What do heating your living room, financial investments, and image processing have in common?

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There is a striking resemblance on the modeling of

- heat &
- option prices in Finance

In both cases the basic object is the same: "the Laplacian" after

Pierre-Simon, marquis de Laplace (1749-1827)

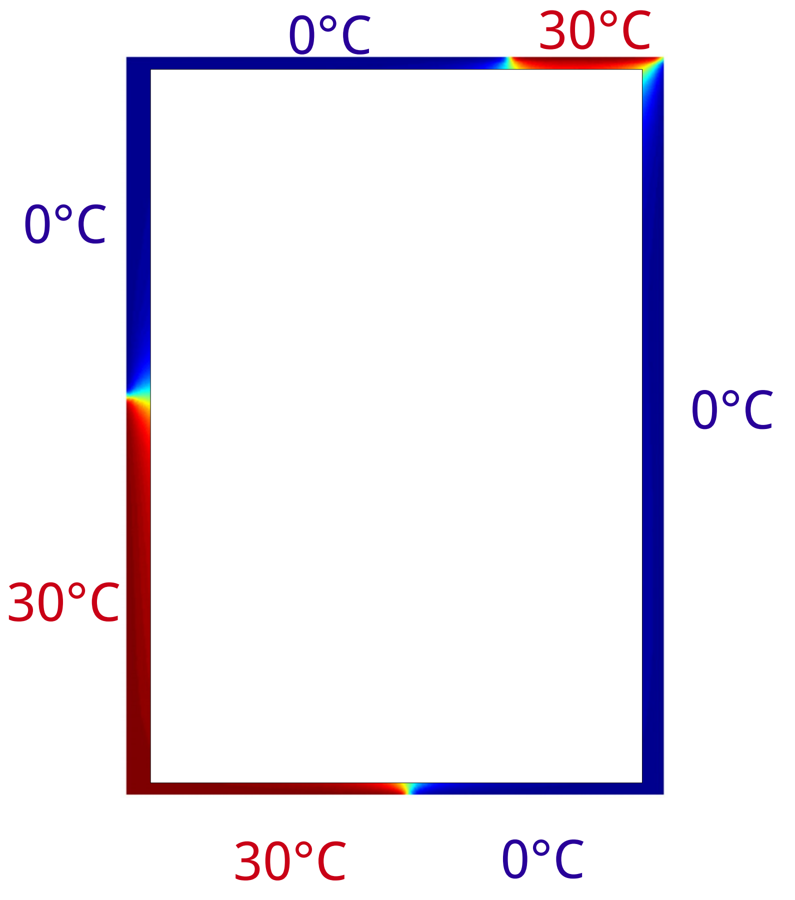
It is responsible for many phenomena in our lives

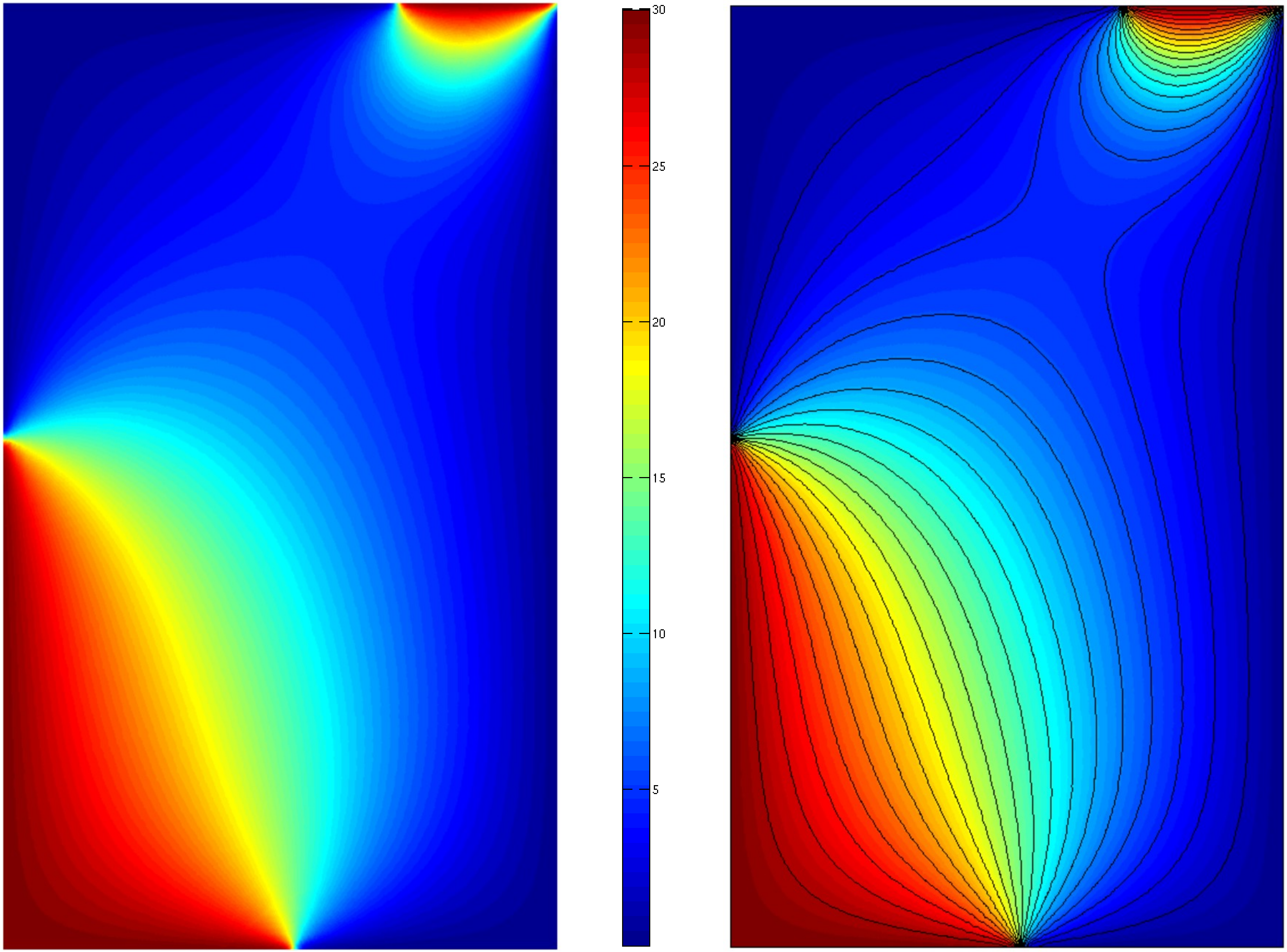
- A first example:

  what is the **temperature** of a certain tile in your living room's floor,

  long after you turn on the wall radiators at 30°C

  while the remaining of the walls are always kept at 0°C ?

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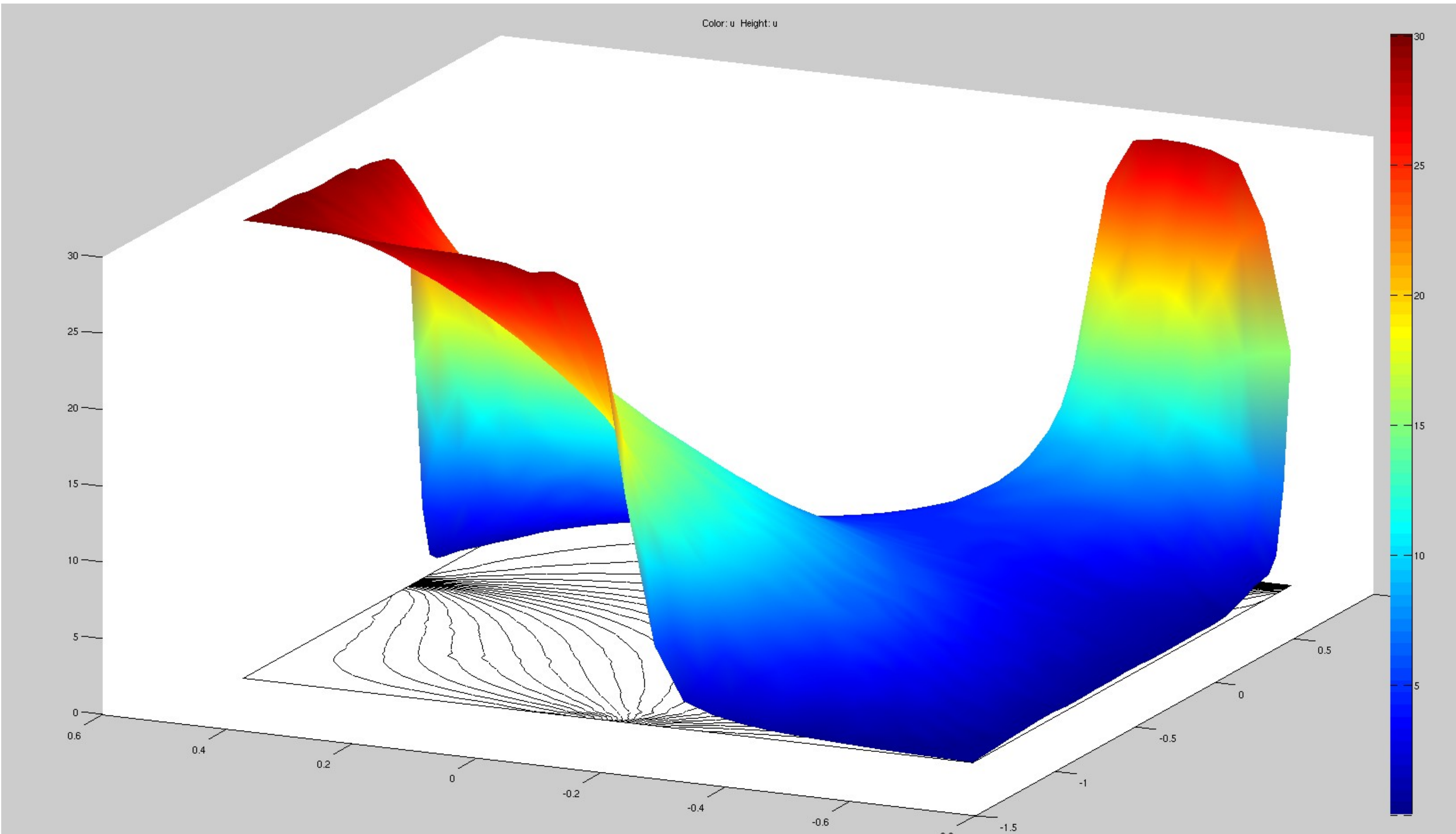

Color: u Height: u
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0.5
0
-0.5
-1
-1.5
30
25
20
15
10
5

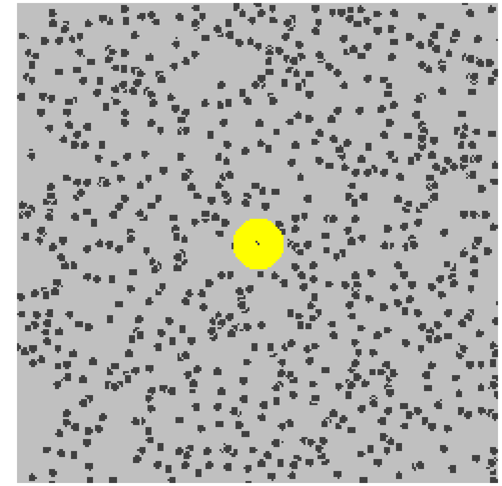
- A second question (on **images**) :
  which color (red level) would you give to the **missing pixels** ?

red
level = 0

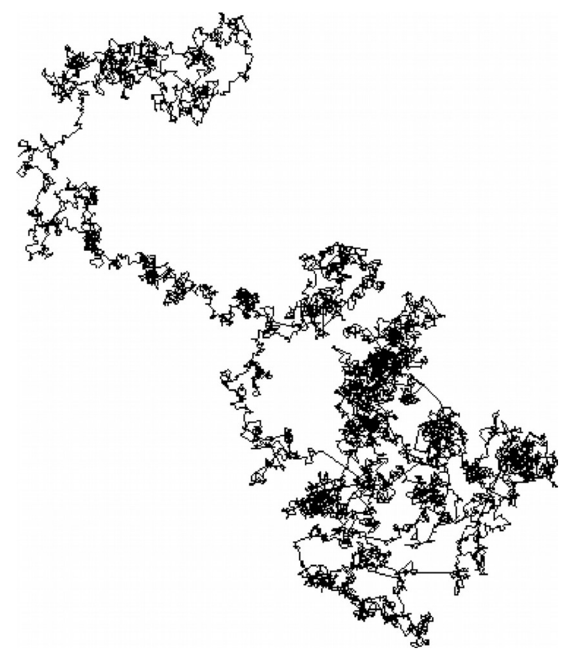
red
level = 30

Robert Brown (1773-1858), biologist

Looking through a microscope at pollen grains in water, he noted that the grains moved randomly through the water



BROWNIAN MOTION

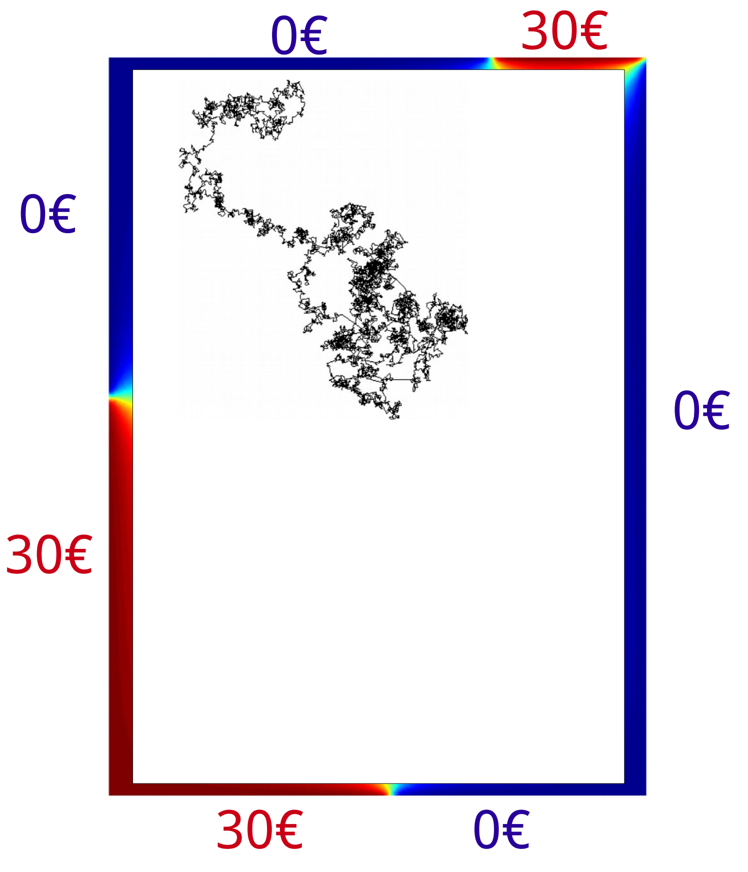


Think also on a large plastic beach ball on the stands of a stadium totally full of people

- A third question (of **finance** type):

what is your **expected gain** when,

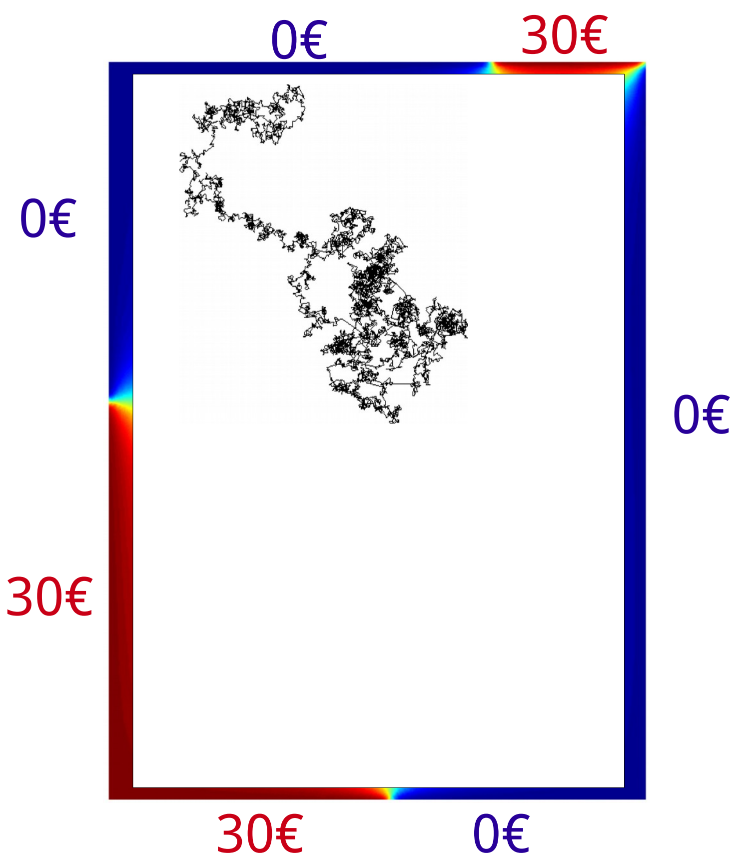
starting always from the same given tile in your living room, you walk randomly and you get 30€ only when you hit a radiator on the first time that you hit your living room's walls (otherwise you get 0€)?

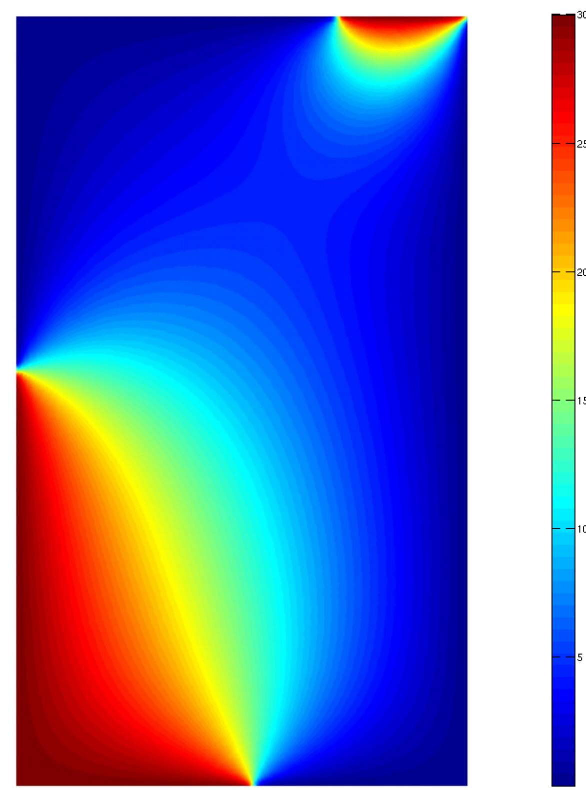


- A third question (of **finance** type)

ANSWER: at every point one has

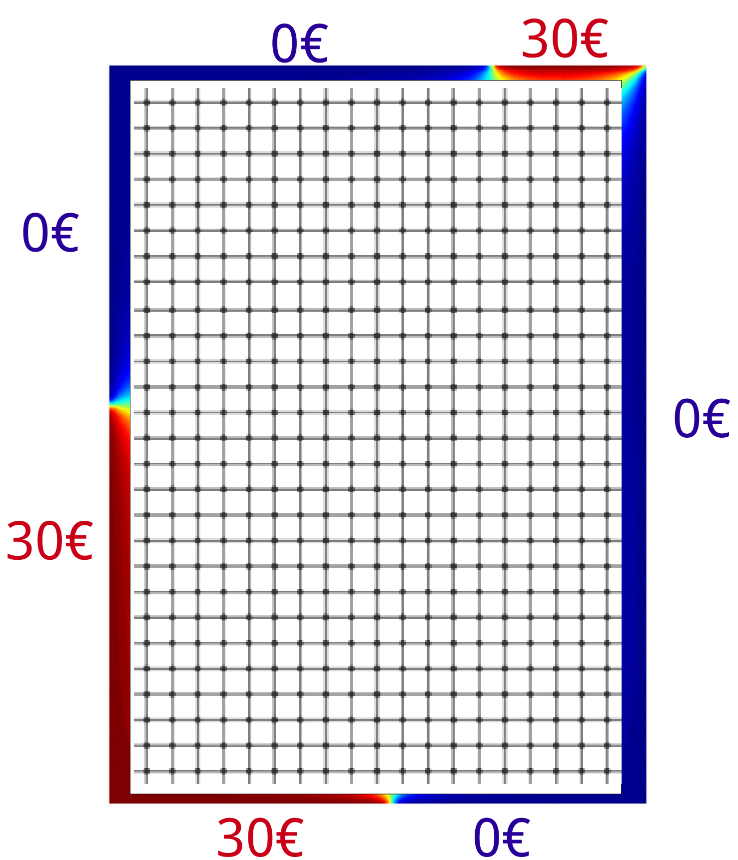
expected gain = temperature !!





How to solve the problem:

- make a **squared lattice** of very small **step-size h**
- Move from a point to either **East, West, North, or South,** each one with **probability 1/4**



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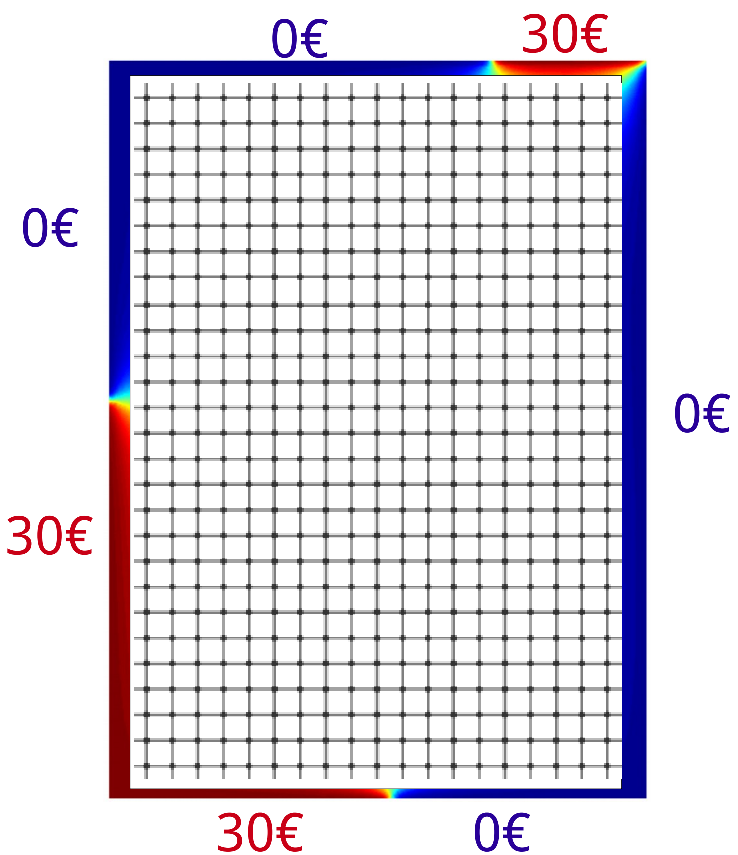
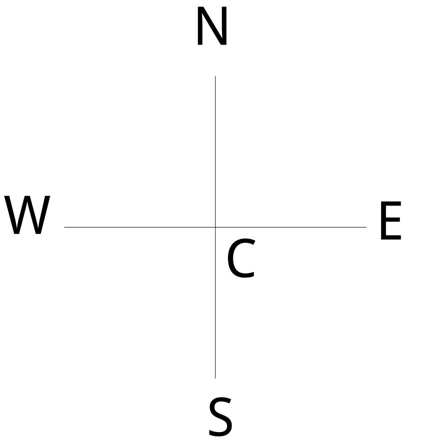




C = starting point of the walk

u(C) = expected gain starting from C

$$u(C) = \frac{1}{4}\{u(E) + u(W) + u(N) + u(S)\}$$

(average)

Some math :

h = step size of the lattice

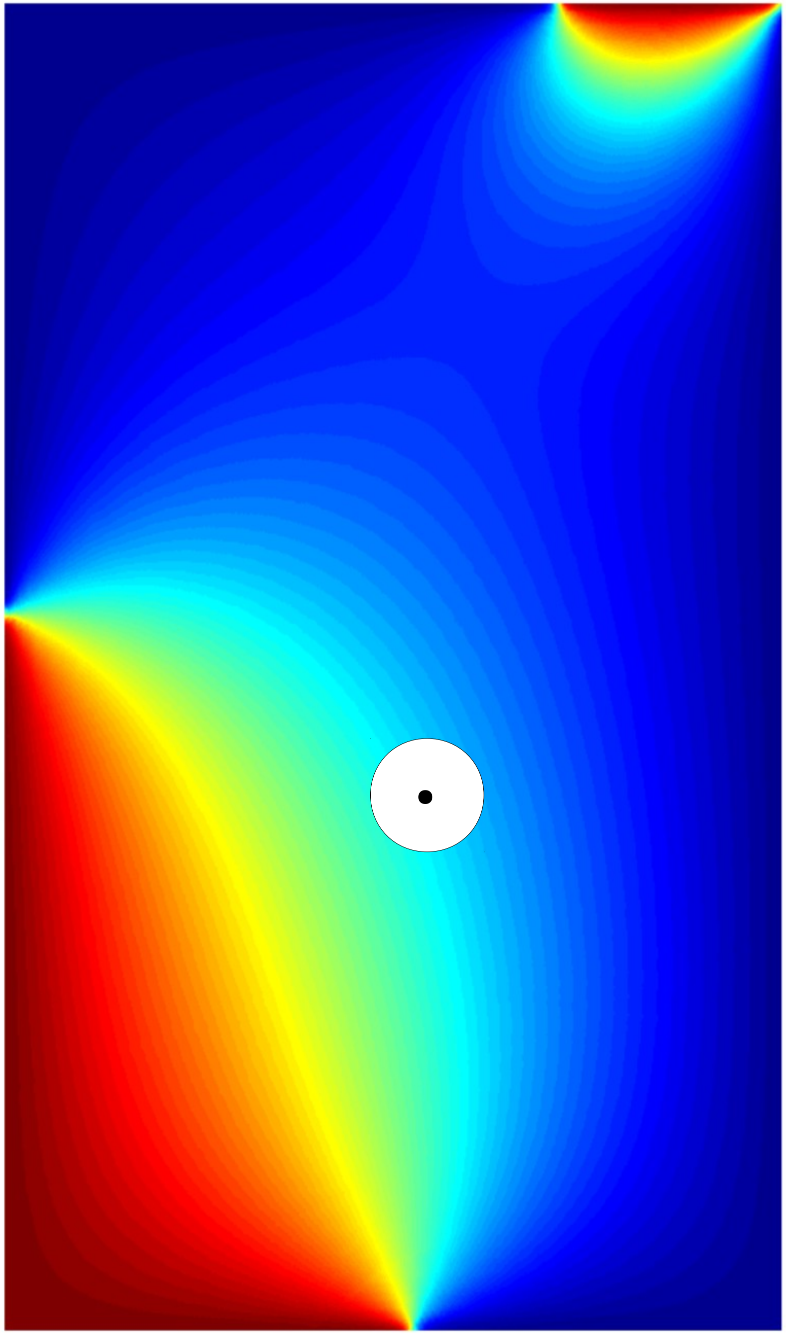
$$u(C) = \frac{1}{4}\{u(E) + u(W) + u(N) + u(S)\}$$

$$u(x,y) = \frac{1}{4}\{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)\}$$

$$\frac{u(x+h,y) + u(x-h,y) - 2u(x,y)}{h^2} + \\ + \frac{u(x,y+h) + u(x,y-h) - 2u(x,y)}{h^2} = 0$$

$$\Delta u\,(x,y) = (\,\partial_{xx}u + \partial_{yy}u\,)\;(x,y) = 0$$

The LAPLACIAN of u = 0



Harmonic functions are characterized by **the mean value property** :

The value of the function
at the center of any circle
=
the **average** of the values
of the function on the circle

OK with HEAT,
and with EXPECTED GAIN !

$$\Delta u = \partial_{xx} u + \partial_{yy} u = 0$$

is called Laplace equation

It is a **Partial Differential Equation** (a PDE)
(also called the equations of Mathematical Physics)

Its solutions are called “harmonic functions”. Together with solutions of the heat or diffusion equation

$$\partial_t u - \Delta u = 0$$

(and other equations of the same type), they model:

- heat (Fourier and Einstein)
- option prices in Finance
- gravitational and electric potentials (Laplace)
- densities of biological or chemical species

Partial Differential Equations. Types :

1. Elliptic : Laplace equation: $\Delta u = \partial_{xx} u + \partial_{yy} u = 0$

2. Parabolic :

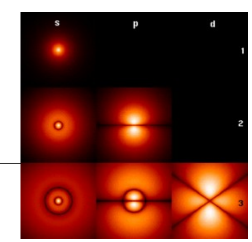
- **Heat** or diffusion equation: $\partial_t u - \Delta u = 0$
- Navier-Stokes (or 1 million $) equations (incompressible viscous **fluids**)

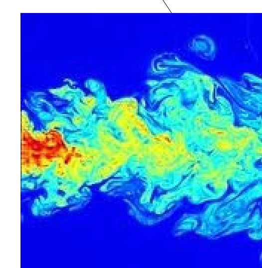
$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot D\mathbf{u} - \nu \Delta \mathbf{u} = -Dp \\ \text{div } \mathbf{u} = 0 \end{cases}$$

3. Hyperbolic :

- **Wave** equation (acoustics, sound-waves) $\partial_{tt} u - \Delta u = 0$

- Schrödinger equation (quantum mechanics) $i\partial_t u + \Delta u = 0$

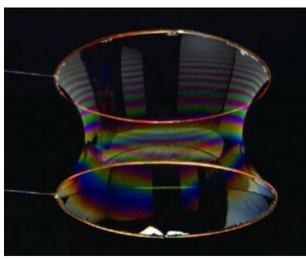




- Euler's equations (incompressible **fluids**)

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot D\mathbf{u} = -Dp \\ \text{div } \mathbf{u} = 0 \end{cases}$$

Some other important PDEs:



a. Linear equations.

1. Laplace's equations: $\Delta u = 0$
2. Helmholtz's equation (involves eigenvalues): $-\Delta u = \lambda u$
3. First-order linear transport equation: $u_t + c\, u_x = 0$
4. Heat or diffusion equation: $u_t - \Delta u = 0$
5. Schrödinger's equation: $i\, u_t + \Delta u = 0$
6. Wave equation: $u_{tt} - c^2\, \Delta u = 0$
7. Telegraph equation: $u_{tt} + d\, u_t - u_{xx} = 0$

b. Nonlinear equations.

1. Eikonal equation: $|Du| = 1$
2. Nonlinear Poisson equation: $-\Delta u = f(u)$
3. Burgers' equation: $u_t + u\, u_x = 0$
4. Minimal surface equation: $\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right) = 0$
5. Monge-Ampère equation: $\det(D^2 u) = f$
6. Korteweg-deVries equation (KdV): $u_t + u\, u_x + u_{xxx} = 0$
7. Reaction-diffusion equation: $u_t - \Delta u = f(u)$

c. System of partial differential equations.

1. Evolution equation of linear elasticity: $u_{tt} - \mu\Delta u - (\lambda+\mu)D(\operatorname{div} u) = 0$
2. System of conservation laws: $u_t + \operatorname{div} F(u) = 0$
3. Maxwell's equations in vaccum: $\begin{cases} \operatorname{curl} E = -B_t \\ \operatorname{curl} B = \mu_0\varepsilon_0 E_t \\ \operatorname{div} B = \operatorname{div} E = 0 \end{cases}$
4. Reaction-diffusion system: $u_t - \Delta u = f(u)$
5. Euler's equations for incompressible, inviscid fluid: $\begin{cases} u_t + u\cdot Du = -Dp \\ \operatorname{div} u = 0 \end{cases}$
6. Navier-Stokes equations for incompressible viscous fluid: $\begin{cases} u_t + u\cdot Du - \Delta u = -Dp \\ \operatorname{div} u = 0 \end{cases}$