Counting noncrossing partitions via Catalan triangles

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Riordan arrays

- The Riordan arrays and the Riordan group were introduced in 1991 by **Shapiro, Getu, and Woan** as lower triangular matrices generated by two formal power series.
- The study of Riordan arrays dates back to
  - the forties with **Jabotinsky**'s work on Bell-type Riordan arrays,
  - the seventies with **Shapiro**'s Catalan triangle and with **Rogers**' introduction of *renewal arrays* as matrices with arithmetic properties similar to those of the Pascal triangle,
  - the eighties when **Barnabei, Brini, and Nicoletti** defined a very general family of *recursive matrices* whose entries are suitably extracted from a pair of formal Laurent series.

Some applications of Riordan arrays

- Applications to combinatorial sums and identities were given by **Sprugnoli**, by **Merlini, Sprugnoli, and Verri** and by **Luzón, Merlini, Morón, and Sprugnoli**.
- Riordan arrays were characterized by **Rogers**, by **He and Sprugnoli**, and by **Merlini, Rogers, Sprugnoli, and Verri**.
- The Riordan subgroups and related concepts were studied by **Shapiro**, by **Peart and Woan**, by **He, Hsu, and Shiue**, by **Cheon, Kim, and Shapiro**.
- For applications to enumerative problems see, e.g., **Baccherini, Merlini, and Sprugnoli** and **Merlini and Verri**.

Riordan arrays – formal definition [Shapiro, Seyoum Getu, Wen-Jin Woan, and Woodson, 1991]

- Consider two formal power series:

  $d(z) = 1 + d_1 z + d_2 z^2 + \cdots$

  $h(z) = h_1 z + h_2 z^2 + \cdots$, where $h_1 \neq 0$.

- Riordan array $\boldsymbol{R} = (d(z), h(z))$:

$$\boldsymbol{R}_{n,k} = [z^n]\, d(z)\, h(z)^k ,$$

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ d_1 & d_1 & 0 & 0 & 0 & \ldots \\ d_2 & h_2 + h_1 d_1 & h_1^2 & 0 & 0 & \ldots \\ d_3 & h_3 + h_2 d_1 + h_1 d_2 & h_1^2 d_1 & h_1^3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

Elementary recursive rule

If $h(z) = h_1 z + h_2 z^2 + h_3 z^3 + \cdots$ then the identity above leads to the following straightforward recursive rule:

$$\begin{aligned} \boldsymbol{R}_{n,k+1} &= [z^n]d(z)h(z)^{k+1} \\ &= [z^n]d(z)h(z)^k \sum_{i=1}^{n-k} h_i z^i \\ &= \sum_{i=1}^{n-k} [z^{n-i}]d(z)h(z)^k h_i \\ &= \sum_{i=k}^{n-1} [z^i]d(z)h(z)^k h_{n-i} \\ &= \sum_{i=k}^{n-1} \boldsymbol{R}_{i,k} h_{n-i} \,. \end{aligned}$$

Riordan group $\mathfrak{Rio}$

- *Riordan group* $\mathfrak{Rio}$ = set of all Riordan arrays;
- Group axioms:
  - Multiplication = matrix multiplication:
$$(d_1(z), h_1(z))\ (d_2(z), h_2(z)) = (d_1(z)\, d_2(h_1(z)), h_2(h_1(z)))\,.$$
  - Identity: $\boldsymbol{I} = (1, z)$
  - Inverse:
$$(d(z), h(z))^{-1} = (1/d(\bar{h}(z)), \bar{h}(z))\,,$$
where $\bar{h}(h(z)) = h(\bar{h}(z)) = z$.

The A-sequence [Merlini, Rogers, Sprugnoli, and Verri, 1997]

Associated with any Riordan array $\boldsymbol{R} = (d(z), h(z)$ there is a power series $A(z) \in \mathbb{C}[[z]]_1$, called the *A-sequence* of $\boldsymbol{R}$:

$$h(z) = \left(\frac{z}{A(z)}\right)^{\langle -1 \rangle} \quad \text{or, equivalently,} \quad A(z) = \frac{z}{h(z)^{\langle -1 \rangle}}.$$

Lagrange array: $\boldsymbol{R} = (1, z\, d(z))$, written $\boldsymbol{R} = d(z)$.

The famous Lagrange inversion formula yields for $\boldsymbol{R} = d(z)$

$$\boldsymbol{R}_{n,k} = [z^{n-k}]d(z)^k = \frac{k}{n}[z^{n-k}]A(z)^n \quad \text{for all} \quad n \geq k \geq 0.$$

The Pascal array $P = 1/(1-z)$

The matrix $\boldsymbol{P}$ stores the well-known Pascal triangle:

$$\boldsymbol{P}_{n,k} = \binom{n-1}{n-k},$$

$$\boldsymbol{P} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & 1 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & 1 & 3 & 3 & 1 & & & \\ & 1 & 4 & 6 & 4 & 1 & & \\ & 1 & 5 & 10 & 10 & 5 & 1 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Aigner array $\boldsymbol{C} = c(z)$, where $c(z) = \frac{1-\sqrt{1-4z}}{2z}$ [Aigner, 2008]

The matrix $\boldsymbol{C}$ stores the widely known *(ordinary) ballot numbers*: $\boldsymbol{C}_{n,k}$, where $\boldsymbol{C}_{n+1,1} = \boldsymbol{C}_{n+1,2} = C_n$ is a Catalan number,

$$\boldsymbol{C} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & 1 & 1 & & & & & \\ & 2 & 2 & 1 & & & & \\ & 5 & 5 & 3 & 1 & & & \\ & 14 & 14 & 9 & 4 & 1 & & \\ & 42 & 42 & 28 & 14 & 5 & 1 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Shapiro array $\boldsymbol{B} = c(z)^2$ [Shapiro, 1976]

$$\boldsymbol{B} = \begin{pmatrix} 1 & & & & & & \\ 1 & & & & & & \\ 2 & 1 & & & & & \\ 5 & 4 & 1 & & & & \\ 14 & 14 & 6 & 1 & & & \\ 42 & 48 & 27 & 8 & 1 & & \\ 132 & 165 & 110 & 44 & 10 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $\boldsymbol{B} = c(z)^2$ then $\boldsymbol{B}$ may be seen as the array obtained from $\boldsymbol{C}$ by extracting coefficients according to the rule $\boldsymbol{B}_{n,k} = \boldsymbol{C}_{n+k,2k}$:

$$\boldsymbol{C} = \begin{pmatrix} \mathbf{1} & & & & & & & \\ & 1 & & & & & & \\ & 1 & \mathbf{1} & & & & & \\ & 2 & \mathbf{2} & 1 & & & & \\ & 5 & \mathbf{5} & 3 & \mathbf{1} & & & \\ & 14 & \mathbf{14} & 9 & \mathbf{4} & 1 & & \\ & 42 & \mathbf{42} & 28 & \mathbf{14} & 5 & \mathbf{1} & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$\boldsymbol{C}^{a,b} = \big(c(rz)^a, z\,c(rz)^b\big)$ [Yang, 2013]

For all $a, b \in \mathbb{N}$, one may consider more general Catalan triangles $\boldsymbol{C}^{a,b}$ obtained from $\boldsymbol{C}$ by extracting all columns of index $a + kb$, for suitable $k \in \mathbb{N}$.

The entry $\boldsymbol{C}^{a,b}_{n,k}$ of $\boldsymbol{C}^{a,b}$ is given by

$$\boldsymbol{C}^{a,b}_{n,k} = \boldsymbol{C}_{n-k+a+bk,a+bk} = \frac{a+bk}{n-k+a+bk}\binom{2(n-k)+a+bk-1}{n-k}.$$

$\boldsymbol{C}^{a,b}$ is the Riordan array considered by Yang when $r = 1$:

$$\boldsymbol{C}^{a,b} = \big(c(z)^a, z\,c(z)^b\big).$$

Note that clearly an array

$$\boldsymbol{R}^{a,b} = (d(z)^a, z\, d(z)^b)$$

can be defined from any generating function $d(z) \in \mathbb{C}[[z]]_0$ and from any integers $a, b \in \mathbb{N}$.

For $d(z) = 1/(1-z)$ we obtain the matrix

$$\boldsymbol{P}^{a,b} = \left(\left(\frac{1}{1-z}\right)^a, z\left(\frac{1}{1-z}\right)^b\right),$$

with

$$\boldsymbol{P}^{a,b}_{n,k} = \boldsymbol{P}_{n-k+a+bk,a+bk} = \binom{n-k+a+bk-1}{n-k}.$$

Consider the formal power series which depends on the parameters q and t in $\mathbb{N}$:

$$c(q,t;z) = \frac{1 - tz \pm \sqrt{(1-tz)^2 - 4qz}}{2qz} = \frac{1}{z}\left(z\frac{1-qz}{1+tz}\right)^{\langle -1 \rangle}.$$

In particular,

$$c(r, c-r; z) = \frac{1 - (c-r)z - \sqrt{1 - 2(c+r) + (c-r)^2 z^2}}{2rz}$$

is the generating function $d_{c,r}(z)$ of He's (c,r)-Catalan triangles.

Generalized Catalan triangles

Accordingly,

- $c(1,0;z) = c(z)$ is the generating function of the Catalan numbers;
- $c(1,1;z)$ is the large Schröder function;
- $c(2,-1;z)$ is the small Schröder function.

For all $a, b \in \mathbb{N}$ define the *generalized Catalan triangles*:

$$\mathbf{C}^{a,b}(q,t) = \left(c(q,t;z)^a, z\, c(q,t;z)^b\right).$$

Some properties

Recursive rule:

$$\boldsymbol{C}^{a,b}_{n,k+1}(q,t) = \sum_{i=k}^{n-1} \boldsymbol{C}^{a,b}_{i,k}(q,t)\, \boldsymbol{C}^{0,b}_{n-i,1}(q,t).$$

Note that if we set $\boldsymbol{C}(q,t) := \boldsymbol{C}^{0,1}(q,t)$ then the array $\boldsymbol{C}^{a,b}(q,t)$ arises from $\boldsymbol{C}(q,t)$ by extracting all columns of index $a+bk$, for a suitable $k \in \mathbb{N}$:

$$\boldsymbol{C}^{a,b}_{n,k}(q,t) = \boldsymbol{C}_{n-k+a+bk,a+bk}(q,t)\,.$$

Some particular cases

(i) $\boldsymbol{C}^{0,1}(1,0) = \boldsymbol{C}$ is Aigner's array of the Ballot numbers;

(ii) $\boldsymbol{C}^{0,2}(1,0) = \boldsymbol{B}$ is Shapiro's Catalan triangle;

(iii) $\boldsymbol{C}^{1,1}(r, c-r) = (d_{c,r}(z), zd_{c,r}(z))$ is He's (c,r)-Catalan triangle;

(iv) $\boldsymbol{C}^{a,b}(r,0) = C[a,b;r]$ for all $a, b \in \mathbb{N}$ is Yang's generalized Catalan matrix;

(v) $\boldsymbol{C}^{1,2}(1,0) = \left(\frac{2k+1}{n+k+1}\binom{2n}{n+k}\right)$ is Radoux's triangle of numbers;

(vi) $\boldsymbol{C}^{a,b}(1,0) = \boldsymbol{C}^{a,b}$ for all $a, b \in \mathbb{N}$;

(vii) $\boldsymbol{C}^{a,b}(0,1) = \boldsymbol{P}^{a,b}$ for all $a, b \in \mathbb{N}$;

(viii) $\boldsymbol{C}^{0,1}(0,t) = \boldsymbol{P}(t) = \left(\binom{n-1}{n-k}t^{n-k}\right)$ is is the well-known generalized Pascal triangle:

$$\boldsymbol{P}(t) = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & t & 1 & & & & \\ & t^2 & 2t & 1 & & & \\ & t^3 & 3t^2 & 3t & 1 & & \\ & t^4 & 4t^3 & 6t^2 & 4t & 1 & \\ & t^5 & 5t^4 & 10t^3 & 10t^2 & 5t & 1 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem

For all $n, k, a, b, q, t \in \mathbb{N}$ we have

$$\begin{aligned} \mathbf{C}^{a,b}_{n,k}(q,t) &= \frac{a+bk}{n-k+a+bk} \sum_{i=0}^{n-k} \binom{n-k+a+bk}{i} \binom{2(n-k)+a+bk-i-1}{n-k-i} t^i \, q^{n-k-i}, \\ &= \frac{a+bk}{n-k+a+bk} \sum_{i=0}^{n-k} \binom{n-k+a+bk}{i} \binom{n-k-1}{n-k-i} (q+t)^i \, q^{n-k-i}. \end{aligned}$$

Given $a, b \in \mathbb{N}$, $\mathbf{C}^{a,b}_{n,k}(q,t)$ is a homogeneous polynomial in q, t with positive integer coefficients.

Generalized Catalan numbers $C^{a,b}(n,i,k)$

For all $a, b, n, k \in \mathbb{N}$ and $0 \leq i \leq n-k$, define

$$C^{a,b}(n,i,k) = \frac{a+bk}{n-k+a+bk}\binom{n-k+a+bk}{i}\binom{2(n-k)+a+bk-i-1}{n-k-i}.$$

For specific values of the parameters the definition specializes to the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$:

- $C^{1,0}(n,0,0) = C_n$,
- $C^{0,1}(n,0,1) = C_{n-1}$.

We have the following compact form:

$$\mathbf{C}^{a,b}_{n,k}(q,t) = \sum_{i=0}^{n-k} C^{a,b}(n,i,k)t^i q^{n-k-i}.$$

Interpolation properties

Setting $i = 0$ or $i = n - k$ in the formula for $C^{a,b}(n, i, k)$ gives

$$C^{a,b}(n, 0, k) = \boldsymbol{C}_{n,k}^{a,b} \quad \text{and} \quad C^{a,b}(n, n-k, k) = \boldsymbol{P}_{n,k}^{a,b}.$$

Therefore, we easily obtain

$$\boldsymbol{C}_{n,k}^{a,b}(q, t) = \boldsymbol{C}_{n,k}^{a,b} q^{n-k} + \cdots + \boldsymbol{P}_{n,k}^{a,b} t^{n-k}.$$

When $a = 0$ and $b = 1$:

$$\boldsymbol{C}_{n,k}(q, t) = \boldsymbol{C}_{n,k} q^{n-k} + \cdots + \boldsymbol{P}_{n,k} t^{n-k}.$$

Generalized Narayana numbers $N^{a,b}(n,i,k)$

For all $a, b, n, k \in \mathbb{N}$ and $0 \leq i \leq n-k$, define

$$N^{a,b}(n,i,k) = \frac{a+bk}{n-k+a+bk}\binom{n-k+a+bk}{i}\binom{n-k-1}{i-1}.$$

This specializes to the well-known Narayana numbers $N(n,i) = \frac{1}{n}\binom{n}{i}\binom{n}{i-1}$:

- $N^{1,0}(n,i,0) = N(n,i)$,
- $N^{0,1}(n,i,1) = N(n-1,i)$.

Generalized Narayana identity

We have

$$\mathbf{C}^{a,b}_{n,k}(q,t) = \sum_{i=0}^{n-k} N^{a,b}(n,i,k)(t+q)^i q^{n-k-i},$$

and a *generalized Narayana identity*.

$$C^{a,b}(n,i,k) = \sum_{j=i}^{n-k} \binom{j}{i} N^{a,b}(n,j,k).$$

Particular cases

- For $a = 1$ and $b = i = k = 0$:

$$C_n = \sum_{j=0}^{n} N(n, j).$$

- For $i = a = 0$ and $b = 1$:

$$C_{n,k} = \frac{k}{n} \sum_{j=0}^{n-k} \binom{n}{j} \binom{n-k-1}{j-1}.$$

- For $i = 0$:

$$C_{n,k}^{a,b} = \sum_{j=0}^{n-k} N^{a,b}(n, j, k).$$

- For $i = a = 0$ and $b = 2$:

$$\boldsymbol{B}_{n,k} = \frac{2k}{n+k} \sum_{j=0}^{n-k} \binom{n+k}{j} \binom{n-k-1}{j-1}.$$

- For $i = a = 0$, $b = 2$, and $k = 1$:

$$C_n = \frac{2}{n+1} \sum_{j=0}^{n-1} \binom{n+1}{j} \binom{n-2}{j-1}.$$

Noncrossing partitions [Kreweras, 1972]

Let $n \in \mathbb{N}$. Recall that a partition τ of $\{1, 2, \ldots, n\}$ is said to be *noncrossing* if and only if there are not $a, b, c, d \in \{1, ..., n\}$ and $B_i, B_j \in \tau$ such that $1 \leq a < b < c < d \leq n$, $\{a, c\} \in B_i$, $\{b, d\} \in B_j$ and $B_i \cap B_j = \varnothing$.

We use of a more compact notation to represent partitions. For instance,

$$\tau = 1\,4\,/2\,3$$

will be preferred to

$$\tau = \{\{1, 4\}, \{2, 3\}\}.$$

Some definitions

$\mathsf{NC}(n)$ = set of all the noncrossing partitions of the totally ordered set $\{1, 2, \ldots, n\}$.

- $\mathsf{NC}(0) = \{\emptyset\}$,
- $\mathsf{NC} = \bigcup_{n \geq 0} \mathsf{NC}(n)$.

Define $\mathsf{sz}, \mathsf{bk} \colon \mathsf{NC} \to \mathbb{N}$:

$$\mathsf{sz}(\tau) = n \text{ if and only if } \tau \in \mathsf{NC}(n),$$

$$\mathsf{bk}(\tau) = b \text{ if and only if } \tau = \{B_1, B_2, \ldots, B_b\}.$$

By convention, $\mathsf{sz}(\emptyset) = 0$ and $\mathsf{bk}(\emptyset) = 0$.

The set NC can be equipped with the following noncommutative associative operation $\otimes \colon \mathrm{NC} \times \mathrm{NC} \to \mathrm{NC}$:

$$\pi \otimes \sigma = \pi \cup \varphi_\pi(\sigma)$$

where φ_π is the unique order preserving bijection associated with π:

$$\varphi_\pi \colon \{1, 2, \ldots, \mathrm{sz}(\sigma)\} \to \{1 + \mathrm{sz}(\pi), \mathrm{sz}(\pi) + 2, \ldots, \mathrm{sz}(\pi) + \mathrm{sz}(\sigma)\}$$

such that $\varphi_\pi(\sigma)$ is such that $\varphi_\pi(i) = i + \mathrm{sz}(\pi)$.

For instance, if $\pi = 1\,2\,5/3\,4$ and $\sigma = 1\,4/2\,3$ then $\mathrm{sz}(\pi) = 5$. Therefore, $\varphi(\sigma) = 6\,9/7\,8$ and

$$1\,2\,5/3\,4 \otimes 1\,4/2\,3 = 1\,2\,5/3\,4/6\,9/7\,8.$$

The maps sz and bk satisfy the following:

$$\mathsf{sz}(\pi \otimes \tau \otimes \cdots \otimes \sigma) = \mathsf{sz}(\pi) + \mathsf{sz}(\tau) + \cdots + \mathsf{sz}(\sigma) \quad \text{for all} \quad \pi, \tau, \ldots, \sigma \in \mathsf{NC}$$

$$\mathsf{bk}(\pi \otimes \tau \otimes \cdots \otimes \sigma) = \mathsf{bk}(\pi) + \mathsf{bk}(\tau) + \cdots + \mathsf{bk}(\sigma) \quad \text{for all} \quad \pi, \tau, \ldots, \sigma \in \mathsf{NC}.$$

Irreducible partitions [Lehner, 2002]

Irreducible partitions: partitions which cannot be factorized into sub-partitions, or partitions of $\{1, \ldots, n\}$ for which 1 and n are in the same block.

Alternatively, a nonempty noncrossing partition $\tau \in \mathrm{NC}$ is said to be *irreducible* if and only if there are not nonempty partitions $\pi, \sigma \in \mathrm{NC}$ such that $\tau = \pi \otimes \sigma$.

Proposition

For all nonempty noncrossing partition $\tau \in \mathrm{NC}$ there exist uniquely determined irreducible partitions $\tau_1, \tau_2, \ldots, \ldots, \tau_d \in \mathrm{NC}$ satisfying

$$\tau = \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_d.$$

By writing $\tau = \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_d$ we always mean $\tau_1, \tau_2, \ldots, \ldots, \tau_d$ irreducible partitions.

Define the following maps dg, sg: NC $\to \mathbb{N}$:

$$\mathrm{dg}(\tau) = d \quad \text{if and only if} \quad \tau = \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_d$$

$$\mathrm{sg}(\tau) = \left|\{i \mid 1 \leq i \leq \mathrm{dg}(\tau),\, \tau_i = \chi\}\right|, \quad \text{where} \quad \mathrm{dg}(\emptyset) = \mathrm{sg}(\emptyset) = 0.$$

We have

$$\mathrm{dg}(\pi \otimes \tau \otimes \cdots \otimes \sigma) = \mathrm{dg}(\pi) + \mathrm{dg}(\tau) + \cdots + \mathrm{dg}(\sigma), \quad \text{for all} \quad \pi, \tau, \ldots, \sigma \in \mathrm{NC},$$

$$\mathrm{sg}(\pi \otimes \tau \otimes \cdots \otimes \sigma) = \mathrm{sg}(\pi) + \mathrm{sg}(\tau) + \cdots + \mathrm{sg}(\sigma) \quad \text{for all} \quad \pi, \tau, \ldots, \sigma \in \mathrm{NC}.$$

- $\mathrm{sg}(\tau) \leq \mathrm{dg}(\tau) \leq \mathrm{bk}(\tau) \leq \mathrm{sz}(\tau)$ for all $\tau \in \mathrm{NC}$.

A generating function for the enumeration of noncrossing partitions

For all $n \geq b \geq d \geq s \geq 0$:

$$\mathrm{NC}(n, b, d, s) = \{\tau \in \mathrm{NC} \,|\, \mathrm{sz}(\tau) = n, \mathrm{bk}(\tau) = b, \mathrm{dg}(\tau) = d, \mathrm{sg}(\tau) = s\},$$

$$\mathrm{nc}(n, b, d, s) = \left|\mathrm{NC}(n, b, d, s)\right|.$$

Consider the generating function:

$$f_{\mathrm{NC}}(z, t, x, w) = \sum_{\tau \in \mathrm{NC}} z^{\mathrm{sz}(\tau)} t^{\mathrm{bk}(\tau)} x^{\mathrm{dg}(\tau)} w^{\mathrm{sg}(\tau)} = \sum_{n \geq b \geq d \geq s \geq 0} \mathrm{nc}(n, b, d, s) z^n t^b x^d w^s.$$

We give an explicit form for $f_{\mathrm{NC}}(z,t,x,w)$.

Theorem

We have

$$f_{\mathrm{NC}}(z,t,x,w) = \frac{1}{1 - xz\big(c(1,t-1;z) + wt - 1\big)},$$

where $c(q,t;z) = \frac{1-tz\pm\sqrt{(1-tz)^2-4qz}}{2qz}$.

Enumerative properties of the polynomials $C_{n,k}^{a,b}(q,t)$ of degree $n-k$

Let

$$\mathrm{NC}^* = \bigcup_{n\geq 0} \bigcup_{S\subseteq\{1,\dots,n\}} \mathrm{NC}(S).$$

Set $\tau \subseteq \pi$ if and only if any block of τ is also a block of π, with $\pi, \tau \in \mathrm{NC}^*$.

Theorem

For all $a, b, n, k \in \mathbb{N}$, *let*

$$\mathrm{NC}^{a,b}(n,k) = \{\tau \mid \tau \in \mathrm{NC}, \mathrm{sz}(\tau) - \mathrm{dg}(\tau) = n-k, \mathrm{dg}(\tau) = a+bk\}.$$

Then,

$$C_{n,k}^{a,b}(q,t) = \sum_{\tau\in\mathrm{NC}^{a,b}(n,k)} \sum_{\substack{\pi\subseteq\tau \\ \chi(\tau)=\chi(\pi)}} q^{\mathrm{sz}(\tau)-\mathrm{dg}(\tau)-\mathrm{bk}(\tau)-\mathrm{bk}(\pi)} t^{\mathrm{bk}(\tau)-\mathrm{bk}(\pi)}.$$

On the enumerative properties of the generalized Catalan and Narayana numbers

Theorem

For all $a, b, n, k \in \mathbb{N}$ and $0 \leq i \leq n-k$ the following holds:

$$C^{a,b}(n,i,k) = \left|\{(\pi,\tau) \mid \tau \in \mathrm{NC}^{a,b}(n,k), \pi \subseteq \tau, \chi(\tau) = \chi(\pi), \mathrm{bk}(\tau) - \mathrm{bk}(\pi) = i\}\right|.$$

and

$$N^{a,b}(n,i,k) = \left|\{\tau \mid \tau \in \mathrm{NC}^{a,b}(n,k), \mathrm{bk}(\tau) - \mathrm{sg}(\tau) = i\}\right|.$$

Particular cases

We now take a closer look at the special cases considered earlier on.

- Case $a = 1$ and $b = i = k = 0$:

  $C_n = |\mathrm{NC}^{1,0}(n{+}1, 1)| = \left|\{\tau \mid \tau \in \mathrm{NC}, \mathrm{sz}(\tau) = n{+}1, \mathrm{dg}(\tau) = 1\}\right| = \left|\mathrm{NC}^{\mathrm{irr}}(n{+}1)\right|,$

  $$N(n, j) = \left|\{\tau \mid \tau \in \mathrm{NC}^{\mathrm{irr}}(n+1), \mathrm{bk}(\tau) = j\}\right|.$$

  The number of irreducible noncrossing partitions of size $n + 1$ yields the sum over the number of irreducible noncrossing partitions of size $n + 1$ with j blocks for $j = 1, \ldots, n + 1$, with $n \geq 0$.

- Case $a = i = 0$ and $b = 1$:

$$\boldsymbol{C}_{n,k} = |\mathrm{NC}^{0,1}(n,k)| = \big|\{\tau \mid \tau \in \mathrm{NC}, \mathrm{sz}(\tau) = n, \mathrm{dg}(\tau) = k\}\big|.$$

  The ballot numbers count the number of noncrossing partitions of size n and degree k.

- Case $i = 0$:

$$\boldsymbol{C}^{a,b}_{n,k} = |\mathrm{NC}^{a,b}(n,k)|.$$

  For $r = 1$ the entries Yang's array count the number of noncrossing partitions of size $n + a + (b-1)k$ and degree $a + bk$.

- Case $i = a = 0$ and $b = 2$:

$$\boldsymbol{B}_{n,k} = |\mathrm{NC}^{0,2}(n,k)| = \big|\{\tau \mid \tau \in \mathrm{NC}, \mathrm{sz}(\tau) = n+k, \mathrm{dg}(\tau) = 2k\}\big|.$$

The entries $\boldsymbol{B}_{n,k}$ of the Shapiro array count the number of noncrossing partitions of size $n+k$ and degree $2k$.

- Case $i = a = 0$, $b = 2$, and $k = 1$:

$$C_n = |\mathrm{NC}^{0,2}(n,1)| = |\{\tau \mid \tau \in \mathrm{NC}, \mathrm{sz}(\tau) = n+1, \mathrm{dg}(\tau) = 2|.$$

The Catalan numbers C_n also count the number of noncrossing partitions of size $n+1$ and degree 2.

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