Multiquadrics method for Couette flow of a yield-stress fluid under imposed torques

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Abstract: *A Lagrangian description of the Couette flow between two coaxial cylinders, of a viscoplastic fluid (i.e. exhibiting a yield stress) under imposed torque is presented. Beyond a value of the shear stress, the viscosity variation is approximated by a layering of two fluid regions with different viscosities such that* $\mu_2 / \mu_1 << 1$*. So the rheological behaviour is described by the model of biviscosity which approaches the Bingham model.*
In this work meshless radial basis function method is used to build an approximation of the PDEs governing the Couette flow. The used technique is based on the application of globally multiquadrics radial basis function to compute the velocity field and the free surface separating the two phases.

Introduction

The original formulation of the constitutive equations for viscoplastic materials was given by Oldroyd [5] and then has been studied extensively for different geometry in [2-7]. However very few numerical, theoretical or experimental works have been dedicated to flows of yield-stress fluids which is due to the difficulties bound up with the free surface separating the solid and gel phases.
We study the Couette flow of an viscoplastic fluid that fills the gap between two circular cylinders when a constant torque C is suddenly applied to the inner cylinder, the outer are being kept motionless.
The fluid is assumed to be at rest and present itself like an inelastic material (gel) at initial time.
By a step change of torque follows an angular velocity at the interior cylinder. The fluid layers at the neighbourhood of the wall of the inner cylinder are going to undergo an angular distortion.
When this exceeds a critical value, the mechanical links between chains of polymers are destroyed, and a change of phase is observed: passage of a gel state to the liquid state.

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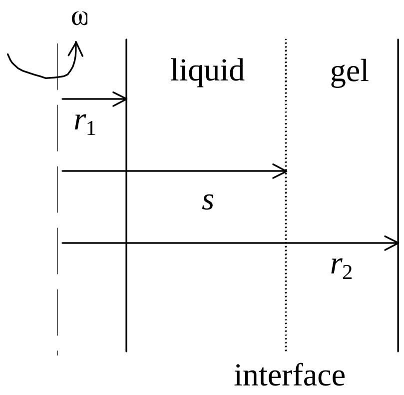

Figure 1 : *Configuration of the problem*

The interface separating the liquid and gel phases is going to evolve in the gap until reaching a stationary state. The situation is sketched in Fig I.
We give here the evolution of the interface and velocity field profiles in the two phases.

Mathematical formulation

The flow is generated by the rotation of the interior cylinder with a small angular velocity. For the biviscosity model, the shear stress is sketched below in Fig II, where τ_s is the yield-stress and τ_c is the critical value. w_1 and w_2 denoted the angular velocity in the liquid phase and in gel phase respectively. Shear stress will be noted τ_l and τ_g.

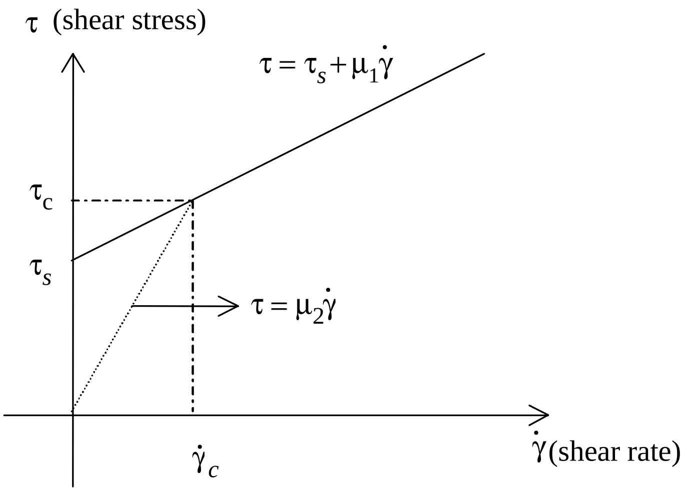

Figure II: *Model of biviscosity*

In what follows we will give the equations governing this type of flow in dimensionless variables, where r is the radial position:

1. Liquid phase

The equations that governs the Liquide phase is

$$r\,\frac{\partial w_1}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_l\right) \qquad \text{for} \quad r_1 < r < s \tag{1}$$

where

$$\tau_l = \tau_s + r\,\frac{\partial w_1}{\partial r} \tag{2}$$

the boundary conditions are as follows:

-At the wall of the inner cylinder ($r = r_1$):

$$I\,\frac{\partial w_1}{\partial t} = C + 2\pi\, r_1^2 L\tau_l \tag{3}$$

where I is the angular momentum and L is a characteristic length of cylinders.

2. Gel phase

For the gel phase we have :

$$r\,\frac{\partial w_2}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_g\right) \qquad \text{for } s < r < r_2 \tag{4}$$

where

$$\tau_g = \mu\, r\,\frac{\partial w_2}{\partial r}\ , \quad \mu = \frac{\mu_1}{\mu_2} \tag{5}$$

With the boundary conditions :
- At the wall of the outer cylinder ($r = r_2$):

$$w_2 = 0 \tag{6}$$

-At the interface liquid-gel ($r = s$), one must have continuity of velocity and shear stress.:

$$w_1 = w_2,\ \tau_l = \tau_s,\ \tau_g = \tau_s. \tag{7}$$

This last relation becomes: $\frac{\partial w_1}{\partial r} = 0$.

By applying the following change of variables in the two phases:

$r = a_1(x - r_1) + r_1$ and $r = a_2(y - r_2) + r_2$, where $a_1 = s - r_1$, $a_2 = r_2 - s$
equation (1) becomes:

$$r\,a_1^2\,\frac{\partial w_1}{\partial t} = \left[3a_1 + a_1\,\dot{a}_1 r(x - r_1)\right]\frac{\partial w_1}{\partial x} + r\,\frac{\partial^2 w_1}{\partial x^2} + \frac{2a_1^2}{r}\tau_s \tag{8}$$

That may be written as :

$$A\,\frac{\partial w_1}{\partial t} = B\frac{\partial w_1}{\partial x} + C\frac{\partial^2 w_1}{\partial x^2} + D \tag{9}$$

While the boundary conditions are:

$$\begin{cases} Ia_1^2 \dfrac{\partial w_1}{\partial t} = a_1^2 \left(C + 2m\tau_s\right) + 2m\ r_1 a_1 \left.\dfrac{\partial w_1}{\partial x}\right|_{x=r_1}, \quad m = \pi r_1^2 L \\ \left.\dfrac{\partial w_1}{\partial x}\right|_{x=r_2} = 0 \end{cases} \tag{10}$$

Therefore w_2 satisfy the system :

$$ra_2^2\ \frac{\partial w_2}{\partial t} = \left[3a_2\,\mu + a_2\ \dot{a}_2\ r(x - r_2)\right]\frac{\partial w_2}{\partial x} + r\frac{\partial^2 w_2}{\partial x^2} \tag{11}$$

That may be written as :

$$P\,\frac{\partial w_2}{\partial t} = Q\frac{\partial w_2}{\partial x} + R\frac{\partial^2 w_2}{\partial x^2} \tag{12}$$

subject to the boundary conditions :

$$\begin{aligned} &\tau_s = \mu s a_2 \left.\frac{\partial w_2}{\partial y}\right|_{y=r_1} \\ &w_2(r_2) = 0 \end{aligned} \tag{13}$$

Continuity of velocity fields, at the interface, gives :

$$w_1(r_2) = w_2(r_1) \tag{14}$$

Solution with Radial basis function method

In this section a implicit scheme is devised using the MQ method to solve the equations (8) and (11) subject to the boundary conditions (10), (13).
Using the notation $w_\alpha^n = w_\alpha\left(r, t^n\right)$, $\alpha = 1, 2$, where $t^{n+1} = t^n + \Delta t$, where Δt is the time step size and following the idea of MQ method [8-10], w_α^n is approximate, at each iteration n, by:

$$w_\alpha^n = \sum_{j=1}^{N} \beta_{\alpha j}^n g_j(x) + \beta_\alpha^{n+1} \tag{15}$$

Where $g_j(x) = g\left(\left|x - x_j\right|\right)$ are the globally multiquadratique function, $g(r) = \left(r^2 + c\right)^{1/2}$ and $c > 0$ is a given shape parameter.

We notice that :

$$\left(\frac{\partial w_\alpha}{\partial x}\right)^n = \sum_{j=1}^{N} \beta_{\alpha j}^n(t) g_j'(x), \qquad \left(\frac{\partial^2 w_\alpha}{\partial x^2}\right)^n = \sum_{j=1}^{N} \beta_{\alpha j}^n(t) g_j''(x) \tag{16}$$

So introducing the form (15) and (16) in the equations (9) and (12) and applying them to N collocation points, we have:

$$A_i\ \frac{w_1^{n+1} - w_1^n}{\Delta t} = B_i \sum_{j=1}^{N} \beta_{1j}^{n+1} g'(x_i) + C_i \sum_{j=1}^{N} \beta_{1j}^{n+1} g''(x_i) + D_i \tag{17}$$

$$P_i \frac{w_2^{n+1} - w_2^n}{\Delta t} = Q_i \sum_{j=1}^{N} \beta_{2j}^{n+1} g'(x_i) + R_i \sum_{j=1}^{N} \beta_{2j}^{n+1} g''(x_i) \tag{18}$$

Additional condition $\sum_{j=1}^{N} \beta_{\alpha j}^{n} = 0$ must be added to equation (17) and (18) to assure the existence and uniquness of the solution.

The interface can be calculated at each time iteration by using the equation (14). The Dichotomie method was used as the optimisation technique for localising the interface position.

Discussion and conclusion

A numerical scheme for computing the velocity field and the interface liquid-gel position of the Couette flow of a yield stress fluid was established. The technique was based on the globally multiquadrics radial basis function. Beside the nonlinearity of the problem arising from the unknown position of the interface, it was also observed that there is a physical singularity at the starting of the flow owed to the imposed torque which complicated more the numerical modelling. Figure III describing the interface location $s(t)$, illustrated this physical singularity and shows the difficulties on convergence of the scheme. It is necessary to note that this curve trend towards a limit position s_l.

For the shape parameter c, we adopt the suggestion from Chon et. al. [9] by choosing c to be a constant times the minimum distance between two collocation points.

It would be easier to implement an algorithm for computing the interface while using an eulerian description of the problem. So it will be necessary to add a kinematical condition carrying on volume fraction.

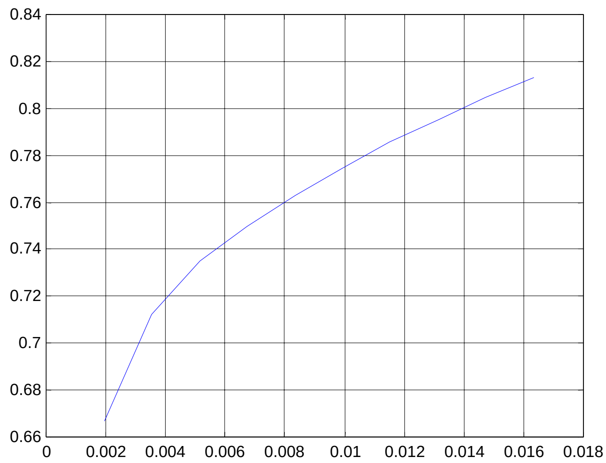

Figure III: *Evolving interface position using 50 collocation points.*

References

[1] Oldroyd J.G. : *A rational formulation of the equations of plastic flow for a Bingham solid. Proc. Camb. Philos. Soc. 43,* 1947.
[2] Alkhatib M.A.M, Wilson S.D.R : *The development of Poiseuille flow of a yield-stress fluid* J. Non-Newtonian .Fluid.Mech,100, pp.1-8, 2001
[3] Bernardin D., Nouar C.: *Transient Couette flows of Oldroyd's viscoelastic fluids under imposed torques,* Journal of Non Newtonian Fluid Mechanics **77,** 1998
[4] Beverly C.R, Tanner R.I: *Numerical analysis of three-dimensional Bingham plastic flow*. J. Non-Newtonian Fluid Mech, 42, 1992
[5] Lipscomb G.C, Denn M.M : *Flow of Bingham fluids in complex geometries.* J. Non-Newtonian .Fluid.Mech,14,. pp. 337-347, 1984
[6] Piau J.M: *Flow of a yield-stress fluid in a long domain. Application to flow on an inclined plane.* J.Rheol., 40, (4), pp. 711-723, 1996
[7] Wilson S.D.R *: The development of Poiseuille flow.* J. Fluid Mech. 38,1969
[8] Kansa E.J : *Multiquadrics- A scattered data approximation scheme with applications to computational dynamics. II Solutions to parabolic, hyperbolic, and elliptic partial differential equations.* Comput. Math. Appl, 19, (6-8), pp. 147-161 1990
[9] Chon Y.C,, Cheung K.F, Mao X.Z, Kansa E.J: *A multiquadric solution for the shallow water equations.* Journal of Hydrology, 125, (5), pp. 524-533 1999
[10] Fasshauer G.E : *Solving partial differential equations by collocation with radial basis functions.* In Surface Fitting and Multiresolution Methods, LeMéhauté A.,Rabut C, Shumaker L, eds Vanderbilt University Press, 1997