Exercise 1 – Wick rotation

Consider the integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\varepsilon)^n}$$

with $k^2 = (k_0)^2 - \vec{k}^2$ and $\Delta > 0$. This integral is convergent for $n \geq 3, n \in \mathbb{N}$.

a) Show that this integral can be brought into the form

$$i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^n} \ ,$$

where $k_E^2 = (k_0)^2 + \vec{k}^2$.

b) Show that

$$\int \frac{d^4k_E}{(2\pi)^4} f(k_E^2) = \frac{1}{8\pi^2} \int_0^\infty dk_E \, k_E^3 \, f(k_E^2) \ .$$

c) Set $n = 3$. Show that

$$\int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E^2 + \Delta)^3} = \frac{1}{32\pi^2 \, \Delta} \ .$$

Hence,

$$\lim_{\varepsilon \to 0} \left(\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\varepsilon)^3} \right) = -\frac{i}{32\pi^2 \Delta} \ .$$

Exercise 2 – Dimensional regularization

Consider the unit sphere in d dimensions ($d \geq 2$). Its $(d-1)$-dimensional surface area equals $2\pi^{d/2}/\Gamma(d/2)$, where Γ denotes the Gamma function.

a) Define the surface area for non-integer dimensions to be given by the above formula. Using this as well as the Euler integral

$$\int_0^1 dx \, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad , \quad \mathrm{Re}\, \alpha > 0, \quad \mathrm{Re}\, \beta > 0,$$

show that

$$\int \frac{d^dk_E}{(2\pi)^d} \frac{1}{(k_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \Gamma(2 - \frac{d}{2}) \left(\frac{1}{\Delta} \right)^{2 - \frac{d}{2}} \ .$$

Note that in deriving this expression we need to take $d < 4$.

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b) Now regularize the integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\varepsilon)^2}$$

by

$$\mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i\varepsilon)^2} ,$$

with $4 - d \to 0_+$. What is the role of μ?

c) The Gamma function $\Gamma(z)$ has a simple pole at $z = 0$,

$$\Gamma(z) = \frac{1}{z} - \gamma + \mathcal{O}(z) ,$$

where γ denotes the Euler-Mascheroni constant. Using this, show that when expanding in powers of $4 - d$,

$$\lim_{\varepsilon \to 0} \left(\mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i\varepsilon)^2} \right) = \frac{i}{16\pi^2} \left(\frac{2}{4-d} + \ln \frac{4\pi e^{-\gamma} \mu^2}{\Delta} + \mathcal{O}(4-d) \right) .$$

Note the presence of the simple pole $1/(4-d)$. We have thus used dimensional regularization to regulate a UV-divergent integral.

d) Similarly, show

$$\int \frac{d^dk_E}{(2\pi)^d} \frac{1}{(k_E^2 + \Delta)} = \frac{1}{(4\pi)^{d/2}} \Gamma(1 - \frac{d}{2}) \left(\frac{1}{\Delta} \right)^{1-\frac{d}{2}} .$$

e) Use this result to show

$$\lim_{\varepsilon \to 0} \left(\mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i\varepsilon)} \right) = \frac{i}{16\pi^2} \Delta \left(\frac{2}{4-d} + \psi(2) + \ln \frac{4\pi \mu^2}{\Delta} + \mathcal{O}(4-d) \right) ,$$

where $\Gamma(-1 + z) = -\frac{1}{z} - \psi(2) + \mathcal{O}(z)$, $\psi(2) = 1 - \gamma$. This result is needed when discussing mass renormalization at 1-loop in $\lambda\phi^4$-theory.