TQFT SEMINARS, 1st
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COUNTING POTENTIALS

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COUNTING MONSTER POTENTIALS

1. THE POTENTIALS. Fix $\alpha > 0$, $L \in \mathbb{N}$, $N \in \mathbb{N}$.
For any monic polynomial $P(x) = \prod_{\kappa=1}^{N}(x - z_{\kappa})$, $P(0) \neq 0$

$$V_{P}(x) = x + \frac{L}{x} - 2 \frac{2}{x} \log P(x^{2\alpha+2}), \ x \in \mathbb{R}$$

GROUND STATE POTENTIAL

SINGULARITIES:

0 REGULAR BRANCH POINT

$\infty$ IRREGULAR SOURCE

$X_{\kappa}$ S.T. $X_{\kappa} = z_{\kappa}$: $V_{P}(x) = \frac{2}{(x - X_{\kappa})^{2} + \Theta(x)}$

RESONANT SINGULARITY: INDEX: $-1, 2$
TRIVIAL MONODROMY

\( -\psi_{\mu}^{\prime\prime} + (V_{\mu}(x) - \varepsilon) \psi(x) = 0, \quad x \in \mathfrak{g}_{\alpha}, \quad \varepsilon \in \mathfrak{g} \)

\( \chi_{n} = 2k \)

\( \chi_{n} \text{ Resonant } \Rightarrow M_{\varepsilon} = \begin{pmatrix} 1 & C(\varepsilon) \\ 0 & 1 \end{pmatrix} \)

DEF: \( V_{\mu} \) has trivial monodromy at \( x_{K} : M_{\varepsilon} = I, \quad \forall \varepsilon \in \mathfrak{g} \)

\( \text{Assume } \varepsilon \text{ has distinct roots. } V_{\mu} \text{ has trivial monodromy at all singularities} \quad 1 \neq \varepsilon \)

BLT SYSTEM

\[ \sum_{1}^{N} \sum_{j \neq k}^{1} \frac{2 \chi_{k} \left( \frac{\chi_{k}^{2} + A \chi_{k} z_{i} + B z_{i}^{2}}{2 \chi_{k} - 2 z_{i}} \right)}{4 \left( \chi_{k} + 1 \right)} - \frac{\chi_{k}^{2}}{4 \left( \chi_{k} + 1 \right)} + \frac{L}{4 \left( \chi_{k} + 1 \right)} + \frac{1 - 4 \chi_{k}^{2}}{16 \left( \chi_{k} + 1 \right)} = 0 \]

\( \chi_{k} = 1, \ldots, N \)
ODE/IM CORRESPONDENCE

Let \( V_p \) be a potential, and \( E_n, n \geq 0 \) such that

\[- \psi''(x) + (V_p - E_n) \psi(x) = 0, \quad \lim_{x \to 0} \psi(x) = \lim_{x \to +\infty} \psi(x) = 0\]

Then \( \{E_n\}_{n=0}^{\infty} \) solves the Bethe Ansatz Equations

\[\begin{align*}
\prod_{n=0}^{\infty} \frac{E_n - e^{\frac{\pi i}{\alpha+1}} E_n}{E_n - e^{\frac{-\pi i}{\alpha+1}} E_n} = -1, \quad n = 0, 1, \ldots
\end{align*}\]

\[L = L(h) = \sum \frac{i \beta(h)}{\alpha+1}, \quad \beta \geq \frac{1}{2}, \quad e^L = \lim_{\alpha \to 0} \lim_{\beta \to 0}
\]

Many generalizations:

(C) \[0 \to \frac{81}{2} \to 9 \to \]

FELIEN, FRITCHE, K., NOYAKA, O., "L.A.M.P. 16, 18."
NOYAKA, O., "L.A.M.P. 20."
1. $6$-vertex lattice site $\to \infty$, Bethe ansatz solution

What about the tail?

$\Rightarrow$ it satisfies the $\infty$-B.A. equations.

2. Integrability in CFT, $C < 1$

$\Delta$ irreducible highest weight representation of $Vir$, $C < 1$

- Ground state: $L_\Delta |\Delta\rangle = \Delta |\Delta\rangle$, $L_n |\Delta\rangle = 0$ for $n > 0$.
- Higher states: choose level $N$, a partition $\lambda = (\lambda_1, \ldots, \lambda_N)$

$|\lambda\rangle = L_{\lambda_1} \cdots L_{\lambda_N} |\Delta\rangle$, $L_0 |\lambda\rangle = (\Delta + N)^{\lambda_1} |\lambda\rangle$

Every state yields a solution of $\infty$-B.A.

$\{ E_n \}_{n=0}^\infty$
BLz CONJECTURE

With \( c = 1 - \frac{6\alpha^2}{\alpha + 1} \) and \( \Delta = \frac{L_0}{4(\alpha + 1)} + \frac{1 - 4\alpha^2}{16(\alpha + 1)} \):

A state \( |\psi\rangle \) of level \( N \), \( \exists \) monstern potential \( V_0 \), \( \Phi = \frac{N}{11(2-2\alpha)} \).

\[
\{ E_n \}^\infty_{n=0} = \{ E_n \}^\infty_{n=0}
\]

WEAK BLz CONJECTURE

Fix \( N \in \mathbb{N} \). For generic \( (\alpha, L) \), the number of monstern potentials with \( \text{"} N \text{" \( \text{roots} \)\text{,"} \) is \( \Phi(N) = \frac{N}{11(2-2\alpha)} \).

\( \text{\"} N \text{\" \( \text{roots} \)} \) = \( \Phi(N) = \prod_{k=1}^{N} (2 - 2\alpha_k) \).
LARGE MOMENTUM LIMIT OF THE BLZ SYSTEM

\[
\sum_{i \neq k} \frac{2z_k^2 + A \beta_k \beta_i^* + z_i^2}{(2z_k - z_i)^3} - \frac{\alpha \beta_k}{4(1 + \alpha)} + \frac{L}{4(\alpha + 1)} - \frac{1 - 4\alpha^2}{16(\alpha + 1)} = 0
\]

\[
B_{N,\alpha} = \left\{ (p, L) \in \mathbb{C}^{N+1} \mid p = \frac{N}{11} (\alpha + 2\alpha), 2^N \neq 0, 2 \neq 0, 2 \text{ solves } (\ast) \right\}
\]

\[
\bar{B}_{N,\alpha} = \text{ closure of } B_{N,\alpha} \text{ in } \mathbb{C}^{N+1}
\]

\[
(p^{(n)}, L^{(n)}) \in \bar{B}_{N,\alpha}, \text{ with } L^{(n)} \to \infty. \text{ Call } z_k^{(n)} \text{ the roots of } \Phi_k^{(n)}
\]

\[
\lim_{n \to \infty} \frac{z_k^{(n)} \alpha}{L} = 1, \quad k = 1, -1, N
\]

The proof is simple but tricky! It took us almost 2 years!
\[ E = 0 \]

RATIONAL EXTENSION OF THE HARMONIC OSCILLATOR

WHERE

\[ \psi''(t) + \left( \frac{L^2}{\alpha^2} \right) \psi(t) - \frac{2}{\alpha^2} \sum_{n=1}^{\infty} \left( \frac{\pi n}{L} \right)^2 \psi(t) = 0 \]

WE GET

\[ \psi(t) = \psi_{in} \left( \frac{t}{T} \right) + \psi_{out} \left( \frac{t}{T} \right) \]

WE EXPAND THE SOLUTIONS EQUATIONS OF THE SYSTEM ABOUT THE AVERAGE ENERGY \( E = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{\pi n}{L} \right)^2 \)

\[ \psi_{in} = \psi_{in}(t) \]

\[ \psi_{out} = \psi_{out}(t) \]

AS \( \psi \rightarrow \psi_{in} \), SOLUTIONS OF \( \psi \)

THESE ARE THE COUPLED EQUATIONS OF THE HARMONIC OSCILLATOR.
OBLOMOV'S THEOREM

Let \( V = (V_1, \ldots, V_d) \), \( D_i \geq V_i \), \( V_i > 0 \), \( \sum_i V_i = N \) be a partition of \( N \)

\[
P^{[V]}(t) = c_V^{-1} W \left[ H_{V_1}(t), H_{V_2}(t), \ldots, H_{V_d}(t) \right]
\]

\[
= \prod_{k=1}^{N} \left( t - \sqrt[k]{t} \right)
\]

E.g. \( V = (N) \), \( P = c_1^{-1} H(t) \)

\[
P^{[V]}(t) = N P^{[V]}(i t)
\]

Theorem: If \( P = \prod_{k=1}^{N} (t - t_k) \) is such that

\[
U_P = t^2 - 2 \frac{\partial^2}{\partial t^2} \ln P(t) = t^2 - \sum_{k=1}^{N} \frac{2}{(t - t_k)^2}
\]

has trivial non-adjacent at \((t_1, \ldots, t_N)\), then

\[
P(t) = P^{[V]}(t) \text{ for some partition } V \text{ of } N
THEOREM N - CONTI 2020

Fix $\alpha > 0$, $N \in \mathbb{N}$.

Let $(P^{(n)}, L^{(n)}) \in B_{N, \alpha}$ with $L^{(n)} \rightarrow \infty$ as $n \rightarrow \infty$.

The sequence $P^{(n)}$ can be split into $J$, $1 \leq J \leq \rho(n)$, subsequences such that, each subsequence associated to a unique partition $\mathbb{U}$ of $N$ such that

$$Z^{(n)}_K = \frac{L^{(n)}}{\alpha} + \frac{(2\alpha + 2)^{3/4}}{\alpha} \sum_{i=1}^{\rho(n)} \left( L^{(n)}_i \right)^{3/4} + O(L^{(n)})^{1/2}$$

where $Z^{(n)}_K$ are the approximately ordered roots of $P^{(n)}$.  

WHERE
Let \( L \to +\infty \) (\( L \) real) and \( V_p, p = p(z; L) \) satisfy the D-asymptotics for some partition \( \nu = (\nu_1, \ldots, \nu_j) \) of \( N \), then

\[
E_{\pi}^{[\nu]} = (1 + \alpha) \left( \frac{L}{\alpha} \right)^{\frac{\alpha}{\alpha + 1}} + \left( 2\alpha + 2 \right) L^{\alpha - \frac{1}{2\alpha + 2}} \left( 2(n-1) + 1 \right) \left( \frac{L}{\alpha} \right)^{\frac{1}{\alpha + 1}}
\]

\( + O\left( L L_{\nu^{-\frac{1}{\alpha + 1}}} \right) \), \( n \in N^{[\nu]} \)

\[
N^{[\nu]} = \{ V_{\nu}, V_{\nu + 1}, \ldots, V_{j+1-1} \}
\]

\( U = 0, N = 0 \)

\[
\begin{array}{cccccc}
1 & 3 & 5 & 7 & 9 & \ldots \leq U = 0.5 \cdot N = 1.5
\end{array}
\]

\[
\begin{array}{cccc}
-1 & 3 & 5 & \ldots
\end{array}
\]
Constructing Monstrous Potential with Given \( y \)-Asymptotics

Does -- for every partition \( \lambda \) of \( \lambda \) exist a unique algebraic family of monstrous potential \( \Phi(\lambda, L) \) satisfying the \( \lambda \)-asymptotics? \( \Phi(2; e L) = \Phi(2, L) \)

Definitely yes, but we are not able to prove it in general.

**Technical Difficulty**: Depending on \( \lambda \), \( \lambda \) or there may be a root of \( \Phi^{(0)}(\lambda) \) with multiplicity greater than \( \lambda \).

Thus making the perturbation series very tricky. We solved it for multiplicity 0, 1, 3, 6.
\[ \mathcal{P} = \frac{t^{3}}{L} (t+n_{N})(t-n_{N}) \]

\[ \mathcal{P}^{(5)}(t) = H_{5}(t) \]

\[ \mathcal{P}(t) = H_{5}(t) \]

\[ N = 6 \quad (3, 2, 1) \]

\[ \mathcal{P} = t^{6} \]