Moving Meshless Methods (I): Moving Element Free Petrov-Galerkin Viscous Method

A. R. Soheili[(1)] and M. Ghorbany[(2)]

Abstract: *Moving Meshless methods are new generation of numerical methods for unsteady partial differential equations that have shock, high gradient region, high oscillatory region, boundary layer .... These methods link the Moving Finite Element method (MFE) by Keith Miller to Meshless methods such as, DEM, EFGM, EFPGM, SPH, RKPH, PUM, h-p Clouds,.... Here grid coordinates are variable, time dependent, unknown and are found together with approximate solution of time dependent PDE. This implies, exertion of indirect or implicit equi-distribution of nodes without use of equi-distribution principles with various monitor functions. Weak form system is an ODE and will be found by Galerkin and Petrov-Galerkin method and its solution by finite difference and method of lines give us approximation and nodal coordinates. Proceeding time steps, nodes move smoothly into the high gradient region and concentrates there, for handling the shock and better approximation. A penalty appended to energy functional for preventing high velocity, colliding and collapsing of nodes, prevention of concentration all the nodes in the shock region, controls their motion and also tend to well conditioning of mass matrix. Numerical solution of heat and burger equation, demonstrate the accuracy of the approximation. Among Meshless methods we only use of EFPG method by T.Belytschko and introduce the Moving Element Free Petrov-Galerkin Viscous Method (MEFPGVM) by $\mathcal{C}^2$ cubic hermite base functions as test functions.*
Keywords: r-refinement, adaptive grids, moving finite element, element free Galerkin and Petrov-Galerkin method.

1 Introduction

Numerical solution of partial differential equations with shock are subject of researchers for many years. Up to this time, there is not any important work on steady shock problems but main works were concentrated on time dependent PDE's. These problems have a wild small region in which the solution have not good activity and this region move with time. Approximation of this region need special techniques. Up to this time,there are two main methods for numerical solution of time dependent PDE's (1) moving mesh methods and (2) moving finite element methods on time dependent or movable nodes. Progressing

[1]Sistan And Baluchestan University, Zahedan,Iran, Address (soheili@hamoon.usb.ac.ir).
[2]Sistan And Baluchestan University, Zahedan,Iran, Address (ghorbany@hamoon.usb.ac.ir).

the meshless methods by finite element people, more of these problems approximated by these methods with some advantages. Two main advantageous of meshless methods are: (1) computational efficiency by deletion the mesh generation process, (2) flexibility in raising the smoothing degree of the approximation, and saving some finite element property such as the locality and easy work on problems with complex region. In 1968 Donald Shepard [13] gave his famous paper, where he presented a new interpolation for irregularly spaced data points. Shepard interpolation was the beginning point for moving least square (MLS) interpolation method given by Lancaster and Salkauskas [9] and Partition of Unity method given by Babuska and Melenk [1]. MLS method is generalization of Shepard interpolation and tend to meshless interpolation and meshless approximation or quasi-interpolation (without Kronecker Delta property). After introducing MLS method, in 1992 Nayroles [12] employed this method for numerical solution of some PDE's using local compact support nonsingular weight functions and got almost good results in spite of important disadvantageous in finding complete derivative of his base functions. His method named Diffuse Element Method (DEM) retain (1) locality of finite element (2) raise smoothing degree of approximation (3) delete the mesh generation time consuming process (4) base functions derivatives are not complete (5) essential boundary condition can not satisfied (6) approximation are based on irregular distribution of nodes (7) approximation smoothing degree directly related to smoothing degree of weight. In 1994, Ted Belytschko and his colleagues [2] generalized DEM and presented element free Galerkin (EFG) method. EFG method has high accuracy in doing on PDE's with shock and more computational task than DEM. In this paper we linked MFE method by Keith Miller [5], [6] and EFPG method by Ted Belytschko [2], [7]. In fact instead of usual piecewise linear hat function in MFE method we employed the base functions built by EFG as approximation or trial functions, and piecewise $\mathcal{C}^2$ cubic hermite base functions as test or weight functions. Penalty part of the MFE are appended to this method because: (1) prevention of colliding and collapsing of the nodes, (2) prevention of concentration of all the nodes in the shock region, (3) prevention of high velocity of nodes, (4) prevention of singularity of mass matrix and making, it to be positive definite, (5) better condition number of mass matrix. Penalty parameters tend us to select Moving Element Free Petrov- Galerkin Viscous Method (MEFPGVM) title for our paper. The paper is as follows: section one will explain MFE method, in section two we will explain EFG and EFPG method and in the next section we will explain our method the hybrid of MFE and EFPG, heat equation and Burger equation will be approximated by this method in section four and concluding remarks will finish this paper.

2 Moving Finite Element

In 1981 Keith Miller, [5], [6] gave his famous paper on 1-D Moving Finite Elements *i.e.* piecewise linear finite elements on unknown, unsteady, time dependent and moveable nodes. He had some expectation of his method and almost succeeded. In 1983 Herbst [4] and his co-worker proved implicit existence and type of equi-distribution in 1-D moving piecewise linear finite element method. Then in 1986, Mitchell [11] employ piecewise $\mathcal{C}^2$ cubic hermite instead of usual approximation base function as test functions, *i.e.* in his method the residual is orthogonal to cubic Hermite base functions with compact support.

Our test example in general form is:

$$u_t = \mathbf{A}u \tag{1}$$

where operator **A** contain only space derivative. For 1-D Burger equation $\mathbf{A}u = -uu_x + (1/R)u_{xx} = -(u^2/2)_x + (1/R)u_{xx} \quad (R \gg 1)$ and for 1-D heat equation $\mathbf{A}u = u_{xx}$ where $0 < x < 1$. We have the node distribution: $0 = x_0(t) < x_1(t) < \ldots < x_{n-1}(t) < x_n(t) = 1$. Assume we have the following approximation

$$\tilde{u}(x,t) = \sum_{j=0}^{n} u_j(t)\phi_j(x,t) \tag{2}$$

where $\{(x_j(t), u_j(t))\}_{j=0}^{n}$ are unknowns. In MFE method the set $\{\phi_j(x,t)\}_{j=0}^{n}$ are usual hat functions on moving nodes and in MEFPGVM instead of hat functions we use of base functions built by EFG method (see section (3)). In fact $\tilde{u}(x,t) \equiv \tilde{u}(t)$, so u_t the partial time derivative of u, will be

$$\tilde{u}_t(x,t) = \sum_{j=0}^{n} (\dot{u}_j(t)\phi_j(x,t) + \dot{x}_j(t)\psi_j(x,t)). \tag{3}$$

dot denotes time derivative and

$$\psi_j(x,t) = \partial\tilde{u}/\partial x_j = -\tilde{u}_x\phi_j(x,t) \tag{4}$$

(see [5], [6]). Under some conditions the base functions set or nonlinear manifold $\{\phi_j(x,t), \psi_j(x,t)\}_{j=0}^{n}$, (see [5], [6]) are linear independent. After substitution equation (2) into PDE (1), the residual will be

$$\mathbf{R} \equiv \sum_{j=0}^{n} \dot{u}_j(t)\phi_j(x,t) + \dot{x}_j(t)\psi_j(x,t) - \mathbf{A}\tilde{u}(x,t) \not\equiv 0 \tag{5}$$

By the Galerkin method *i.e.* the following inner product:

$$\begin{aligned} (\mathbf{R}, \phi_i(x,t)) &= 0 \\ (\mathbf{R}, \psi_i(x,t)) &= 0, \qquad i = 0,1,\ldots,n \end{aligned} \tag{6}$$

or by the Petrov-Galerkin method the ODE system will be found.

$$\begin{aligned} (\mathbf{R}, S_i(x,t)) &= 0 \\ (\mathbf{R}, T_i(x,t)) &= 0, \qquad i = 0,1,\ldots,n \end{aligned} \tag{7}$$

the set $\{S_j(x,t)\}_{j=0}^{n}$ and $\{T_j(x,t)\}_{j=0}^{n}$ are cubic hermite base function in the following form:

$$\begin{aligned} S_i(x,t) &= \phi_i^2(x,t)(3 - 2\phi_i(x,t)) \\ T_i(x,t) &= \phi_i^2(x,t)(x - x_i(t)) \qquad i = 0,1,\ldots,n \end{aligned} \tag{8}$$

where $\phi_i(x,t)$'s are hat functions. In reference [4] Herbst showed existence of equi-distribution principle in MFE method. In Galerkin method the following equi-distribution relations are satisfied,

$$(x_i(t) - x_{i-1}(t))u_{xx}(x_i^-, t) = (x_{i+1}(t) - x_i(t))u_{xx}(x_i^+, t) + O(h^2), \quad i = 1,2,\ldots,n-1 \tag{9}$$

and for Petrov-Galerkin method using cubic Hermite base functions (8) as test functions, the following equi-distribution principle are satisfied,

$$(x_i(t)-x_{i-1}(t))^2\tilde{u}_{xx}(x_i^-,t)=(x_{i+1}(t)-x_i(t))^2\tilde{u}_{xx}(x_i^+,t)+O(h^3), \quad i=1,2,\dots,n-1 \qquad (10)$$

Final ODE system is in the following form

$$\underline{\mathbf{A}}(t)\dot{\underline{\mathbf{u}}}(t)=\underline{\mathbf{u}}(t) \qquad (11)$$

where $\underline{\mathbf{A}}(t)$ is a tridiagonal mass matrix, $\underline{\mathbf{u}}(t)=[u_1(t),x_1(t),u_2(t),x_2(t),\dots,u_{n-1}(t),x_{n-1}(t)]^T$ and dot denotes time derivative. This dynamical system can be solved by method of lines and various ODE method. [8]

3 Element Free Galerkin and Petrov-Galerkin Method

Let $\mathbf{P}^T(x)=\{1,x,x^2,\dots,x^{(m-1)}\}$ be a 1-D m dimensional polynomial base set. Local approximation on fixed point $y\in(0,1)$ is defined by:

$$\tilde{u}_y(x,t)=\mathbf{P}^T(x).\mathbf{a}(y,t). \qquad (12)$$

The coefficient vector $\mathbf{a}(y,t)=[a_1(y,t),\dots,a_m(y,t)]^T$ is unknown and are found by MLS method [9]. This method needs a suitable weight function. Sometime we can say that this weight is an approximation of Kronecker Delta function. Some of the singular and nonsingular weight functions in standard or radial form are:

- $w(x)=|x|^{-2}$
- $w(x)=\begin{cases}(1-|x/r|^{2\times k_1})^{2\times k_2} & |x|<r\\ 0 & otherwise\end{cases}$
  where k_1,k_2 are integers and r is radius of support.
- $w(x)=\exp^{(-\lambda|x|^{2k})}$, λ is a parameter and k is an integer.
- $w(x)=\begin{cases}2/3-4|x|^2+4|x|^3 & |x|\le 1/2\\ 4/3-4|x|+4|x|^2-4/3|x|^3 & 1/2<|x|\le 1\\ 0 & |x|>1\end{cases}$
  this weight is cubic spline

(see [9] for singular and [12] for nonsingular). Singular weight function tend to interpolation, but have difficulties in computational working with. Nonsingular weight functions haven't interpolation property but can build good approximation, however lack of interpolation property by using nonsingular weights, will show itself on exertion of boundary condition and need some methods for handling and correcting it such as Lagrange multiplier [2] and penalty methods [10]. *Accuracy of a meshless method is directly depend on selected weight function* . Minimization of the following discrete weighted local error functional around a fixed point $y\in(0,1)$ by MLS method (see [9])

$$\begin{aligned}\mathbf{J}(\mathbf{a}(\mathbf{y},\mathbf{t})) &= \|u(.,t)-\tilde{u}_y(.,t)\|_w^2 \qquad (13)\\ &= \sum_{j=0}^{n} w(y-x_j(t))(u_j(t)-\tilde{u}_y(x_j(t),t))^2\\ &= \sum_{j=0}^{n} w(y-x_j(t))(u_j(t)-\mathbf{P}^T(x_j(t)).\mathbf{a}(y,t))^2\end{aligned}$$

with respect to coefficient vector $\mathbf{a}(y,t)$ will tend us to the following system

$$\mathbf{A}(y,t)\mathbf{a}(y,t) = \mathbf{F}(y,t).\mathbf{U}(t) \tag{14}$$

where

$$\begin{aligned}
\mathbf{A}(y,t) &= \mathbf{B}(t)\mathbf{W}(y,t)\mathbf{B}(t)^T \\
\mathbf{F}(y,t) &= \mathbf{B}(t)\mathbf{W}(y,t) \\
\mathbf{B}(t) &= \{x_j(t)^{i-1}\} \quad i = 1,2,\ldots,m \quad j = 0,1,\ldots,n \\
\mathbf{W}(y,t) &= diag(w(y-x_0(t)), w(y-x_1(t)),\ldots,w(y-x_n(t))) \\
\mathbf{U}(t) &= [u(x_0(t)), u(x_1(t)),\ldots,u(x_n(t))]^T
\end{aligned}$$

then the local approximation (12) will change in the following form

$$\tilde{u}_y(x,t) = \mathbf{\Phi}_y^T(x,t).\mathbf{U}(t) \tag{15}$$

and the global approximation will be

$$\tilde{u}(x,t) = \Phi^T(x,t).\mathbf{U}(t) = \sum_{j=0}^{n} \phi_j(x,t)u_j(t). \tag{16}$$

where the vector base function is:

$$\Phi^T(x,t) = \mathbf{P}^T(x).\mathbf{A}^{-1}(x,t).\mathbf{F}(x,t). \tag{17}$$

and $\phi_j(x,t),\ j = 0,1,\ldots,n$ are elements of it. Smoothing degree of this shape function directly related to smoothing degree of weight function, so we can find first derivative of base function in the following form:

$$\Phi'(x) = (\mathbf{P}^T)'(x)\mathbf{A}^{-1}(x)\mathbf{F}(x) + \mathbf{P}^T(x)\mathbf{A}^{-1}(x)(\mathbf{F}'(x) + -\mathbf{A}'(x).\mathbf{A}^{-1}(x)\mathbf{F}(x)) \tag{18}$$

(see [2], [3]). Here we can not see explicit form of base functions or its derivative, but we can use of them in our work and draw it. Interpolation property is not satisfied or bases have not Kronecker Delta property. Here in moving element free Petrov-Galerkin (EFPG) method we employ the EFG base functions for approximating the problem as the trial or approximation functions and piecewise $\mathcal{C}^2$ cubic hermite as the test functions.

4 Moving Element Free Petrov-Galerkin Method

Combination of the previous two methods *i.e.* MFE and EFG with piecewise $\mathcal{C}^2$ cubic hermite as the test functions, make new powerful and flexible generation for numerical solution of time dependent PDE's. Let the approximation (16) in PDE (1), then the $\mathcal{L}^2(0,1)$ norm residual functional by EFG base function with penalty and regularizing term for node movement will be built.

$$\mathbf{J}(t) = \|\mathbf{R}\|^2_{\mathbf{L}^2(0,1)} + \sum_{j=0}^{n} (\varepsilon_j(\dot{x}_{j+1}(t) - \dot{x}_j(t)) - \eta_j)^2 \tag{19}$$

where residual $\mathbf{R}$ is similar to (5) in which $\{\phi_j(x,t)\}_{j=0}^n$ was built by EFG method and are elements of (17) and $\psi_j(x,t) \equiv \partial\tilde{u}(x,t)/\partial x_j$, ε_j and η_j are experimental small positive constants. ε_j is viscosity parameters and η_j try to take care the first term of penalty near it (see [5] and [6] [4]). Minimizing this functional with respect to $\dot{x}_j(t)$, $\dot{u}_j(t)$, $j = 0, 1, \ldots, n$, will give us an ODE system with time derivative similar to (11). This method is MEFGVM and weighted residual form of this functional by cubic hermite weights with regularizing terms lead us to Moving Element Free Petrov-Galerkin with viscosity method or MEFPGVM.

5 Numerical Examples

We want to show power of the method by approximating the following two 1-D example:

Example 1 1-*D heat equation*

$$u_t = u_{xx} \qquad 0 < x < 1$$
$$u(0,t) = u(1,t) = 0, \quad u(x,0) = \sin(\pi x)$$

with its exact solution, $u(x,t) = \sin(\pi x)\exp(-\pi^2 t)$. *Figure (2) shows approximation, error and x motion of nodes with respect to time. Here viscosity parameter is equal to* 0.2

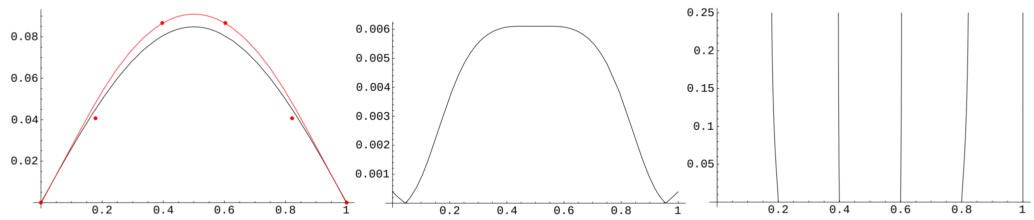

Figure 1: Approximation, error and x motion respectively

Example 2 1-*D Burger equation*

$$u_t + (u^2/2)_x = (1/R)u_{xx}, \quad 0 < x < 1$$

where $u(x,t) = (\mu + \lambda + (\mu - \lambda)\exp(\lambda\xi/\varepsilon)/(1 + \exp(\lambda\xi/\varepsilon))$ *is exact solution.* $\xi = x - \mu t - \beta$ *and here* $\lambda = 0.4, \beta = 0.16, \mu = 0.5$ *and* $R \gg 1$ *is a constant. Here* $R = 15$, *time step is* 0.0025, *final time is* 0.51 *Sec. Figure (2) shows both approximation and exact plot, error plot or* $|\tilde{u}(x, 0.51) - u(x, 0.51)|$ *and motion of x coordinates with respect to time.*

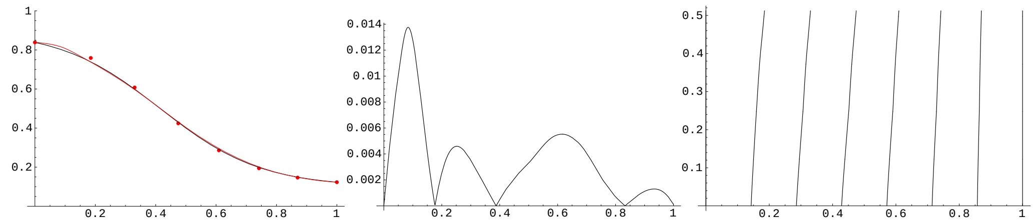

Figure 2: Approximation, error and x motion respectively

6 Conclusion

- This method can be extended to another meshless method.
- Mathematical analysis of this method similar to other meshless methods are meager.
- In this method, equi-distribution principle maybe exist and is a good work
- Computational volume is almost high but can be optimized
- Use of $\mathcal{C}^2$ piecewise cubic hermite, needs a typical mesh of elements in the domain. This is a disadvantage
- Computation of matrix inverse in EFG method is time consuming with errors
- Exertion of essential boundary conditions needs another techniques for handling

References

[1] I. Babuska, J. M. Melenk, ”The Partition of Unity Method”, June 5, 1996

[2] T. Belytschko,Y. Y. Lu,L. Gu, ”Element-Free Galerkin methods”,Int. J. Numr. Meth. Eng., 37, pp:229-256

[3] C. Armando Duarte, ”A Review of Some Meshless Methods to Solve Partial Differential Equations ”,1995 , TICAM Report 95-06

[4] B. M. Herbst, S. W. Schoombie, A. R. Mitchell, ” Equidistributing principles in moving finite element methods ”, J. Comp. Appl. Math. 9, 1983, pp377-389

[5] Keith Miller, Robert Miller, ” Moving Finite Elements. I ”, SIAM J. Numer. Anal., Vol. 18, No. 6, December 1981

[6] Keith Miller, ” Moving Finite Elements. II ”, SIAM J. Numer. Anal., Vol.18, No. 6, December 1981

[7] Y. Krongauz, T. Belytschko, ”A Petrov-Galerkin Diffuse Element Method and its comparison to EFG ”, Comp. Mech. 19 (1997), pp:327-333

[8] J. D. Lambert, ”Numerical Methods For Ordinary Differential Systems”, 1991, John Wiley & Sons Ltd.

[9] P. Lancaster, K. Salkauskas, ”Surfaces Generated by Moving Least Squares Methods”, Mathematics of Computation, Vol. 37, N 155, Jul. 1981

[10] K. H. Leem, S. Oliveira, D.E. Stwart, ” Some Numerical Results from Meshless Linear Systems ”, $http://www.math.uiowa.edu/ftp/comp-math-rep/report-140.pdf$

[11] A. R. Mitchel, B. M. Herbst, ” Adaptive Grids in Petrov Galerkin Computations ”, Accuracy estimates in finite element computations, Edited by I. Babuška, O. C. Zienkiewicsz, J. Gago. and E. R. de A. Oliveira

[12] B. Nayroles, G. Touzot, P. Villon, ”Generalizing the finite element method: Diffuse approximation and diffuse elements ”,Comp. Mech. (1992) 10, pp:307-318

[13] Donald Shepard, ”A two-dimensional interpolation function for irregularly-spaced data ”, Proceeding-1968, ACM National Conference