Project – The partition function of the closed bosonic string compactified on a torus T^d

The partition function of the closed bosonic string compactified on a torus T^d is given by

$$Z(\tau,\bar\tau) = {\tau_2}^{-(24-d)/2}\,|\eta(\tau)|^{-48} \sum_{(p_L,p_R)\in\Gamma_{d,d}} \bar q^{\frac{1}{2}p_L^2}\, q^{\frac{1}{2}p_R^2} \quad , \quad \tau=\tau_1+i\tau_2\in\mathcal{H}\,, \tag{1}$$

where $|\eta(\tau)|^{-48}$ is the bosonic oscilator contribution, and ${\tau_2}^{-(24-d)/2}$ is the contribution from the transverse non-compact momenta. $\Gamma_{d,d}$ denotes an even, self-dual Lorentzian lattice, the Narain lattice, with

$$\begin{aligned} p_L &= \left(\sqrt{\frac{\alpha'}{2}}\,m_i + \frac{1}{\sqrt{2\alpha'}}\,g_{ij}\,n^j - \frac{1}{\sqrt{2\alpha'}}\,b_{ij}\,n^j\right) e^{*i} \\ p_R &= \left(\sqrt{\frac{\alpha'}{2}}\,m_i - \frac{1}{\sqrt{2\alpha'}}\,g_{ij}\,n^j - \frac{1}{\sqrt{2\alpha'}}\,b_{ij}\,n^j\right) e^{*i}\,, \end{aligned} \tag{2}$$

and $g_{ij}=e_i\cdot e_j$ the metric on the lattice $\Lambda=\{\sum_{i=1}^d n^i\,e_i | n^i\in\mathbb{Z}\}$.

The one-loop vacuum amplitude is then

$$\int_{\mathcal{F}} \frac{d^2\tau}{4\,\tau_2^2}\, Z(\tau,\bar\tau)\,, \tag{3}$$

where $\mathcal{F}$ denotes the fundamental domain of $SL(2,\mathbb{Z})$.

a) Begin by briefly reviewing and outlining the derivation of (1).

b) Show that $p_{L,R}^2$ are invariant under T-duality,

$$\mathbf{m}\leftrightarrow\mathbf{n} \quad , \quad \frac{1}{\alpha'}\,(\mathbf{g}+\mathbf{b}) \leftrightarrow \alpha'\,(\mathbf{g}+\mathbf{b})^{-1} \tag{4}$$

(Literature: section 2.4 of `Target Space Duality in String Theory`, arXiv:hep-th/9401139)

c) Show that (1) is invariant under modular transformations, i.e. under $\tau\mapsto\tau+1$, $\tau\mapsto -1/\tau$. The latter requires peforming a Poisson resummation on the lattice.

(Literature:
chapters 9.6 and 10.4 of `Basic Concepts of String Theory` by Blumenhagen, Lüst, Theisen;
section 7.4 of `Les Houches Lectures on Fields, Strings and Duality`, arXiv:hep-th/9703136)

d) Consider the one-loop partition function

$$Z_{T^d}(\tau,\bar\tau) = |\eta(\tau)|^{-2d} \sum_{(p_L,p_R)\in\Gamma_{d,d}} \bar q^{\frac{1}{2}p_L^2}\, q^{\frac{1}{2}p_R^2} \quad , \quad \tau=\tau_1+i\tau_2\in\mathcal{H}\,, \tag{5}$$

which is related to the one in (1) in an obvious way. Perform a Poisson resummation over m_i, to obtain an expression of the form

$$Z_{T^d}(\tau,\bar\tau) = \sum_{k,n\,\in\Lambda} e^{-S_{k,n}}\,. \tag{6}$$

Identify the form of $S_{k,n}$. This is the path-integral computation of the partition function.

(Literature: section 7.6 of `Les Houches Lectures on Fields, Strings and Duality`, arXiv:hep-th/9703136)