Introduction to deformation quantization.

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1. Fedosov type deformation quantization

1.1. Basic definition.

Definition 1.1.1. *Let (M,ω) be a symplectic manifold.*

(1) *A* deformation quantization *of M (cf.* [BFFLS]*) is a formal power series*

$$f * g = fg + \sum_{k=1}^{\infty} (i\hbar)^k D_k(f,g) \tag{1.1}$$

where $D_k : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$ are bilinear bidifferential expressions, $*$ is associative, $f * 1 = 1 * f = f$, and

$$\{f,g\} = D_1(f,g) - D_1(g,f) \tag{1.2}$$

is the Poisson bracket defined by the symplectic structure.

(2) *An* isomorphism *between $*$ and $*'$ is a formal series*

$$T(f) = f + \sum_{k=1}^{\infty} (i\hbar)^k T_k(f) \tag{1.3}$$

where $T(f*g) = T(f) *' T(g)$ and T_k are differential operators on $C^\infty(M)$.

Example 1.1.2. For $M = \mathbb{R}^{2n}$ and $\omega = d\xi \wedge dx$, put

$$f * g = \exp(\frac{i\hbar}{2}(\partial_\xi \partial_y - \partial_\eta \partial_x)) f(x,\xi) g(y,\eta)|_{x=y,\xi=\eta} \tag{1.4}$$

(the Moyal product). This is a deformation quantization of $\mathbb{R}^{2n}$.

1.2. Infinitesimal version.

We will denote by W (the Weyl algebra) the algebra $\mathbb{C}[[\widehat{x}^1, \ldots, \widehat{x}^n, , \widehat{\xi}^1, \ldots, \widehat{\xi}^n, \hbar]]$ with the Moyal product (1.4) (we always denote formal variables by $\widehat{x}$, $\widehat{\xi}$). One can identify W with the ring of operators of the form $\sum A_{\alpha\beta} \widehat{x}^\alpha (i\hbar \frac{\partial}{\partial \widehat{x}})^\beta$ on $\mathbb{C}[[\widehat{x}, \hbar]]$. This identification takes $\widehat{x}^\alpha \widehat{\xi}^\beta$ to the symmetrized product $\widehat{x}^\alpha (i\hbar \frac{\partial}{\partial \widehat{x}})^\beta$ (the Weyl identification). Note that, if one puts

$$|\widehat{x}^k| = |\widehat{\xi}^k| = 1,\ |\hbar| = 2, \tag{1.5}$$

then W becomes a direct product of its graded components

$$W = \prod_{k \geq 0} W_k$$

Put

$$\mathfrak{g} = \frac{1}{i\hbar} W / \frac{1}{i\hbar} \mathbb{C}[[\hbar]] \tag{1.6}$$

with the bracket $a * b - b * a$. This Lie algebra is isomorphic to the algebra of continuous derivations of W via $a \mapsto \mathrm{ad}(a)$. The Lie algebra $\mathfrak{g}$ splits into the product of its graded components

$$\mathfrak{g} = \prod_{k \geq -1} \mathfrak{g}_k$$

Note that $\mathfrak{g}_{-1} = \frac{1}{i\hbar}\mathbb{C}^{2n}$ and $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{C})$. Also, the group $\mathrm{Sp}(2n, \mathbb{C})$ acts on W by linear changes of coordinates. This action preserves the product. Its infinitesimal action coincides with the adjoint action of $\mathfrak{g}_0$.

Put

$$\widetilde{\mathfrak{g}} = \frac{1}{i\hbar}W \tag{1.7}$$

with the bracket $a * b - b * a$. One has

$$\widetilde{\mathfrak{g}} = \prod_{k \geq -2} \widetilde{\mathfrak{g}}_k$$

where $\widetilde{\mathfrak{g}}_{-2} = \frac{1}{i\hbar}\mathbb{C}$, $\widetilde{\mathfrak{g}}_{-1} = \frac{1}{i\hbar}\mathbb{C}^{2n}$ and

$$\widetilde{\mathfrak{g}}_0 = \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathbb{C}$$

canonically. The subalgebra $\mathfrak{sp}(2n, \mathbb{C})$ is formed by $\frac{1}{i\hbar}q(\widehat{x}, \widehat{\xi})$ where q are quadratic functions.

1.3. **Fedosov connections.** For any symplectic manifold M, we form a bundle of associative algebras $W = W_M$ and the bundles of Lie algebras $\mathfrak{g} = \mathfrak{g}_M$, $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_M$. We define them to be the bundles associated to the Sp-equivariant algebras W, $\mathfrak{g}$, etc. *A Fedosov connection* is an operator

$$\nabla : \Omega^{\bullet}(M, W) \to \Omega^{\bullet+1}(M, W)$$

such that:

(1)
$$\nabla = A_{-1} + \nabla_0 + A_1 + \dots$$
where $A_k \in \Omega^1(M, \mathfrak{g}_k)$ and ∇_0 is a connection in TM preserving ω;

(2)
$$A_{-1} \in \Omega^1(M, \mathfrak{g}_{-1}) = \Omega^1(M, T_M^*)$$
is minus the map $T_M \to T_M^*$ defined by ω;

(3) $\nabla^2 = 0$

A lifting of ∇ is an expression

$$\widetilde{\nabla} = \widetilde{A}_{-1} + \nabla_0 + \widetilde{A}_0 + \widetilde{A}_1 + \dots$$

where ∇_0 is the same connection which is now viewed as $\widetilde{\mathfrak{g}}_0$-valued, $\widetilde{A}_k \in \Omega^1(M, \widetilde{\mathfrak{g}}_k)$, and the image of $\widetilde{A}_k$ under the map induced by the projection $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ is A_k. (For the sake of uniformity we put $A_0 = 0$).

The following theorem is essentially due to Fedosov [Fe]. For expositions closer to ours, cf. [BNT], [NT1]. Another approach, which is valid for algebraic varieties, is contained in [BK].

Theorem 1.3.1. *1) For any Fedosov connection ∇,*

$$\mathbb{A}_M = \mathbb{A}_M^{\nabla} = \ker(\nabla : C^{\infty}(M, W) \to \Omega^1(M, W))$$

is an algebra which is isomorphic to $C^{\infty}(M)[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$-module. The resulting product on $C^{\infty}(M)[[\hbar]]$ is a deformation quantization.

2) Any ∇ admits a lifting $\widetilde{\nabla}$, and

$$\widetilde{\nabla}^2 = \theta = \frac{1}{i\hbar}\omega + \sum_{k=0}^{\infty}(i\hbar)^k\theta_k \tag{1.8}$$

where θ_k are closed two-forms. For two different liftings of ∇, the forms θ are cohomologous.

3) Deformation quantizations $\ker(\nabla)$ *and* $\ker(\nabla')$ *are isomorphic if and only if the curvatures of their liftings are cohomologous.*

4) Given two lifted Fedosov connections with the same curvature form θ, there is an isomorphism between $\ker(\nabla)$ *and* $\ker(\nabla')$ *which is canonical up to a canonical inner isomorphism.*

5) Any deformation quantization is isomorphic to $\ker(\nabla)$ *for some ∇.*

6) For any closed θ such as in (1.8)*, there is a Fedosov connection ∇ with a lifting $\widetilde{\nabla}$ such that $\widetilde{\nabla}^2 = \theta$.*

1.4. **Groups of automorphisms of W and gauge transformations.** Put $\widetilde{\mathfrak{g}}_{\geq 1} = \prod_{k\geq 1} \widetilde{\mathfrak{g}}_k$. This is a pronilpotent Lie algebra, so one can define

$$\widetilde{G}_{\geq 1} = \exp \widetilde{\mathfrak{g}}_{\geq 1}$$

Put also

$$\overline{G}_{\geq 0} = \mathrm{Sp}(2n, \mathbf{R}) \ltimes \widetilde{G}_{\geq 1} \tag{1.9}$$

$$\widetilde{G}^N_{\geq 0} = \widetilde{\mathrm{Sp}}^N(2n, \mathbf{R}) \ltimes \widetilde{G}_{\geq 1} \tag{1.10}$$

Note that $\widetilde{G}_{\geq 1}$ acts on $\widetilde{\mathfrak{g}}$-valued connections by gauge transformations. One can show that

Lemma 1.4.1. *Two Fedosov connections ∇, ∇' define isomorphic star products if and only if they are gauge equivalent, if and only if they have gauge equivalent liftings. Two lifted Fedosov connections are gauge equivalent if and only if their curvature forms are equal.*

This is the key part of the proof of the statements 3 and 4 of theorem 1.3.1. For example, to prove 4, observe that a gauge equivalence defines an isomorphism of corresponding bundles with connection, hence of the algebras of horizontal elements; two gauge equivalences of lifted connections differ by a gauge auto-equivalence, which is by definition an invertible section of the bundle W which is horizontal under ∇, hence an invertible element of $\ker(\nabla)$.

1.5. **Gelfand-Fuks and all that.** Let $\mathcal{P}$ denote the unitary frame bundle of T^*M (associated to a reduction of the structure group from Sp(2n) to U(n)). Let $(\mathbb{L}^*, \delta)$ be a (profinite) complex of $(\mathfrak{g}, U)$-modules. Given Fedosov connection, we set

$$\Omega(M, \mathcal{L}^*) = (\Omega(\mathcal{P}) \otimes \mathbb{L})_{basic} = \{\omega \in (\Omega(\mathcal{P}) \otimes \mathbb{L})^U \mid \forall_{H\in\mathfrak{u}} \iota_H \omega = 0\}$$

and form the complex

$$(\Omega(M, \mathcal{L}^*), \nabla + \delta).$$

The **Gelfand-Fuks map** is a map of complexes:

$$GF : (C^*_{Lie}(\mathfrak{g}, U; \mathbb{L}^*), \partial_{Lie} + \delta) \to (\Omega(M, \mathcal{L}^*), \nabla + \delta)$$

constructed as follows. Let $l \in {C^k}_{Lie}(\mathfrak{g}, U; \mathbb{L}^*)$ and $X_1, \ldots, X_k \in Vect(M)$. Let $\tilde{X}_1, \ldots, \tilde{X}_k$ be arbitrary lifts to $T\mathcal{P}$. Then

$$GF(l)(X_1, \ldots X_k) = l(A(\tilde{X}_1), \ldots, A(\tilde{X}_k)).$$

Example 1.5.1. (1) Suppose that $\mathbb{L} = \mathbb{C}$.
Then
$$(\Omega(M, \mathcal{L}^*), \nabla + \delta) = (\Omega(M), d),$$
the standard de Rham complex of M, and
$$(\mathcal{S}(\mathfrak{u}^*)^U \overset{Chern-Weil}{\longrightarrow} H^*_{Lie}(\mathfrak{g}, U) \overset{GF}{\longrightarrow} H^*_{DR}(M)$$
coincides with the Chern-Weil construction of the Chern classes associated to the unitary structure of the tangent bundle of M.
Let θ be the 2-cohomology class of $\mathfrak{g}$ associated to the extension
$$0 \to \frac{1}{i\hbar}\mathbb{C}^{2n} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0.$$
Then $GF(\theta)$ is the characteristic class (curvature) of the formal deformation provided by Fedosov connection ∇.

(2) Suppose that $\mathbb{L} = W$.
Then
$$C^\infty(M)[[\hbar]] \overset{quis}{\to} (\Omega(M, W), \nabla),$$
and the product on the right hand side induces the *-product of Fedosov construction. In fact, locally, think of the W-valued forms as $f(x, \xi, \widehat{x}, \widehat{\xi})dx^n d\xi^m$, and of ∇ as
$$\sum(\partial_x + \partial_{\widehat{x}})dx + (\partial_\xi + \partial_{\widehat{\xi}})d\xi.$$
Then, as easily seen, $(\Omega(M, W), \nabla)$ is acyclic in positive dimension and the 0'th cohomology coincides with the set of functions (formal power series) of the form $f(x + \widehat{x}, \xi + \widehat{\xi})$ (think of $\widehat{x}$ and $\widehat{\xi}$ as infinitesimal increments of x and ξ).

The following will be very useful later on. We set
$$\mathcal{L}^* = \mathrm{Ker}\nabla|_{(\Omega^0(M, \mathcal{L}^*)}.$$
This is naturally a sheaf of complexes on M, and, very often, $(\Omega(M, \mathcal{L}^*)$ is its acyclic resolution.

1.6. **The jet bundle J_M.** For any manifold M of dimension m, let J be the bundle of jets of C^∞ functions.

In invariant terms, set, for an open subset $U \subset M$, set
$$\mathcal{J}(U) = Hom_{C^\infty(U)}(Diff(U), C^\infty(U)).$$
Then $U \to \mathcal{J}(U)$ form a locally free sheaf with the associated bundle J.

If $M = \cup U_\alpha$ is an open cover and a coordinate system $x^1, \dots, x^m$ is chosen on every U_α, then we identify $J|U_\alpha$ with $U_\alpha \times \mathbb{C}[[\widehat{x}]]$ where we denote by $\mathbb{C}[[\widehat{x}]]$ the algebra $\mathbb{C}[[\widehat{x}^1, \dots, \widehat{x}^m]]$. The transition functions of the bundle J are
$$G_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}\,\mathbb{C}[[\widehat{x}]]$$
defined as follows:
$$G_{\alpha\beta}(x) : \widehat{x} \mapsto g_{\alpha\beta}(\phi_\beta(x) + \widehat{x}) - g_{\alpha\beta}(\phi_\beta(x)) \tag{1.11}$$
where
$$\phi_\alpha : U_\alpha \hookrightarrow \mathbb{R}^N$$
are the coordinate embeddings and
$$g_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1}$$

The jet bundle is filtered by powers of the ideal $(\widehat{x}^1, \dots, \widehat{x}^m)$, and the associated graded bundle of algebras is $S[T^*_M]$, the symmetric algebra of the cotangent bundle. Using the fact that C^∞_M is an acyclic sheaf, one shows that, noncanonically,

$$J_M \xrightarrow{\sim} S[[T^*_M]] \tag{1.12}$$

(the completion of the symmetric algebra $S[T^*_M]$) as bundles of algebras.

If a deformation quantization is given on M, then $J_M[[\hbar]]$ becomes a bundle of algebras. Locally, using the grading as in (1.5), put

$$F^k J[[\hbar]] = \prod_{p\geq k} J[[\hbar]]_p$$

The transition functions and the product preserve this filtration. The completed associated graded bundle of algebras is the Weyl bundle W. As above, one can show that

$$J_M[[\hbar]] \xrightarrow{\sim} W_M \tag{1.13}$$

as bundles of algebras.

For any M there is the canonical flat connection

$$\nabla_{\mathrm{can}} : \Omega^\bullet(M, J) \to \Omega^{\bullet+1}(M, J) \tag{1.14}$$

which preserves the product.

Abstractly,

$$(\nabla_{\mathrm{can},X} f)(D) = Xf(D) - f(XD).$$

In coordinates,

$$(\nabla_{\mathrm{can}} f)(x, \widehat{x}) = \sum_{k=1}^{m} (\frac{\partial f}{\partial x^k} + \frac{\partial f}{\partial \widehat{x}^k}) dx^k$$

The kernel of $\nabla_{\mathrm{can}}|C^\infty(M, J)$ is canonically isomorphic to the algebra $C^\infty(M)$.

If a deformation quantization is given on M, the image of ∇_{can} under the isomorphism (1.13) becomes a Fedosov connection. This proves the assertion 5) of theorem 1.3.1.

1.7. **Differential operators and the deformation quantization of $T^*(X)$.** Let X be a manifold. For the sheaf of rings $D_X^{\frac{1}{2}}$ of differential operators on half-densities on X, let $F_p D_X^{\frac{1}{2}}$ be the filtration by order. Let

$$\mathcal{R} D_X^{\frac{1}{2}} = \bigoplus_{p\geq 0} \hbar^p F_p D_X^{\frac{1}{2}} \tag{1.15}$$

be the Rees ring. Let $X = \cup_\alpha U^0_\alpha$ be an open cover. A choice of coordinates on U^0_α identifies $\mathcal{R} D^{\frac{1}{2}}_{U^0_\alpha}$ with the ring $C^\infty_{U^0_\alpha}[\xi^1, \dots, \xi^n, \hbar]$ where

$$\xi^k = i\hbar \frac{\partial}{\partial x^k}$$

The latter ring can be identified with the ring $C^{\infty,\mathrm{poly}}_{\pi^{-1}U^0_\alpha}$ where $\pi : T^*X \to X$ is the projection and $C^{\infty,\mathrm{poly}}$ stands for the sheaf of C^∞ functions which are polynomial along the fibers. We use the Weyl identification, analogous to one that was used in the discussion after (1.4).

We get

$$\phi_\alpha : \mathcal{R} D^{\frac{1}{2}}_{U^0_\alpha} \xrightarrow{\sim} C^{\infty,\mathrm{poly}}_{\pi^{-1}U^0_\alpha}$$

One checks that the product on $\mathcal{R}D^{\frac{1}{2}}_{U^0_\alpha}$ induces on the right hand side a star product which we denote by $*_\alpha$. This star product extends to $C^\infty_{\pi^{-1}U_\alpha}$. Furthermore, $G_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1}$ extend to isomorphisms between $*_\alpha$ and $*_\beta$ in the sense of definition 1.1.1. Using partitions of unity, one constructs automorphisms T_α of $*_\alpha$ such that $G_{\alpha\beta} = T_\alpha T_\beta^{-1}$. This allows to define a star product on T^*X.

Let us recall how one identifies the above deformation in terms of the classification theorem 1.3.1.

Proposition 1.7.1. *The characteristic class θ of this deformation is $\frac{1}{i\hbar}\omega$ $(=0)$.*

Proof. Start with an open cover $\{\pi^{-1}(U^0_\alpha)\}$ of $T^*(X)$. A choice of coordinates $x_\alpha = (x^1_\alpha, \dots, x^n_\alpha)$ on U^0_α determines a coordinate system (x_α, ξ^α) on $\pi^{-1}(U^0_\alpha)$. The transition functions between two different coordinate systems are

$$\begin{aligned} x_\alpha &= g_{\alpha\beta}(x_\beta) \\ \xi^\alpha &= {}^t g'_{\alpha\beta}(x_\beta)^{-1} \xi^\beta \end{aligned} \tag{1.16}$$

(${}^t g'_{\alpha\beta}(x_\beta)$ stands for the transposed Jacobi matrix).

Now consider the bundle $\pi^* \,\mathrm{jets}\, \mathcal{R}D^{\frac{1}{2}}_X$ with the canonical connection $\pi^*\nabla_{\mathrm{can}}$. (To construct the bundle of jets one acts as in 1.6). This is a bundle of algebras with the fiber

$$W_{\mathrm{fin}} = \mathbb{C}[[\widehat{x}]][\widehat{\xi}, \hbar] \tag{1.17}$$

(again, we use the Weyl identification of the two sides). Its transition functions are given explicitly as follows:

$$\widehat{x} \mapsto g_{\alpha\beta}(x_\beta + \widehat{x}) - x_\alpha \tag{1.18}$$

$$\widehat{\xi} \mapsto^t g'_{\alpha\beta}(x_\beta + \widehat{x})^{-1} \widehat{\xi} \tag{1.19}$$

Because of the presence of the half-densities, the product in the second equation is the commutative product, not the composition in the Weyl algebra (as always, we use the Weyl identification between functions and operators). The canonical connection is, in coordinates, given by

$$\pi^*\nabla_{\mathrm{can}} = d - \sum \frac{\partial}{\partial \widehat{x}^k} dx^k,$$

without a similar ξ term. To correct that, apply the gauge transformation

$$\sigma_\alpha = \exp \mathrm{ad} \frac{1}{i\hbar} \xi^\alpha \widehat{x}^\alpha \in \mathrm{Aut}(W_{\mathrm{fin}}) \tag{1.20}$$

We get a new bundle of algebras whose transition functions are

$$\widehat{x} \mapsto g_{\alpha\beta}(x_\beta + \widehat{x}) - x_\alpha \tag{1.21}$$

$$\widehat{\xi} \mapsto^t g'_{\alpha\beta}(x_\beta + \widehat{x})^{-1} (\xi^\beta + \widehat{\xi}) - \xi^\alpha \tag{1.22}$$

Again, the multiplication in the second formula is the commutative multiplication of power series. Therefore, these transition functions coincide with the transition functions of the jet bundle J_{T^*X} (compare with (1.16)). Note also that these transition functions admit a canonical lifting to $\widetilde{G}_{\geq 0}$. Indeed, consider the group K of formal automorphisms of the trivial line bundle on $\mathbb{R}^n$ acting by

$$f(\widehat{x}) \mapsto p(\widehat{x}) f(g(\widehat{x})) |\mathrm{det} g'(\widehat{x})|^{\frac{1}{2}} \tag{1.23}$$

where g is a formal diffeomorphism of the form $g : \widehat{x} \mapsto a\widehat{x}+o(\widehat{x})$, $a \in \mathrm{GL}(n, \mathbb{R})$, and $p(\widehat{x}) \in \mathbb{C}[[\widehat{x}]]$, $p(0) = 0$. The group K maps into $G_{\geq 0}$ as follows. We can represent W as an algebra of operators on $\mathbb{C}[[\widehat{x}, \hbar]]$ by identifying, as above, $\widehat{x}^n\widehat{\xi}^m$ with the symmetrized product $\widehat{x}^n(i\hbar\frac{\partial}{\partial \widehat{x}})^m$. Then K acts on these operators by conjugation. To check that there is a canonical lifting $K \to \widetilde{G}_{\geq 0}$, observe that

$$K = \mathrm{GL}(n, \mathbb{R}) \ltimes K_{\geq 1}$$

where $K_{\geq 1}$ is the group of elements for which $a = 1$. But GL maps to Sp, and this map lifts to $\widetilde{\mathrm{Sp}}$; on the other hand, $K_{\geq 1}$ has a canonical lifting to $\widetilde{G}_{\geq 1}$. Indeed,

$$K_{\geq 1} = \exp(\mathfrak{k}_{\geq 1})$$

where

$$\mathfrak{k}_{\geq 1} = \{\frac{1}{i\hbar}P(\widehat{x})\widehat{\xi} + Q(\widehat{x}) | P(\widehat{x}) = o(\widehat{x});\ Q(0) = 0\}$$

and this pronilpotent Lie algebra is a subalgebra of $\widetilde{\mathfrak{g}}_{\geq 1}$, not just of $\mathfrak{g}_{\geq 1}$.

As for the connection, $\pi^*\nabla_{\mathrm{can}}$ becomes a Fedosov connection after the gauge transformation σ_α. It has a flat lifting which has an extra summand $\frac{i}{\hbar}\xi dx$. Subtract it, and we get a Fedosov connection, together with a lifting whose curvature is $\frac{1}{i\hbar}\omega$. This proves the proposition.

1.8. **Gelfand-Fuks again.** Some more local-to-global examples. Let

$$(\mathbb{L}^*, \delta) = (CC_*(W), b + B)$$

be the cyclic periodic complex of the Weyl algebra. Suppose moreover that (M, ω) is a symplectic manifold, ∇ is a $\mathfrak{g}$-valued Fedosov connection and A_M is the formal deformation of M associated to ∇. Since $(CC_*(W), b + B)$ is a module over $\mathfrak{g} = Der(W)$, Gelfand-Fuks construction provides the associated complex of sheaves $(\mathcal{L}^*, b + B)$. As easily seen,

$$(\mathcal{L}^*, b + B) = \mathrm{jets}_\Delta(CC_*(A_M), b + B),$$

the jets along the diagonal of the periodic cyclic complex of the deformation algebra.

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