Deformed Airy kernel determinants: from KPZ tails to initial data for KdV

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The KPZ equation (Kardar-Parisi-Zhang '86) is a stochastic PDE given by

$$\partial_T \mathcal{H}(T,X) = \frac{1}{2}\partial_X^2 \mathcal{H}(T,X) + \frac{1}{2}(\partial_X \mathcal{H}(T,X))^2 + \xi(T,X)$$

where $\xi(T,X)$ is "space-time white noise".

Physical interpretation:

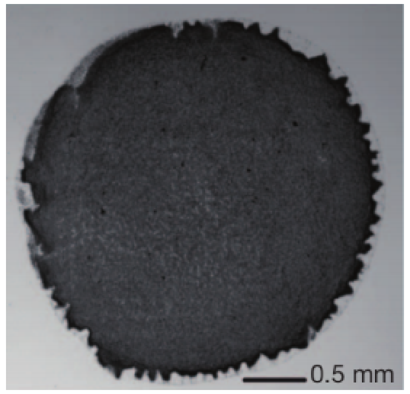
relaxation + nonlinear slope-dependent growth + random forcing

Allows to model various types of interface growth:

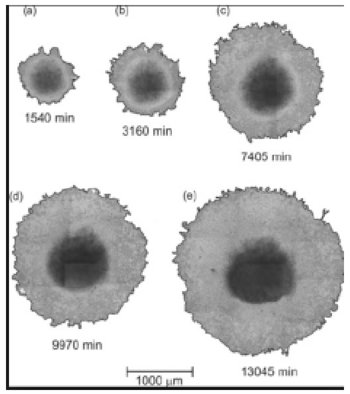
bacterial growth, coffee stains, forest fires, burning paper ...

Applications: universal model for interface growth

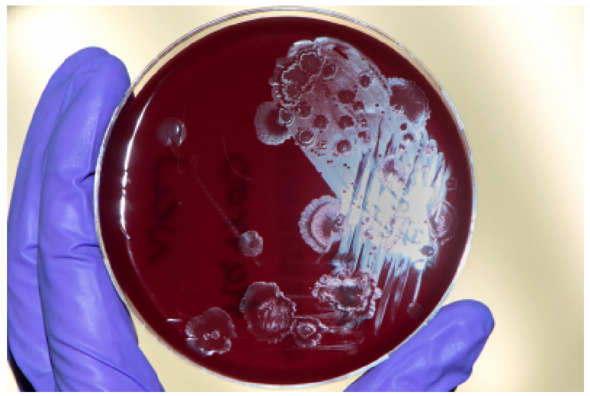
Burning paper



Coffee Stains



Tumour growth (?)



Bacterial growth

(pictures from Quastel, Introduction to KPZ)

Cole-Hopf solutions

Solutions can be defined through Cole-Hopf transformation

$$\mathcal{H}(T, X) = \log Z(T, X),$$

which relates KPZ to the stochastic heat equation (Bertini-Giacomin '97)

$$\partial_T Z(T, X) = \frac{1}{2}\partial_X^2 Z(T, X) + Z(T, X)\xi(T, X)$$

for which the solution theory is classical.

More general notion of solution, without passing via the stochastic heat equation, established by Hairer '11.

Narrow wedge initial data

One of the physically relevant solutions is the one with narrow wedge initial data, which means formally that

$$\mathcal{H}(0, X) = \log Z(0, X) \text{ with } Z(0, X) = \delta_{X=0}.$$

$\mathcal{H}(T, X)$ typically behaves roughly like a parabola $\frac{X^2}{2T}$ becoming narrow as $T \to 0$.

Invariance

The probability distribution of $\mathcal{H}(T, X) - \frac{X^2}{2T}$ is independent of X, so it suffices to study the probability distribution of $\mathcal{H}(T, 0)$.

KPZ solution with narrow wedge initial data

Scaling

For asymptotics, it is convenient to scale $\mathcal{H}(T,0)$ in the following way:

$$\Upsilon_T = \frac{\mathcal{H}(2T,0) + \frac{T}{12}}{T^{1/3}}.$$

Tail estimates

Important to control the tails of the probability distribution of Υ_T:

$$\mathbb{P}\left(\Upsilon_T \leq x\right) \quad \text{for } x \to \pm\infty.$$

Exact solution

An exact expression for the Laplace transform of the solution to the stochastic heat equation was established by Amir-Corwin-Quastel '10

written as a multiplicative statistic in the Airy point process
or
written as the Fredholm determinant of an Airy integral operator.

Similar but not completely rigorous results around the same time by Sasamoto-Spohn, Dotsenko and Calabrese-Le Doussal-Rosso.

These results rely on similar expressions obtained by Tracy-Widom '08 in asymmetric exclusion processes, which can be seen as discrete approximations of KPZ.

Airy point process

The Airy point process is a determinantal point process on the real line with correlation kernel given by

$$K^{\mathrm{Ai}}(u,v) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v)}{u-v} = \int_0^{+\infty} \mathrm{Ai}(u+r)\mathrm{Ai}(v+r)dr.$$

It has almost surely a largest point and an infinite number of points $\zeta_1 \geq \zeta_2 \geq \dots$.

Models largest eigenvalues of random matrices near soft edges (GUE, Wigner matrices ...).

Fredholm determinant

Multiplicative statistics of determinantal point processes are Fredholm determinants:

$$\mathbb{E}_{\mathrm{Ai}}\left[\prod_{j=1}^{\infty}(1-\sigma(\zeta_j))\right] = \det\left(1-\sigma K^{\mathrm{Ai}}\right)$$

where

$$\det(1-\sigma K^{\mathrm{Ai}}) = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\int_{\mathbb{R}^k}\det\left[\sigma(x_i)K^{\mathrm{Ai}}(x_i,x_j)\right]_{i,j=1}^{k} dx_1\ldots dx_k$$

KPZ and the Airy point process

Borodin-Gorin '16 proved, reformulating the results of Amir-Corwin-Quastel '10, that

$$R(s,T) := \mathbb{E}_{\mathrm{KPZ}}\left[\mathrm{e}^{-\mathrm{e}^{T^{1/3}(\Upsilon_T+s)}}\right] = \mathbb{E}_{\mathrm{Ai}}\left[\prod_{j=1}^{\infty} \frac{1}{1+\mathrm{e}^{T^{1/3}(\zeta_j+s)}}\right].$$

Thinned Airy point process

$R(s,T)$ is the probability that the thinned Airy process, obtained by removing each particle ζ_j independently with probability $\frac{1}{1+\mathrm{e}^{T^{1/3}(\zeta_j+s)}}$, is empty.

Fredholm determinant

We have

$$R(s,T)=\mathbb{E}_{\mathrm{KPZ}}\left[\mathrm{e}^{-\mathrm{e}^{T^{1/3}(\Upsilon_T+s)}}\right]=\det(1-\sigma(T^{1/3}(x+s))K^{\mathrm{Ai}}),$$

with

$$\sigma(r)=\frac{1}{1+e^{-r}},$$

and the study of the tails of the probability distribution of Υ_T reduces to the asymptotic analysis of a Fredholm determinant.

Characterization as a Fredholm determinant

Alternative Fredholm determinant representation

$$R(s,T) = \det(1 - K_T^{\mathrm{Ai}})_{L^2(-s,+\infty)},$$

where K_T^{Ai} is the finite temperature Airy kernel

$$K_T^{\mathrm{Ai}}(u,v) = \int_{-\infty}^{+\infty} \sigma(T^{1/3}r)\mathrm{Ai}(u+r)\mathrm{Ai}(v+r)dr, \quad \sigma(r) = \frac{1}{1+e^{-r}}.$$

Large T limit

$$\lim_{T\to\infty} K_T^{\mathrm{Ai}}(u,v) = \int_{0}^{+\infty} \mathrm{Ai}(u+r)\mathrm{Ai}(v+r)dr,$$

and $R(s,T)$ converges to the Tracy-Widom distribution

Asymptotic analysis

Asymptotic analysis of Fredholm determinants

$$\det(1-K) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^k} \det\left(K(x_i, x_j)\right)_{i,j=1,\ldots,k} dx_1 \ldots dx_k$$

- ✓ Easy if the norm of the operator K is small
- ✓ Hard otherwise: Riemann-Hilbert method developed by Its-Izergin-Korepin-Slavnov '90 if the kernel K is of integrable form

$$K(x,y) = \frac{\sum_{j=1}^{k} f_j(x) h_j(y)}{x-y}, \qquad \sum_{j=1}^{k} f_j(x) h_j(x) = 0.$$

Fredholm representation of $R(s,T)$

$$R(s,T) = \det(1 - \sigma(T^{1/3}(u+s))K^{\mathrm{Ai}}(u,v)),$$

Integrable form with $k=2$:

$$f(x) = \begin{pmatrix} -i\sigma(T^{1/3}(x+s))\mathrm{Ai}'(x) \\ \sigma(T^{1/3}(x+s))\mathrm{Ai}(x) \end{pmatrix} \quad h(x) := \begin{pmatrix} -i\mathrm{Ai}(x) \\ \mathrm{Ai}'(x) \end{pmatrix}$$

Logarithmic derivatives of $R(s,T)$ can be expressed in terms of a 2×2 matrix Riemann-HIlbert problem.

(a) $\Psi : \mathbb{C} \backslash (i\mathbb{R} \cup \mathbb{R}) \to \mathbb{C}^{2\times 2}$ is analytic

(b) Jump relations

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ (1-\sigma(T^{1/3}(\zeta+s)))^{-1} & 1 \end{pmatrix} \zeta \in i\mathbb{R}^{\pm},$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1-\sigma(T^{1/3}(\zeta+s)) \\ -(1-\sigma(T^{1/3}(\zeta+s)))^{-1} & 0 \end{pmatrix} \zeta < 0,$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 1-\sigma(T^{1/3}(\zeta+s)) \\ 0 & 1 \end{pmatrix} \zeta > 0.$$

(c) Asymptotic behavior as $\zeta \to \infty$

$$\Psi(\zeta) = \left(I + \mathcal{O}\left(\frac{1}{\zeta}\right)\right) \zeta^{\frac{1}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \mathrm{e}^{-\frac{2}{3}\zeta^{3/2}\sigma_3}.$$

General deformed Airy kernel determinants

Set $s = xt^{-1/3}$ and $T = t^{-2}$, and define

$$Q_\sigma(x,t) = \det\left(1 - \sigma\left(t^{-2/3}u + x/t\right)K^{\mathrm{Ai}}(u,v)\right),$$

where

- ✓ $\sigma : \mathbb{R} \to [0,1]$ is locally L^2, non-decreasing and it is piecewise C^∞; $\gamma = \lim_{r\to+\infty}\sigma(r) \in [0,1]$,
- ✓ for any $x \in \mathbb{R}, t > 0$, the kernel is integrable,
- ✓ there exist $c_1, c_2, c_3 > 0$ such that

$$\left|\sigma(y) - \gamma\chi_{(0,+\infty)}(y)\right| \le c_1 \mathrm{e}^{-c_2|y|}, \quad |\sigma'(y)| \le \frac{c_3}{|y|^2+1}.$$

Theorem (Cafasso-C-Ruzza '20)

The function $u_\sigma(x,t) := \partial_x^2 \log Q_\sigma(x,t) + \frac{x}{2t}$ solves the KdV equation

$$\partial_t u_\sigma + 2u_\sigma \partial_x u_\sigma + \frac{1}{6}\partial_x^3 u_\sigma = 0, \text{ and}$$

$$u_\sigma(x,t) = -\frac{1}{t}\int_{\mathbb{R}} \phi_\sigma^2(r;x,t)\, d\sigma(r) + \frac{x}{2t},$$

where ϕ_σ solves the Schrodinger equation with potential $2u_\sigma$,

$$\partial_x^2 \phi_\sigma(z;x,t) = (z - 2u_\sigma(x,t))\,\phi_\sigma(z;x,t),$$

and has asymptotic behavior
$\phi_\sigma(z;x,t) \sim t^{1/6}\mathrm{Ai}\left(t^{2/3}z - xt^{-1/3}\right)$ as $z \to \infty$ with $|\arg z| < \pi - \delta$.

Consequence

$$\partial_x^2 \log Q_\sigma(x,t) = -\frac{1}{t}\int_{\mathbb{R}} \phi_\sigma^2(r;x,t)\, d\sigma(r),$$

where ϕ_σ satisfies the integro-differential Painlevé II equation

$$\partial_x^2 \phi_\sigma(z;x,t) = \left(z - \frac{x}{t} + \frac{2}{t}\int_{\mathbb{R}} \phi_\sigma^2(r;x,t)\, d\sigma(r)\right) \phi_\sigma(z;x,t).$$

(cf. Amir-Corwin-Quastel '10)

KPZ, KdV and KP

The connection with KdV is not surprising, given recent results of Quastel-Remenik '19, and Le Doussal '20 who established a more general relation between KPZ and the KP hierarchy, and old but not so well-known results by Poppe-Sattinger '88 relating Fredholm determinants with the KP hierarchy.

Scattering and inverse scattering theory is understood (among others) for KdV solutions decaying at $\pm\infty$, but not for the solutions under consideration.

Can we understand the small t behavior of the KdV solutions $u_\sigma(x,t)$? This encodes information about the large T and tail asymptotics for the KPZ solution Υ_T.

Theorem (Cafasso-C-Ruzza '20)

1. For any $t_0 > 0$, there exist $M, c > 0$ such that

$$u_\sigma(x,t) = \frac{x}{2t} + \mathcal{O}\left(\mathrm{e}^{-c\frac{|x|}{t^{1/3}}}\right) x \leq -Mt^{1/3}, 0 < t < t_0.$$

2. There exists $\epsilon > 0$ such that for any $M > 0$,

$$u_\sigma(x,t) = \frac{x}{2t} - t^{-2/3} y_\gamma^2\left(-xt^{-1/3}\right) + \mathcal{O}(1), |x| \leq Mt^{1/3}, 0 < t < \epsilon,$$

where y_γ is the Ablowitz-Segur solution of Painlevé II.

3. If $\gamma = 1$, there exist $\epsilon, M > 0$ such that for any $K > 0$,

$$u_\sigma(x,t) = v_\sigma(x)\left(1 + \mathcal{O}\left(x^{-1}t^{1/3}\right)\right), Mt^{1/3} \leq x \leq K, 0 < t < \epsilon,$$

where $v_\sigma(x)$ is a function of $x > 0$, independent of t.

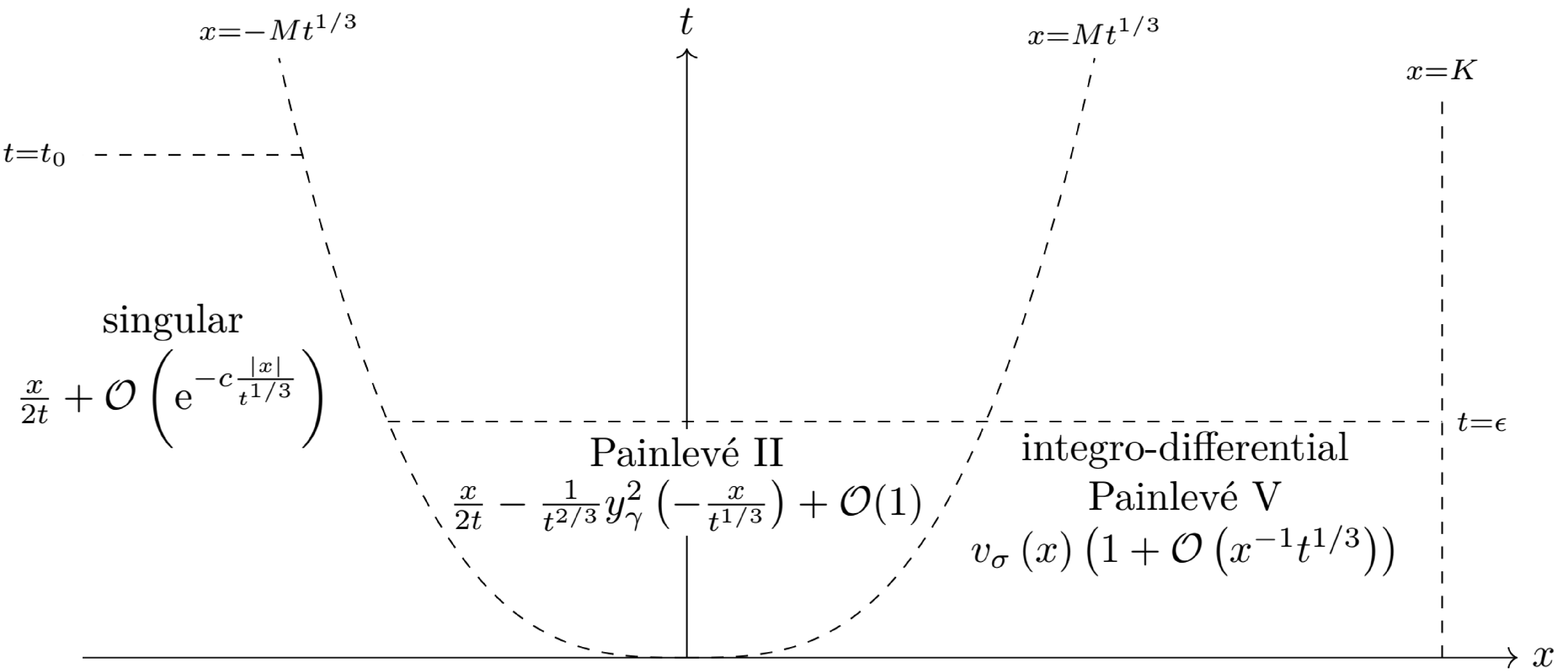


Figure 1: Phase diagram showing the different types of small t asymptotic behavior for $u_\sigma(x,t)$.

Painlevé equations

Ablowitz-Segur solution of Painlevé II equation characterized by

$$y_\gamma''(s) = sy_\gamma(s) + 2y_\gamma(s)^3, \; y_\gamma(s) \sim \sqrt{\gamma}\mathrm{Ai}(s), \; s \to +\infty.$$

v_σ satisfies an integro-differential Painlevé V equation, and has asymptotics

$$v_\sigma(x) = \frac{1}{8x^2} + \frac{1}{2}\int_{\mathbb{R}} \left(\chi_{(0,+\infty)}(r) - \sigma(r)\right) dr + \mathcal{O}\left(x^2\right), \qquad \text{as } x \to 0.$$

Scattering theory was also understood for KdV solutions behaving like $x/(2t)$ via the Cylindrical KdV equation (Its-Sukhanov), but the solutions we consider behave only like this for negative x, not for positive x.

Theorem (Cafasso-C-Ruzza '20)

1. For $x \leq -Mt^{1/3}$, $\log Q_\sigma(x,t) = \mathcal{O}\left(e^{-c\frac{|x|}{t^{1/3}}}\right)$.
2. For $|x| \leq Mt^{1/3}$ and for $0 < t < \epsilon$,

$$\log Q_\sigma(x,t) = \log F_{\mathrm{TW}}\left(-xt^{-1/3}\right) + \mathcal{O}\left(t^{1/3}\right)$$

3. If $\gamma = 1$, for $Mt^{1/3} \leq x \leq K$ and for $0 < t < \epsilon$,

$$\log Q_\sigma(x,t) = -\frac{x^3}{12t} - \frac{1}{8}\log(xt^{-1/3}) + \frac{\log 2}{24} + \log\zeta'(-1) + \int_0^x (x-\xi)\left(v_\sigma(\xi) - \frac{1}{8\xi^2}\right)d\xi + \mathcal{O}(x^{-1}t^{1/3}).$$

Consequences

Precise estimates for KPZ tail probabilities

$$\mathbb{P}_{\mathrm{KPZ}}(\Upsilon_T \leq s) \text{ as } s \to \pm\infty, T \to \infty$$

(via standard probability estimates using Laplace transform).

Large gap asymptotics in models for finite temperature free fermions (or MNS matrix models).

Compare with other results

Corwin-Ghosal '18 established upper and lower bounds for $\log \mathbb{P}_{\mathrm{KPZ}}(\Upsilon_T \le -s)$ for large s, T using rigidity of the points in the Airy point process, and estimates on the spectrum of the stochastic Airy operator.

Tsai '18, Le Doussal '20 obtained the large deviation rate function in the scaling limit where $sT^{-2/3} \to y \in (0, \infty)$.

Lin and Tsai '20 obtained the large deviation rate function in the scaling limit where $T \to 0$.

Bothner '20 developed an operator-valued Riemann-Hilbert approach for finite temperature kernels K_T^{Ai}.

Outlook

Riemann-Hilbert approach is powerful to obtain precise asymptotics in various regimes.

To do: asymptotics for x large, understanding integro-differential Painlevé equations, deformations of other determinants (Bessel, sine).

The end

Thank you for your attention!