Exercise 1 – The modular group $SL(2,\mathbb{Z})$

The special linear group $SL(2,\mathbb{R})$ acts on the complex upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} | \operatorname{Im}\tau > 0\}$ by fractional linear transformations

$$\tau \mapsto M \cdot \tau = \frac{a\tau + b}{c\tau + d} \quad , \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \; . \tag{1}$$

a) Verify:

1. $M \cdot \tau \in \mathcal{H} \quad \forall M \in SL(2,\mathbb{R}) \,, \tau \in \mathcal{H}$
2. $(M_1 M_2) \cdot \tau = M_1 \cdot (M_2 \cdot \tau) \quad \forall M_1, M_2 \in SL(2,\mathbb{R}) \,, \tau \in \mathcal{H}$

b) The set $\bar{\mathcal{F}} = \{\tau \in \mathbb{C} | \quad |\tau| \geq 1, |\operatorname{Re}\tau| \leq \frac{1}{2}\}$ denotes the closure of the fundamental domain for the modular group $SL(2,\mathbb{Z})$. Show that any point $\tau \in \mathcal{H}$ can be mapped into $\bar{\mathcal{F}}$ by an application of T- and S-transformations, where

$$T \cdot \tau = \tau + 1 \quad , \quad S \cdot \tau = -1/\tau \; . \tag{2}$$

Exercise 2 – Eisenstein series of weight $k > 2$

Let $k \in \mathbb{Z}$. A modular form of weight k is a function $f : \mathcal{H} \to \mathbb{C}$ that is holomorphic on $\mathcal{H} \cup \{\infty\}$, and that satisfies

$$f(M \cdot \tau) = (c\tau + d)^k f(\tau) \quad \forall \tau \in \mathcal{H} \,, \forall M \in SL(2,\mathbb{Z}) \; . \tag{3}$$

Choosing $M = T$, we have $f(\tau + 1) = f(\tau) \quad \forall \tau \in \mathcal{H}$, and hence f is a periodic function of period 1. Introducing $q = e^{2\pi i \tau}$, it has the Fourier development, valid in the open complex unit disc,

$$f(\tau) = \sum_{n=0}^{\infty} a_n \, q^n \; . \tag{4}$$

Consider the following series for even $k \geq 4$ ($k \in \mathbb{N}$),

$$G_k(\tau) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{m,n \in \mathbb{Z} \text{ with } (m,n) \neq (0,0)} \frac{1}{(m\tau + n)^k} \quad , \quad \tau \in \mathcal{H} \; . \tag{5}$$

This series, called Eisenstein series, is absolutely and locally uniformly convergent for $k > 2$.

a) Show that $G_k(\tau)$ is a modular form of weight k.

b) Show that the Fourier expansion of the Eisenstein series is

$$G_k(\tau) = \frac{(k-1)! \, \zeta(k)}{(2\pi i)^k} + \sum_{n=1}^{\infty} q^n \, \sigma_{k-1}(n) \; , \tag{6}$$

where $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ and where $\sigma_{k-1}(n)$ denotes the sum of the $(k-1)$st power of the positive divisors of n, i.e. $\sigma_{k-1}(n) = \sum_{r|n} r^{k-1}$. *Hint:* Use $\frac{\pi}{\tan(\pi\tau)} = \sum_{m\in\mathbb{Z}} \frac{1}{\tau+m}$, $\tau \in \mathbb{C}\backslash\mathbb{Z}$.

Exercise 3 – The Eisenstein series of weight 2

The Eisenstein series of weight 2 is defined by

$$G_2(\tau) = \frac{\zeta(2)}{(2\pi i)^2} + \sum_{n=1}^{\infty} q^n \, \sigma_1(n) \; . \tag{7}$$

a) Show that $G_2(\tau)$ equals

$$G_2(\tau) = -\frac{1}{4\pi^2} \left[\frac{1}{2} \sum_{n\neq 0} \frac{1}{n^2} + \frac{1}{2} \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^2} \right] , \tag{8}$$

where one first carries out the summation over n, and then over m. Note that this series is not absolutely convergent, and hence one cannot interchange the order of summation to obtain $G_2(-1/\tau) = \tau^2 \, G_2(\tau)$.

b) Define $G_{2,\epsilon}(\tau)$ by

$$G_{2,\epsilon}(\tau) = -\frac{1}{4\pi^2} \left[\frac{1}{2} \sum_{m,n\in\mathbb{Z} \text{ with } (m,n)\neq(0,0)} \frac{1}{(m\tau+n)^2 \, |m\tau+n|^{2\epsilon}} \right] , \tag{9}$$

which converges absolutely. How does it transform under modular transformations $\tau \mapsto M\cdot\tau$ with $M \in SL(2,\mathbb{Z})$?

c) It can be shown that

$$G_2^*(\tau) \equiv \lim_{\epsilon\to 0} G_{2,\epsilon}(\tau) = G_2(\tau) + \frac{1}{8\pi \, \mathrm{Im}\tau} \; . \tag{10}$$

Using this, deduce the transformation behaviour

$$G_2(M\cdot\tau) = (c\tau+d)^2 \, G_2(\tau) + \frac{i}{4\pi} \, c(c\tau+d) \; . \tag{11}$$

c) Show that the Dedekind η function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \tag{12}$$

satisfies

$$\frac{d}{d\tau} \ln \eta(\tau) = -2\pi i \, G_2(\tau) \; . \tag{13}$$

d) Using the above, deduce the transformation law

$$\eta(-1/\tau) = \sqrt{-i\,\tau} \, \eta(\tau) \; . \tag{14}$$