = E(\iota)^{-1} E(\eta) E(\varepsilon^n)^{-1} E(\iota)$$

It is clear that $E(h)$ is well-defined and that $E(kk(\alpha)) = E(\alpha)$. □

The preceding result can be interpreted differently, see also [20], [1]. For this, note first that kk_0 can be regarded as a category, whose objects are the m-algebras, and where the morphism set between $\mathfrak{A}$ and $\mathfrak{B}$ is given by $kk_0(\mathfrak{A}, \mathfrak{B})$. This category is additive in the sense that the morphism set between two objects forms an abelian group and that the product of morphisms is bilinear.
We denote the natural functor from the category of m-algebras to the category kk_0, which is the identity on objects, also by kk_0.

Corollary 21.2 *Let F be a functor from the category of m-algebras to an additive category C, such that $F(\beta \circ \alpha) = F(\alpha) \cdot F(\beta)$, for any two homomorphisms $\alpha : A_1 \to A_2$ and $\beta : A_2 \to A_3$ between m-algebras.*
We assume that for each $\mathfrak{B}$, the contravariant functor $C(F(\,\cdot\,), F(\mathfrak{B}))$ and the covariant functor $C(F(\mathfrak{B}), F(\,\cdot\,))$ on the category of m-algebras satisfy the properties (E1), (E2), (E3). Then there is a unique covariant functor F' from the category kk_0 to C, such that $F = F' \circ kk_0$.

Remark 21.3 *Property (E3) implies that any such functor F' is automatically additive:*

$$F'(h+g) = F'(h) + F'(g)$$

As a consequence of the preceding corollary we get a bilinear multiplicative transformation from kk_0 to HP_0—the bivariant Chern-Connes character.

Corollary 21.4 *There is a unique (covariant) functor $ch : kk_0 \to HP_0$, such that $ch(kk_0(\alpha)) = HP_0(\alpha) \in HP_0(\mathfrak{A}, \mathfrak{B})$ for every morphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ of m-algebras.*

Proof. This follows from 21.2, since HP_0 satisfies properties (E1), (E2) and (E3). □

The Chern-Connes-character ch is compatible with the exterior product on kk_0 as well, see 20.6(b), and the corresponding product on HP_0, see [15], p.86.
It remains to extend ch to a multiplicative transformation from the $\mathbb{Z}/2$-graded theory kk_* to HP_* and to study the compatibilty with the boundary maps in the long exact sequences associated to an extension for kk and HP.
For the special case of the free extension

$$E_u : \qquad 0 \to J\mathfrak{A} \to T\mathfrak{A} \to \mathfrak{A} \to 0$$

we put, using the notation in 15.3,

$$x_{\mathfrak{A}} = ch(E_u) = \delta'(1_{\mathfrak{A}}) = -\delta(1_{J\mathfrak{A}}) \ \in HP_1(\mathfrak{A}, J\mathfrak{A})$$

The boundary maps δ and δ' in the long exact sequences associated to E_u are given by left and right multiplication by $x_{\mathfrak{A}}$, see 15.3.
Let now u be an element in $kk_1(\mathfrak{A}, \mathfrak{B})$. By definition we have $kk_1(\mathfrak{A}, \mathfrak{B}) = kk_0(J\mathfrak{A}, \mathfrak{B})$. Let u_0 be the element in $kk_0(J\mathfrak{A}, \mathfrak{B})$ that corresponds to u under this isomorphism.
We set

$$ch(u) = \sqrt{2\pi i}\, x_{\mathfrak{A}} \cdot ch(u_0) \qquad \in \ HP_1(\mathfrak{A}, \mathfrak{B})$$

Theorem 21.5 *The thus defined Chern-Connes character $ch : kk_* \to HP_*$ is multiplicative, i.e., for $u \in kk_i(\mathfrak{A}, \mathfrak{B})$ and $v \in kk_j(\mathfrak{B}, \mathfrak{C})$ we have*

$$ch(u \cdot v) = ch(u) \cdot ch(v)$$

For m-algebras, the bivariant character ch constructed here is a far reaching generalization of the Chern character from K-Theory and K-homology in Sections 18 and 19.
If e is an idempotent in $M_n\mathfrak{A}$,then e can be viewed as a homomorphism from $\mathbb{C}$ to $\mathfrak{K} \otimes \mathfrak{A}$, thus as an element h_e of $kk_0(\mathbb{C}, \mathfrak{A})$. It is not difficult to see that for the class $ch([e])$ constructed in section 18 one has

$$ch([e]) = ch(h_e)$$

Similarly any invertible element u in $M_n\mathfrak{A}^\sim$ defines an element h_u in $kk_1(\mathbb{C},\mathfrak{A})$ and we have $ch([u]) = ch(h_u)$
Similarly, any p-summable Fredholm module over $\mathfrak{A}$ gives an element of $kk_*(\mathfrak{A},\ell^p(H))$.

Corollary 17.2 shows that $\ell^p\hat{\otimes}\mathfrak{B}$ and $\ell^1\hat{\otimes}\mathfrak{B}$ are equivalent in HP_0 for each $p \geq 1$. On the other hand, it is easy to see that $\ell^1\hat{\otimes}\mathfrak{B}$ is in HP_0 equivalent to $\mathfrak{B}$, see for instance [17]. We thus obtain

Corollary 21.6 *The m-algebra $\ell^p\hat{\otimes}\mathfrak{B}$ is equivalent, in HP_0, to $\mathfrak{B}$ for each $p \geq 1$. The Chern-Connes-character gives a transformation $ch^{(p)}\colon kk_*(\mathfrak{A},\ell^p\hat{\otimes}\mathfrak{B}) \to HP_*(\mathfrak{A},\mathfrak{B})$ such that*

$$ch^{(p)}(x \cdot kk(\iota^{(p)})) = ch(x) \qquad \textit{für } x \in kk_*(\mathfrak{A},\mathfrak{B})$$

where $\iota^{(p)}$ denotes the canonical inclusion $\mathfrak{B} \to \ell^p\hat{\otimes}\mathfrak{B}$.

Comparing the functorial properties, one sees that this transformation generalizes the construction of cyclic cohomology classes associated to p-summable Fredholm modules in Section 19. It also generalizes previous constructions of a bivariant Chern-Connes charcter in special cases, [27], [45], [34], [35].

22 Entire cyclic cohomology

In the study of discrete subgroups of Lie groups and of quantum field theory, one is naturally lead to consider Fredholm modules that are not finitely summable but only θ-summable:

Definition 22.1 *A Fredholm module (F,π,H) is called θ-*summable *iff the (graded) commutators $[F,\pi(a)]$ are contained in the operator ideal*

$$\mathrm{Li}^{1/2}(H) := \{T \in \mathcal{L}(H) \mid \mu_n(T) = O\big((\ln n)^{-1/2}\big) \textit{ for } n \to \infty\}.$$

Here $\mu_n(T)$ denotes the nth singular value of T.

In periodic cyclic cohomology, one can only express the Chern character of a finitely summable Fredholm module. Entire cyclic cohomology was introduced to handle θ-summable Fredholm modules as well [5].

Definition 22.2 *Let A be a Banach algebra.*
Let K be the $\mathbb{Z}/2$-graded vector space of all linear maps $\phi\colon \Omega A \to \mathbb{C}$ that satisfy the following growth condition: For all bounded subsets $S \subseteq A$, there is a constant C_S such that

$$|\phi(a_0 da_1 \dots da_n)| \leq C_S, \qquad |\phi(da_1 \dots da_n)| \leq C_S \tag{20}$$

for all $a_0,\dots,a_n \in S$. We equip K with the boundary $\partial^*\colon K \to K$ induced from the boundary maps of $X(TA)$. That is, $\partial^*(\phi) := \phi \circ \beta$ for even ϕ and $\partial^*(\phi) := -\phi \circ \delta$ for odd ϕ. The entire cyclic cohomology *$HE^*(A)$ is defined to be the homology of the complex K.*

If we use $B \pm b$ instead of the X-complex boundary, we have to modify the growth condition above, replacing (20) by

$$[n/2]! \cdot |\phi(a_0 da_1 \dots da_n)| \le C_S, \quad [n/2]! \cdot |\phi(da_1 \dots da_n)| \le C_S, \tag{21}$$

where $[n/2] = k$ if $n = 2k$ or $n = 2k+1$. This yields a chain homotopic complex [32]. Definition 22.2 can be carried over literally to locally convex algebras with separately continuous multiplication. However, since we only use the bounded subsets of A, it is more appropriate to define entire cyclic cohomology for *complete (convex) bornological algebras*. These are algebras equipped with a collection $\mathcal{S}$ of subsets satisfying some reasonable axioms like

$$S, T \in \mathcal{S},\ \lambda \in \mathbb{C} \Longrightarrow S \cdot T,\ S + T,\ \lambda \cdot S \in \mathcal{S}$$

Convexity means that if $S \in \mathcal{S}$, so is the absolutely convex hull of S. If $S \subseteq A$ is absolutely convex, then the linear span A_S of S carries a unique semi-norm with unit ball S. We call S *completant* iff A_S is a Banach space. We call A *separated* iff A_S is a normed space for all S and we call A *complete* iff each $S \in \mathcal{S}$ is contained in a completant absolutely convex set $T \in \mathcal{S}$. A complete bornological vector space is automatically separated. In the following, we will drop the words convex and separated from our notation and assume all bornological vector spaces to be convex and separated. The collection $\mathcal{S}$ is called the *bornology* of A. We call $S \subseteq A$ *small* iff $S \in \mathcal{S}$. See [21] for an introduction to bornological vector spaces.
Here are some important examples of bornologies. If V is just a vector space, we can equip it with the finest possible bornology: We let $S \in \mathcal{S}$ iff S is a bounded subset of a finite dimensional vector subspace of V. If V is a (quasi)complete locally convex vector space, then the collection of bounded subsets of V is a complete bornology. Another reasonable complete bornology is the collection of all precompact or relatively compact subsets. (If V is (quasi)complete, then a subset is precompact iff it is relatively compact iff it is contained in a compact subset.) Actually, the precompact bornology tends to be better behaved than the bounded one. For local cyclic cohomology, it is essential to use the precompact bornology.
Let A be a bornological algebra. Then there is an obvious bornology on ΩA such that the space K of Definition 22.2 is equal to the space of bounded linear functionals on ΩA. Namely, the bornology generated by the sets $\bigcup_{n\ge 0} S(dS)^n \cup (dS)^{n+1}$. That is, a subset of ΩA is small if and only if it is contained in the absolutely convex hull of a set of the form $\bigcup_{n\ge 0} S(dS)^n \cup (dS)^{n+1}$. The operators b, d, B, κ and the multiplication of differential forms are bounded with respect to this bornology. Identifying TA and $X(TA)$ with $\Omega^{\mathrm{ev}} A$ and ΩA as in Proposition 10.7, we obtain canonical bornologies on TA and $X(TA)$. We always equip $\Omega A \cong X(TA)$ and TA with these bornologies.
If V is a bornological vector space, we write V^{c} for its *completion*. Moreover, we write $\mathcal{L}(X, Y)$ for the space of bounded linear maps between two bornological vector spaces X and Y.

Definition 22.3 *Let A and B be bornological algebras. We define* entire cyclic homology *and* bivariant entire cyclic homology *by*

$$HE_*(B) := H_*\big(X(TB)^c\big),$$
$$HE_*(A,B) := H_*\Big(\mathcal{L}\big(X(TA)^c, X(TB)^c\big)\Big).$$

It is necessary to complete $X(TB)$ because the complex $X(TB)$ itself is always acyclic. We have $HE_*(A,\mathbb{C}) \cong HE^*(A)$ and $HE_*(\mathbb{C},A) \cong HE_*(A)$ because the complex $X(T\mathbb{C})^c$ is chain homotopic to the complex $\mathbb{C}$ with $\mathbb{C}$ in degree 0 and 0 in degree 1.
The composition of chain maps gives rise to an associative *composition product* in bivariant entire cyclic homology. A bounded homomorphism $f\colon A \to B$ induces a bounded chain map $f_*\colon X(TA)^c \to X(TB)^c$. Thus we obtain an element $[f_*] \in HE_0(A,B)$. Composition with $[f_*]$ yields the functoriality of the entire cyclic theories.
Since $X(TA)^c$ is contained in the complex $\hat{X}(TA)$ defining the periodic cyclic theories, there are natural transformations $HP^*(A) \to HE^*(A)$ and $HE_*(A) \to HP_*(A)$. Entire cyclic (co)homology has the same stability, homotopy invariance, and exactness properties as periodic cyclic (co)homology [32].
To formulate homotopy invariance and stability, we need the completed bornological tensor product $\hat{\otimes}$. It is defined by a universal property analogous to the universal property of the projective tensor product: Bounded linear maps $V \hat{\otimes} W \to X$ correspond bijectively to bounded bilinear maps $V \times W \to X$. For Banach spaces equipped with the bounded bornology, $V \hat{\otimes} W$ is nothing but the projective tensor product, again equipped with the bounded bornology. If V and W are Fréchet spaces equipped with the precompact bornology, then $V \hat{\otimes} W$ will be isomorphic to the projective tensor product $V \hat{\otimes}_\pi W$ equipped with the precompact bornology [32]. A slightly weaker statement holds if V and W carry the bounded bornology.
For a complete bornological algebra, we define $B[0,1] := \mathbb{C}[0,1] \hat{\otimes} B$, where $\mathbb{C}[0,1]$ carries the bounded bornology (which is equal to the precompact bornology). This gives rise to a notion of smooth homotopy. Entire cyclic (co)homology is smoothly homotopy invariant, that is, Theorem 12.3.(a) holds with HE instead of HP. Similarly, entire cyclic homology is stable with respect to the trace class operators $\ell^1(H)$. That is, the canonical embedding $A \to A \hat{\otimes} \ell^1(H)$ is invertible in HE_0. These two assertions can be proved as for the periodic theory. The excision theorem also holds [39], [32]:

Theorem 22.4 *Bivariant entire cyclic homology satisfies excision in both variables for extensions with a bounded linear section.*

That is, if $0 \to S \to P \to Q \to 0$ is an extension of complete bornological algebras with a bounded linear section and if A is another complete bornological algebra, then there are two natural six-term exact sequences as in Theorem 15.1, only with

HE instead of HP. However, the proof of Theorem 22.4 is considerably more difficult than the proof of Theorem 15.1. We will explain the strategy of the proof below. The formulas for the Chern character in K-theory in Section 18 actually yield elements of $X(TA)^{\mathrm{c}}$. Hence these formulas define a Chern character

$$\mathrm{ch}\colon K_*(A) \to HE_*(A).$$

The Chern character $K_*(A) \to HP_*(A)$ is the composition of this character with the natural map $HE_*(A) \to HP_*(A)$.
There are two apparently different definitions for the Chern character of a θ-summable Fredholm module, due to Connes [5] and Jaffe-Lesniewski-Osterwalder [22], respectively. It is an important fact that both approaches yield homologous cocycles [6]. In [22], an entire cochain is written down explicitly and it is checked that it is a cocycle. However, it is not clear why this particular formula should be the right one. Connes's construction is more conceptual. Presumably, it can be explained using the excision theorem, as in Section 19. However, Connes's construction is technically quite involved, so that we cannot discuss it here.
Both constructions of the Chern character work with *unbounded* Fredholm modules in the following sense.

Definition 22.5 *Let A be a bornological algebra. Let H be a Hilbert space, let $\pi\colon A \to \mathcal{L}(H)$ be a bounded representation of A, and let D be an unbounded self-adjoint operator on H. We call the triple (D, π, H) an* unbounded odd Fredholm module *over A iff $(1 + D^2)^{-1}$ is compact, $D\pi(a)$ is densely defined for all $a \in A$, $[D, \pi(a)]$ extends to a bounded operator on H for all $a \in A$, and the map $A \ni a \mapsto [D, \pi(a)] \in \mathcal{L}(H)$ is bounded.*
If H is a graded Hilbert space, $\pi(a)$ has degree 0 for all $a \in A$, and D has degree 1, we call the triple (D, π, H) an unbounded even Fredholm module.
We call (D, π, H) θ-summable *iff $\exp(-D^2) \in \ell^1(H)$.*

In applications, it frequently happens that $\exp(-\beta D^2) \in \ell^1(H)$ for all $\beta > 0$. This stronger assumption is made in [5] and [22], although it is not necessary to obtain the Chern character.
Given an unbounded Fredholm module (D, π, H), we get an associated bounded Fredholm module as follows. If D is injective, we may put $F := \operatorname{sign} D$. Otherwise, we first modify (D, π, H) so that D becomes injective and then put $F := \operatorname{sign} D$. In the odd case, we may simply add to D the projection onto the kernel of D. In the even case, this does not work because the resulting operator will no longer have degree 1. In fact, there may be no injective self-adjoint degree 1 operator on $\ker D$. Therefore, we replace H by the direct sum $H \oplus \ker D$ and extend $\pi(a)$ and D to be 0 on the summand $\ker D$. We equip the summand $\ker D$ with the opposite grading. This insures that the operator

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on $\ker D \oplus \ker D$ has degree 1. Adding this operator to $D \oplus 0$, we get an unbounded Fredholm module $(D', \pi \oplus 0, H \oplus \ker D)$ with $\ker D' = \{0\}$.
If we start with a θ-summable unbounded Fredholm module in the sense of Definition 22.5, the resulting bounded Fredholm module will be θ-summable as well [41]. Conversely, if A is countably generated, then any bounded θ-summable Fredholm module arises in this way from a suitable unbounded θ-summable Fredholm module [7]. This construction only yields $\exp(-D^2) \in \ell^1(H)$, not $\exp(-\beta D^2) \in \ell^1(H)$ for all $\beta > 0$.
Let (D, π, H) be an unbounded, even, θ-summable Fredholm module with grading operator γ. The associated *JLO-cocycle* is defined as follows. Let

$$\Delta_k := \{(s_0, \dots, s_k) \in [0,1]^{k+1} \mid s_0 + \dots + s_k = 1\}$$

be the standard k-simplex and let $\int_{\Delta_k}$ denote integration over Δ_k with respect to the usual Lebesgue measure $ds_0 \dots ds_{k-1}$. The JLO-cocycle is defined by the formula

$$\tau(a_0 da_1 \dots da_{2n}) \\ := \int_{\Delta_{2n}} \operatorname{tr}\bigl(\gamma a_0 e^{-s_0 D^2}[D, a_1] e^{-s_1 D^2} \cdots [D, a_{2n}] e^{-s_{2n} D^2}\bigr) \quad (22)$$

for $n \geq 0$. An involved computation shows that $\tau \circ (B + b) = 0$ and that τ satisfies the growth condition (21). Hence τ defines an element of $HE^0(A)$. For odd Fredholm modules, almost the same formula works, see [7]. As expected, the Chern character for Fredhom modules is compatible with the Chern character in K-theory $\mathrm{ch}\colon K_*(A) \to HE_*(A)$. That is, the analogues of Theorem 19.2 and 19.3 hold:

Theorem 22.6 *Let $x \in K_*(A)$ and let y be a θ-summable Fredholm module with the same parity as x. Then the pairing between the Chern characters $\langle \mathrm{ch}\, x, \mathrm{ch}\, y\rangle$ equals the index pairing between K-theory and K-homology.*

Viewing entire cyclic cohomology as a version of periodic cyclic cohomology that contains, in addition, certain infinite dimensional cycles, we expect that $HE^*(A) = HP^*(A)$ for "finite dimensional" algebras. A positive result in this direction is the following [28]:

Theorem 22.7 *Let A be a Banach algebra of finite Hochschild homological dimension. Then $HE^*(A) \cong HP^*(A)$.*

However, there are quasi-free Fréchet algebras for which entire and periodic cyclic cohomology differ.
Entire and periodic cyclic cohomology have many properties in common. An explanation for this is that we can describe both $HP^0(A)$ and $HE^0(A)$ using traces on "nilpotent" extensions of A. For the purposes of periodic cyclic cohomology, a

nilpotent extension of A is an extension of algebras $N \to E \to A$ in which the kernel N is nilpotent in the sense that $N^n = 0$ for some $n \in \mathbb{N}$. A trace on E gives rise to an element of $HP^0(E)$ and hence of $HP^0(A)$ because Goodwillie's theorem asserts that $HP^0(A) \cong HP^0(E)$. Moreover, all elements of $HP^0(A)$ arise in this way. It is also possible to describe when two traces give rise to the same element in $HP^0(A)$, but we will not explain this here.
We get entire cyclic cohomology by weakening the notion of nilpotence.

Definition 22.8 *A bornological algebra A is called* analytically nilpotent *iff $S^\infty := \bigcup_{n\geq 1} S^n$ is small for all $S \in \mathcal{S}(A)$.*
An analytically nilpotent extension *of A is an extension $N \to E \to A$ of complete bornological algebras with a bounded linear section and analytically nilpotent kernel N.*

Let A be a complete bornological algebra that is analytically nilpotent and let $a \in A$. If the power series $\sum c_j z^j$ has non-zero radius of convergence, then the series $\sum c_j a^j$ converges in A. This means that all elements of A have "spectral radius zero". A typical example of an analytically nilpotent algebra is JA or its completion $(JA)^{\mathrm{c}}$. Indeed, the bornology on TA is the coarsest one for which JA is analytically nilpotent and the canonical linear map $A \to TA$ is bounded.
Consequently, the extension $(JA)^{\mathrm{c}} \to (TA)^{\mathrm{c}} \to A$ is universal among the analytically nilpotent extensions in the following sense: If $N \to E \to A$ is an analytically nilpotent extension of A, then there is a morphism of extensions from $(TA)^{\mathrm{c}} \to A$ to $E \to A$, that is, a commutative diagram

$$\begin{array}{ccccc} (JA)^{\mathrm{c}} & \longrightarrow & (TA)^{\mathrm{c}} & \longrightarrow & A \\ \downarrow & & \downarrow & & \| \\ N & \longrightarrow & E & \longrightarrow & A \end{array}$$

In addition, this morphism is unique up to smooth homotopy.

Definition 22.9 *Let $l\colon A \to B$ be a linear map between bornological algebras. For $S \in \mathcal{S}(A)$, let*

$$\omega_l(S,S) := \{l(xy) - l(x)l(y) \mid x, y \in S\} \subseteq B.$$

We say that l is almost multiplicative *or that l has* analytically nilpotent curvature *iff $\bigcup_{n\geq 1} \omega_l(S,S)^n$ is small for all $S \in \mathcal{S}(A)$.*

It is an important fact that almost multiplicative linear maps form a category, that is, compositions of almost multiplicative linear maps are again almost multiplicative. This is closely related to the observation that the class of analytically nilpotent algebras is closed under extensions. That is, if $A \to B \to C$ is an extension with analytically nilpotent A and C, then B is analytically nilpotent as well. Working with almost multiplicative linear maps, it is easy to obtain the following analogue of Goodwillie's theorem:

Theorem 22.10 *Let $N \to E \xrightarrow{p} A$ be an extension of complete bornological algebras with a bounded linear section. Suppose that N is analytically nilpotent. Then $[p_*] \in HE_0(E, A)$ is invertible. Therefore, $p^*\colon HE^0(A) \to HE^0(E)$ is an isomorphism.*

Another consequence of the composability of almost multiplicative maps is that any extension $N \to E \to TA$ with analytically nilpotent kernel N splits by a bounded homomorphism. Thus $(TA)^{\mathrm{c}}$ is analytically quasi-free:

Definition 22.11 *A complete bornological algebra R is called* analytically quasi-free *iff any analytically nilpotent extension $N \to E \to R$ splits by a bounded homomorphism.*

Analytical quasi-freeness is a much stronger condition than quasi-freeness and is not related to homological algebra. Examples of analytically quasi-free algebras besides $(TA)^{\mathrm{c}}$ are $\mathbb{C}$ and the algebra of Laurent series $\mathbb{C}[u, u^{-1}]$, equipped with the finest possible bornology.

Theorem 22.12 *Let $N \to R \to A$ be an extension of complete bornological algebras with a bounded linear section. If N is analytically nilpotent and R is analytically quasi-free, then the complexes $X(R)^{\mathrm{c}}$ and $X(TA)^{\mathrm{c}}$ are homotopy equivalent via bounded chain maps and homotopies.*

We now sketch the proof of the Excision Theorem 22.4 in [32]. Let $S \to P \to Q$ be an extension with bounded linear section. The basic idea of the proof is to play with the *left* ideal $L \subseteq TP$ that is generated by $S \subseteq TP$. It turns out that while the two-sided ideal generated by S does not have good properties, the ideal L behaves very nicely. First, there is an explicit free L-bimodule resolution of $\tilde{L}$ of length 1 related to the embedding $L \subseteq TP$. It follows that L is quasi-free. An analysis of this resolution shows that the complex $X(L)$ is homotopy equivalent to the kernel of the canonical map $X(TP) \to X(TQ)$. Secondly, L is not just quasi-free but analytically quasi-free. This is more difficult to show. One can construct a bounded homomorphism $L \to TL$ using the universal property of TP. The kernel of the canonical projection $L \to S$ is contained in JP and therefore analytically nilpotent. Hence Theorem 22.12 shows that $X(TS)$ and $X(L)$ are homotopy equivalent. Since $X(L)$ is homotopy equivalent to the kernel of the canonical map $X(TP) \to X(TQ)$, the Excision Theorem follows.

23 Local cyclic cohomology

Neither periodic nor entire cyclic cohomology yield good results for big algebras like C^*-algebras. If A is a nuclear C^*-algebra or, more generally, an amenable Banach algebra, then $HE^1(A) = HP^1(A) = 0$ and $HE^0(A) \cong HP^0(A) \cong HH^0(A)$ is the space of traces on A [28]. To obtain reasonable results for C^*-algebras as well, one has to use local cyclic cohomology [38].

The definition of local cyclic cohomology is based on entire cyclic cohomology. The first modification with respect to the entire theory is that we equip a Banach (or Fréchet) algebra A with the bornology of all *precompact* subsets. This change of bornology can still be handled within the framework of section 22. The second modification is that we view a bornological vector space as an inductive system and replace the ordinary cohomology by a "local" cohomology that is defined in the setting of inductive systems.
Let X be a bornological vector space. Then the absolutely convex small subsets of X form a directed set with respect to inclusion. Recall that if $S \subseteq X$ is an absolutely convex small subset, then its linear span X_S carries a canonical norm. The map $S \mapsto X_S$ defines an inductive system of normed spaces with the additional property that all the structure maps are injective. Its limit (in the category of bornological vector spaces) is isomorphic to X. Thus any bornological vector space can be written in a canonical way as an inductive limit of normed spaces. This construction gives rise to an equivalence between the category of bornological vector spaces and bounded linear maps and the category of inductive systems of normed spaces with *injective* structure maps and morphisms of inductive systems [32].
If we apply this construction to a $\mathbb{Z}/2$-graded complex of bornological vector spaces, the resulting inductive system is (isomorphic to) an inductive system of $\mathbb{Z}/2$-graded complexes of normed spaces. In the following, we view $X(TA)$ as an inductive system of $\mathbb{Z}/2$-graded complexes of normed spaces in this way. *We still write $X(TA)$ for this inductive system to simplify notation.*
The space of bounded linear functionals on an inductive system $(C_i)_{i\in I}$ is equal to the projective limit of the dual spaces C_i'. The projective limit functor has the following undesirable property: Even if all the complexes C_i' are contractible, it may happen that the cohomology of the inductive limit is non-zero. We would like to have another cohomology functor H^*_{loc} that is *local* in the following sense: if $H^*(C_i) = 0$ for all $i \in I$, then $H^*_{\text{loc}}\big((C_i)_{i\in I}\big) = 0$ as well.
To define H^*_{loc}, we observe the following. The category Ho(Ind) of inductive systems of $\mathbb{Z}/2$-graded complexes of normed spaces with homotopy classes of chain maps as morphisms is a triangulated category. The subclass $\mathcal{N}$ of inductive systems of *contractible* complexes is a "null system" in Ho(Ind). Thus the corresponding quotient category Ho(Ind)$/\mathcal{N}$ is again a triangulated category. The decisive property of this quotient category is that in it all elements of $\mathcal{N}$ become isomorphic to the zero object.
In the following definition, we view $\mathbb{C}$ as a constant inductive system of complexes with zero boundary as usual.

Definition 23.1 *Let $K = (K_i)_{i\in I}$ and $L = (L_j)_{j\in J}$ be two inductive systems of complexes. Define $H_*^{\text{loc}}(K, L)$ to be the space of morphisms $K \to L$ in the category* Ho(Ind)$/\mathcal{N}$. *Let*

$$H^*_{\text{loc}}(K) := H_*^{\text{loc}}(K, \mathbb{C}), \qquad H_*^{\text{loc}}(L) := H_*^{\text{loc}}(\mathbb{C}, L).$$

One can compute $H^{\mathrm{loc}}_*(K,L)$ via an appropriate projective resolution of K (see [38]). An analysis of this resolution yields that $H^{\mathrm{loc}}_*(K,L)$ is a limit of a spectral sequence whose E^2-term involves the homologies $H_*\bigl(\mathcal{L}(K_i,L_j)\bigr)$ and the derived functors of the projective limit functor $\varprojlim$. For countable inductive systems, the derived functors $R^p\varprojlim$ with $p \geq 2$ vanish. Hence the spectral sequence degenerates to a Milnor $\varprojlim^1$-exact sequence

$$0 \longrightarrow {\varprojlim}^1_I \varinjlim_J H_{*-1}\bigl(\mathcal{L}(K_i,L_j)\bigr) \longrightarrow H^{\mathrm{loc}}_*(K,L) \\ \longrightarrow \varprojlim_I \varinjlim_J H_*\bigl(\mathcal{L}(K_i,L_j)\bigr) \longrightarrow 0$$

if I is countable. In the local theory we have this exact sequence for arbitrary countable inductive systems (K_i). In particular, for $K = \mathbb{C}$ we obtain

$$H^{\mathrm{loc}}_*(L) = \varinjlim H_*(L_j).$$

Since the inductive limit functor is exact, $H^{\mathrm{loc}}_*(L)$ is equal to the homology of the inductive limit of L.

The completion of an inductive system of normed spaces $(X_i)_{i\in I}$ is defined entrywise: $(X_i)^{\mathrm{c}} := (X_i^{\mathrm{c}})_{i\in I}$.

Definition 23.2 *Let A and B be complete bornological algebras. We define* local cyclic cohomology, local cyclic homology, *and* bivariant local cyclic homology *by*

$$HE^*_{\mathrm{loc}}(A) := H^*_{\mathrm{loc}}\bigl(X(TA)\bigr), \qquad HE^{\mathrm{loc}}_*(B) := H^{\mathrm{loc}}_*\bigl(X(TB)^{\mathrm{c}}\bigr),$$
$$HE^{\mathrm{loc}}_*(A,B) := H^{\mathrm{loc}}_*\bigl(X(TA)^{\mathrm{c}}, X(TB)^{\mathrm{c}}\bigr) \cong H^{\mathrm{loc}}_*\bigl(X(TA), X(TB)^{\mathrm{c}}\bigr).$$

As usual, we have $HE^{\mathrm{loc}}_*(A,\mathbb{C}) \cong HE^*_{\mathrm{loc}}(A)$ and $HE^{\mathrm{loc}}_*(\mathbb{C},A) \cong HE^{\mathrm{loc}}_*(A)$. The composition of morphisms gives rise to a product on HE^{loc}_*.

If we view a bornological vector space X as an inductive system (X_i) with injective structure maps, then the structure maps of the completion $(X_i)^{\mathrm{c}}$ need not be injective any more, so that $(X_i)^{\mathrm{c}}$ may differ from the inductive system that corresponds to the bornological vector space X^{c}. Hence we have to be careful in what category we form the completion $X(TB)^{\mathrm{c}}$. For the local theory, it is crucial to take the completion in the category of inductive systems. A consequence is that, in general, $HE^{\mathrm{loc}}_*(B)$ may be different from $HE_*(B)$. However, in applications it often turns out that the two completions coincide. This holds, for instance, if B is a Fréchet algebra equipped with the precompact bornology or if B is nuclear in the sense of Grothendieck. In these two cases we have $HE_*(B) \cong HE^{\mathrm{loc}}_*(B)$ and a natural map $HE_*(A,B) \to HE^{\mathrm{loc}}_*(A,B)$.

The main advantage of local cyclic cohomology is that it behaves well when passing to "smooth" subalgebras.

Definition 23.3 *Let A be a complete bornological algebra with bornology $\mathcal{S}(A)$, let B be a Banach algebra with closed unit ball U, and let $j\colon A \to B$ be an injective bounded homomorphism with dense image. We call A a* smooth subalgebra *of B iff $S^\infty := \bigcup S^n \in \mathcal{S}(A)$ whenever $S \in \mathcal{S}(A)$ and $j(S) \subseteq rU$ for some $r < 1$.*

Let $A \subseteq B$ be a smooth subalgebra. Then an element $a \in A$ that is invertible in B is already invertible in A. Hence A is closed under holomorphic functional calculus. In the setting of topological algebras, smooth subalgebras are defined in [2] using norms with special properties. If $A \subseteq B$ is a smooth subalgebra in the sense of [2], then A equipped with the precompact bornology is a smooth subalgebra in the sense of Definition 23.3.

Whereas the inclusion of a smooth subalgebra $A \to B$ induces an isomorphism $K_*(A) \cong K_*(B)$ in K-theory, the periodic or entire cyclic theories of A and B may differ drastically. However, local theory behaves like K-theory in this situation:

Theorem 23.4 *Let $A \subseteq B$ be a smooth subalgebra. Assume that A and B are both endowed with the bornology of precompact subsets. Let $j\colon A \to B$ be the inclusion and let $j_* \in HE_0^{\mathrm{loc}}(A, B)$ be the corresponding element in the bivariant local cyclic homology.*

If B has Grothendieck's approximation property, then j_* is invertible. More generally, j_* is invertible if the identity map on B can be approximated uniformly on small subsets of B by maps of the form $j \circ l$ with $l\colon B \to A$ bounded and linear.

Bivariant local cyclic homology is exact for extensions with a bounded linear section, homotopy invariant for smooth homotopies and stable with respect to tensor products with the trace class operators $\ell^1(H)$. The proofs are the same as for entire cyclic cohomology. Using Theorem 23.4, we can strengthen these properties considerably:

Theorem 23.5 *Let A be a C^*-algebra. The functors $B \mapsto HE_*^{\mathrm{loc}}(A, B)$ and $B \mapsto HE_*^{\mathrm{loc}}(B, A)$ are split exact, stable homotopy functors on the category of C^*-algebras. For separable C^*-algebras, there is a natural bivariant Chern character*

$$\mathrm{ch}\colon KK_*(A, B) \to HE_*^{\mathrm{loc}}(A, B).$$

The Chern character is multiplicative with respect to the Kasparov product on the left and the composition product on the right hand side.

If both A and B are in the bootstrap class satisfying the universal coefficient theorem, then the Chern character gives rise to isomorphisms

$$HE_*^{\mathrm{loc}}(A, B) \cong KK_*(A, B) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathrm{Hom}\big(K_*(A), K_*(B)\big) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Proof. Since $C^\infty([0,1], A) \subseteq C([0,1], A)$ is a smooth subalgebra, Theorem 23.4 and smooth homotopy invariance imply continuous homotopy invariance. The subalgebra $M_\infty(A) = M_\infty \hat{\otimes} A$ defined as in (1) is a smooth subalgebra of the C^*-algebraic stabilization of A. Hence Theorem 23.4 and stability with respect to M_∞ imply C^*-algebraic stability.

The existence of the bivariant Chern character follows from these homological properties by the universal property of Kasparov's KK-theory.

The last assertion is trivial for $A = B = \mathbb{C}$. The class of C^*-algebras for which it holds is closed under KK-equivalence, inductive limits and extensions with completely positive section. Hence it contains all C^*-algebras satisfying the universal coefficient theorem (see [1]). $\square$

Examples of C^*-algebras satisfying the universal coefficient theorem are commutative C^*-algebras and the C^*-algebras of amenable groupoids [44].
Assume that Γ is a discrete torsion free group that is the fundamental group of a manifold M of non-positive sectional curvature. Then M is a model for $B\Gamma$, so that $\mathbb{C}[\Gamma]$ has finite homological dimension. We have

$$HE_*^{\mathrm{loc}}\big(\ell^1(\Gamma)\big) \cong HP_*\big(\mathbb{C}[\Gamma]\big) \cong H_*(\Gamma, \mathbb{C}).$$

However, the local cyclic homology of $C^*_{\mathrm{red}}\Gamma$ is still unknown.

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