Homotopy Quantum Field Theories

Alexis Virelizier
(University of Lille)

Topological Quantum Field Theory Seminar
Técnico Lisboa - September 11, 2020

Homotopy Quantum Field Theories

Alexis Virelizier
(University of Lille)

Joint work with ***Vladimir Turaev***

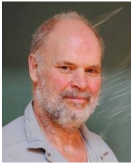
Topological Quantum Field Theory Seminar
Técnico Lisboa - September 11, 2020

Homotopy Quantum Field Theories

Alexis Virelizier
(University of Lille)

Joint work with ***Vladimir Turaev***

Topological Quantum Field Theory Seminar
Técnico Lisboa - September 11, 2020



Dedicated to the memory of Vaughan Jones

Homotopy quantum field theories (HQFTs)

Idea: TQFTs for manifolds endowed with maps to a fixed target topological space X (with base point $*$)

The category $X\text{-}\mathrm{Cob}_n$ is a symmetric monoidal category

- an object is a pair (Σ, f)
  - Σ closed oriented pointed $(n-1)$-manifold
  - $f\colon (\Sigma, \Sigma_\bullet) \to (X, *)$ pointed map
- a morphism $f\colon (\Sigma_1, f_1) \to (\Sigma_2, f_2)$ is equiv. class of (M, f)
  - M an oriented n-cobordism $\Sigma_1 \to \Sigma_2$
  - h an homotopy class $M \to X$ with $h_{|\Sigma_i} = f_i$

Homotopy quantum field theories (HQFTs)

Idea: TQFTs for manifolds endowed with maps to a fixed target topological space X (with base point $*$)

The category $X\text{-}\mathrm{Cob}_n$ is a symmetric monoidal category

- an object is a pair (Σ, f)
  - Σ closed oriented pointed $(n-1)$-manifold
  - $f\colon (\Sigma, \Sigma_\bullet) \to (X, *)$ pointed map
- a morphism $f\colon (\Sigma_1, f_1) \to (\Sigma_2, f_2)$ is equiv. class of (M, f)
  - M an oriented n-cobordism $\Sigma_1 \to \Sigma_2$
  - h an homotopy class $M \to X$ with $h_{|\Sigma_i} = f_i$

Homotopy quantum field theories (HQFTs)

Idea: TQFTs for manifolds endowed with maps to a fixed target topological space X (with base point $*$)

The category $X\text{-}\mathrm{Cob}_n$ is a symmetric monoidal category

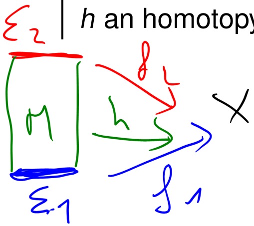
- an object is a pair (Σ, f)
  - Σ closed oriented pointed $(n-1)$-manifold
  - $f\colon (\Sigma, \Sigma_\bullet) \to (X, *)$ pointed map
- a morphism $f\colon (\Sigma_1, f_1) \to (\Sigma_2, f_2)$ is equiv. class of (M, f)
  - M an oriented n-cobordism $\Sigma_1 \to \Sigma_2$
  - h an homotopy class $M \to X$ with $h_{|\Sigma_i} = f_i$

Homotopy quantum field theories (HQFTs)

Idea: TQFTs for manifolds endowed with maps to a fixed target topological space X (with base point $*$)

The category $X\text{-}\mathrm{Cob}_n$ is a symmetric monoidal category

- an object is a pair (Σ, f)
  - Σ closed oriented pointed $(n-1)$-manifold
  - $f\colon (\Sigma, \Sigma_\bullet) \to (X, *)$ pointed map
- a morphism $f\colon (\Sigma_1, f_1) \to (\Sigma_2, f_2)$ is equiv. class of (M, f)
  - M an oriented n-cobordism $\Sigma_1 \to \Sigma_2$
  - h an homotopy class $M \to X$ with $h_{|\Sigma_i} = f_i$



Homotopy quantum field theories (HQFTs)

Idea: TQFTs for manifolds endowed with maps to a fixed target topological space X (with base point $*$)

The category $X\text{-}\mathrm{Cob}_n$ is a symmetric monoidal category

- an object is a pair (Σ, f)
  - Σ closed oriented pointed $(n-1)$-manifold
  - $f\colon (\Sigma, \Sigma_\bullet) \to (X, *)$ pointed map
- a morphism $f\colon (\Sigma_1, f_1) \to (\Sigma_2, f_2)$ is equiv. class of (M, f)
  - M an oriented n-cobordism $\Sigma_1 \to \Sigma_2$
  - h an homotopy class $M \to X$ with $h_{|\Sigma_i} = f_i$

  $(M, f) \sim (M', f')$ if $\exists$ o.p. diffeo $\phi\colon M \to M'$ such that $h'\phi = h$
- $\circ =$ gluing $\quad \otimes = \amalg \quad \mathbb{1} = (\emptyset, \emptyset \to X)$

Homotopy quantum field theories (HQFTs)

Idea: TQFTs for manifolds endowed with maps to a fixed target topological space X (with base point $*$)

The category $X\text{-}\mathrm{Cob}_n$ is a symmetric monoidal category

- an object is a pair (Σ, f)
  - Σ closed oriented pointed $(n-1)$-manifold
  - $f\colon (\Sigma, \Sigma_\bullet) \to (X, *)$ pointed map

- a morphism $f\colon (\Sigma_1, f_1) \to (\Sigma_2, f_2)$ is equiv. class of (M, f)
  - M an oriented n-cobordism $\Sigma_1 \to \Sigma_2$
  - h an homotopy class $M \to X$ with $h_{|\Sigma_i} = f_i$

  $(M, f) \sim (M', f')$ if $\exists$ o.p. diffeo $\phi\colon M \to M'$ such that $h'\phi = h$

- $\circ =$ gluing $\qquad \otimes = \amalg \qquad \mathbb{1} = (\emptyset, \emptyset \to X)$

Homotopy quantum field theories (HQFTs)

Idea: TQFTs for manifolds endowed with maps to a fixed target topological space X (with base point $*$)

The category $X\text{-}\mathrm{Cob}_n$ is a symmetric monoidal category

- an object is a pair (Σ, f)
  - Σ closed oriented pointed $(n-1)$-manifold
  - $f\colon (\Sigma, \Sigma_\bullet) \to (X, *)$ pointed map
- a morphism $f\colon (\Sigma_1, f_1) \to (\Sigma_2, f_2)$ is equiv. class of (M, f)
  - M an oriented n-cobordism $\Sigma_1 \to \Sigma_2$
  - h an homotopy class $M \to X$ with $h_{|\Sigma_i} = f_i$

  $(M, f) \sim (M', f')$ if $\exists$ o.p. diffeo $\phi\colon M \to M'$ such that $h'\phi = h$
- $\circ =$ gluing $\quad \otimes = \amalg \quad \mathbb{1} = (\emptyset, \emptyset \to X)$

Homotopy quantum field theories (HQFTs)

A *n*-dim HQFT with target *X* is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Data:

- $\Bbbk$-vector spaces $\tau\Big($ $\xrightarrow{f} X\Big)$
- $\Bbbk$-linear maps $\tau\Bigg($ $\xrightarrow{h} X\Bigg) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$
- isomorphisms $\tau\big((\Sigma, f) \amalg (\Sigma', f')\big) \simeq \tau(\Sigma, f) \otimes_{\Bbbk} \tau(\Sigma', f')$
- an isomorphism $\tau(\emptyset) \simeq \Bbbk$

Axioms: compatibilities with $\circ$, $\amalg$, and the symmetries

Homotopy quantum field theories (HQFTs)

A *n*-dim HQFT with target X is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Data:

- $\Bbbk$-vector spaces $\tau\Big($  $\xrightarrow{f} X\Big)$

- $\Bbbk$-linear maps $\tau\Big($ $\xrightarrow{h} X\Big) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$

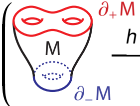
- isomorphisms $\tau\big((\Sigma, f) \amalg (\Sigma', f')\big) \simeq \tau(\Sigma, f) \otimes_{\Bbbk} \tau(\Sigma', f')$
- an isomorphism $\tau(\emptyset) \simeq \Bbbk$

Axioms: compatibilities with $\circ$, $\amalg$, and the symmetries

A *n-dim HQFT with target X* is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Data:

- $\Bbbk$-vector spaces $\tau\Big($  $\xrightarrow{f} X\Big)$
- $\Bbbk$-linear maps $\tau\Big($  $\xrightarrow{h} X\Big) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$
- isomorphisms $\tau\big((\Sigma, f) \amalg (\Sigma', f')\big) \simeq \tau(\Sigma, f) \otimes_{\Bbbk} \tau(\Sigma', f')$
- an isomorphism $\tau(\emptyset) \simeq \Bbbk$

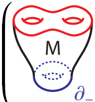
Axioms: compatibilities with $\circ$, $\amalg$, and the symmetries

Homotopy quantum field theories (HQFTs)

A *n*-dim HQFT with target X is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Data:

- $\Bbbk$-vector spaces $\tau\Big($  $\xrightarrow{f} X\Big)$
- $\Bbbk$-linear maps $\tau\left(\right.$  $\left.\xrightarrow{h} X\right) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$
- isomorphisms $\tau\big((\Sigma, f) \amalg (\Sigma', f')\big) \simeq \tau(\Sigma, f) \otimes_{\Bbbk} \tau(\Sigma', f')$
- an isomorphism $\tau(\emptyset) \simeq \Bbbk$

Axioms: compatibilities with $\circ$, $\amalg$, and the symmetries

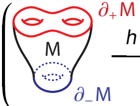
Homotopy quantum field theories (HQFTs)

A *n*-dim HQFT with target *X* is a symmetric monoidal functor

$$\tau \colon X\text{-}\,\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Data:

- $\Bbbk$-vector spaces $\tau\Big($ 

$\xrightarrow{f} X\Big)$
- $\Bbbk$-linear maps $\tau\Big($ 

$\xrightarrow{h} X\Big) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$
- isomorphisms $\tau\big((\Sigma, f) \amalg (\Sigma', f')\big) \simeq \tau(\Sigma, f) \otimes_{\Bbbk} \tau(\Sigma', f')$
- an isomorphism $\tau(\emptyset) \simeq \Bbbk$

Axioms: compatibilities with $\circ$, $\amalg$, and the symmetries

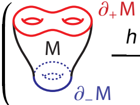
Homotopy quantum field theories (HQFTs)

A *n*-dim HQFT with target *X* is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Data:

- $\Bbbk$-vector spaces $\tau\Big($  $\xrightarrow{f} X\Big)$

- $\Bbbk$-linear maps $\tau\Bigg($  $\xrightarrow{h} X\Bigg) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$

- isomorphisms $\tau\big((\Sigma, f) \amalg (\Sigma', f')\big) \simeq \tau(\Sigma, f) \otimes_{\Bbbk} \tau(\Sigma', f')$
- an isomorphism $\tau(\emptyset) \simeq \Bbbk$

Axioms: compatibilities with $\circ$, $\amalg$, and the symmetries

Homotopy quantum field theories (HQFTs)

A *n-dim HQFT with target X* is a symmetric monoidal functor

$$\tau \colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\mathbb{k}}$$

Basic properties:

- $X = \{\mathrm{pt}\} \rightsquigarrow$ TQFT
- M closed oriented n-manifold, $h \in [M, X]$
  $\tau(M, h) \in \mathrm{End}_{\mathbb{k}}(\tau(\emptyset)) \simeq \mathbb{k}$ is a numerical invariant of h
- $\tau(\Sigma, f)$ is finite-dimensional and $\tau(\Sigma, f)^* \simeq \tau(-\Sigma, f)$
- τ induces finite-dimensional representation of
  $$\mathrm{MCG}(\Sigma, f) = \left\{\phi \colon \Sigma \to \Sigma \text{ o.p. diffeo } \middle| f\phi = f\right\}_{/\mathrm{isotopy}}$$
- $X\text{-}\mathrm{Cob}_n$ only depends (up to equivalence) on the n-homotopy type of X

Homotopy quantum field theories (HQFTs)

A *n-dim HQFT with target X* is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Basic properties:

- $X = \{\mathrm{pt}\} \rightsquigarrow$ TQFT
- M closed oriented n-manifold, $h \in [M, X]$
  $\tau(M, h) \in \mathrm{End}_{\Bbbk}(\tau(\emptyset)) \simeq \Bbbk$ is a numerical invariant of h
- $\tau(\Sigma, f)$ is finite-dimensional and $\tau(\Sigma, f)^* \simeq \tau(-\Sigma, f)$
- τ induces finite-dimensional representation of
  $$\mathrm{MCG}(\Sigma, f) = \left\{\phi\colon \Sigma \to \Sigma \text{ o.p. diffeo } \middle| f\phi = f\right\}_{/\mathrm{isotopy}}$$
- $X\text{-}\mathrm{Cob}_n$ only depends (up to equivalence) on the n-homotopy type of X

Homotopy quantum field theories (HQFTs)

A *n-dim HQFT with target X* is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Basic properties:

- $X = \{\mathrm{pt}\} \rightsquigarrow$ TQFT
- M closed oriented n-manifold, $h \in [M, X]$
  $\tau(M, h) \in \mathrm{End}_{\Bbbk}(\tau(\emptyset)) \simeq \Bbbk$ is a numerical invariant of h
- $\tau(\Sigma, f)$ is finite-dimensional and $\tau(\Sigma, f)^* \simeq \tau(-\Sigma, f)$
- τ induces finite-dimensional representation of
  $$\mathrm{MCG}(\Sigma, f) = \left\{\phi\colon \Sigma \to \Sigma \text{ o.p. diffeo } \middle| f\phi = f\right\}_{/\text{isotopy}}$$
- $X\text{-}\mathrm{Cob}_n$ only depends (up to equivalence) on the n-homotopy type of X

Homotopy quantum field theories (HQFTs)

A *n*-dim HQFT with target X is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Basic properties:

- $X = \{\mathrm{pt}\} \rightsquigarrow$ TQFT
- M closed oriented n-manifold, $h \in [M, X]$
  $\tau(M, h) \in \mathrm{End}_{\Bbbk}(\tau(\emptyset)) \simeq \Bbbk$ is a numerical invariant of h
- $\tau(\Sigma, f)$ is finite-dimensional and $\tau(\Sigma, f)^* \simeq \tau(-\Sigma, f)$
- τ induces finite-dimensional representation of
  $$\mathrm{MCG}(\Sigma, f) = \left\{\phi\colon \Sigma \to \Sigma \text{ o.p. diffeo } \middle| f\phi = f\right\}_{/\mathrm{isotopy}}$$
- $X\text{-}\mathrm{Cob}_n$ only depends (up to equivalence) on the n-homotopy type of X

Homotopy quantum field theories (HQFTs)

A *n-dim HQFT with target X* is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Basic properties:

- $X = \{\mathrm{pt}\} \rightsquigarrow$ TQFT
- M closed oriented n-manifold, $h \in [M, X]$
  $\tau(M, h) \in \mathrm{End}_{\Bbbk}(\tau(\emptyset)) \simeq \Bbbk$ is a numerical invariant of h
- $\tau(\Sigma, f)$ is finite-dimensional and $\tau(\Sigma, f)^* \simeq \tau(-\Sigma, f)$
- τ induces finite-dimensional representation of
  $$\mathrm{MCG}(\Sigma, f) = \left\{\phi\colon \Sigma \to \Sigma \text{ o.p. diffeo } \middle| f\phi = f\right\}_{/\mathrm{isotopy}}$$
- $X\text{-}\mathrm{Cob}_n$ only depends (up to equivalence) on the n-homotopy type of X

Homotopy quantum field theories (HQFTs)

A *n-dim HQFT with target X* is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Basic properties:

- $X = \{\mathrm{pt}\} \rightsquigarrow$ TQFT
- M closed oriented n-manifold, $h \in [M, X]$
  $\tau(M, h) \in \mathrm{End}_{\Bbbk}(\tau(\emptyset)) \simeq \Bbbk$ is a numerical invariant of h
- $\tau(\Sigma, f)$ is finite-dimensional and $\tau(\Sigma, f)^* \simeq \tau(-\Sigma, f)$
- τ induces finite-dimensional representation of
  $$\mathrm{MCG}(\Sigma, f) = \Big\{\phi\colon \Sigma \to \Sigma \text{ o.p. diffeo } \Big| f\phi = f\Big\}_{/\mathrm{isotopy}}$$
- $X\text{-}\mathrm{Cob}_n$ only depends (up to equivalence) on the n-homotopy type of X

HQFTs of dimension 1

There are bijective correspondences between:

1. 1-dimensional HQFTs with target X
2. finite-dimensional representations of $\pi_1(X)$
3. finite-dimensional flat vector bundles over X

Rk: HQFTs may be seen as higher-dimensional generalizations of finite-dimensional flat vector bundles

HQFTs of dimension 1

There are bijective correspondences between:

1. 1-dimensional HQFTs with target X
2. finite-dimensional representations of $\pi_1(X)$
3. finite-dimensional flat vector bundles over X

Rk: HQFTs may be seen as higher-dimensional generalizations of finite-dimensional flat vector bundles

HQFTs of dimension 1

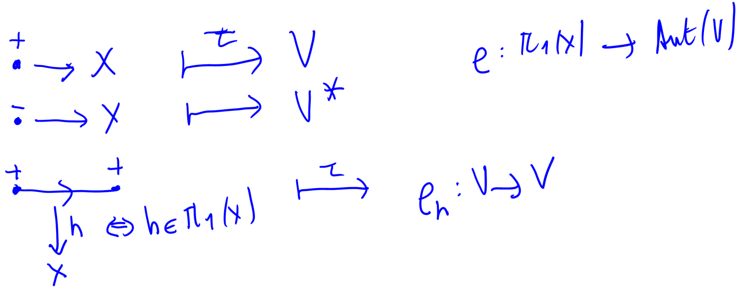
There are bijective correspondences between:

1. 1-dimensional HQFTs with target X
2. finite-dimensional representations of $\pi_1(X)$
3. finite-dimensional flat vector bundles over X

Rk: HQFTs may be seen as higher-dimensional generalizations of finite-dimensional flat vector bundles

There are bijective correspondences between:

1. 1-dimensional HQFTs with target X
2. finite-dimensional representations of $\pi_1(X)$
3. finite-dimensional flat vector bundles over X



Rk: HQFTs may be seen as higher-dimensional generalizations of finite-dimensional flat vector bundles

Cohomological HQFTs

$\theta \in H^n(X, \Bbbk^*) \quad \rightsquigarrow \quad n$-dim HQFT τ^θ with target X

τ^θ is characterized by :

- M closed oriented n-manifold, $h \in [M, X]$

  $$\tau^\theta(M, h) = \langle h^*(\theta), [M] \rangle \in \Bbbk$$

  where $[M] \in H_n(M, \mathbb{Z})$ is the fundamental class of M

- Σ closed oriented $(n-1)$-manifold, $f \colon \Sigma \to X$

  $\tau^\theta(\Sigma, f)$ is one-dimensional

Rk: τ^θ can be explicitly defined using singular chains representing the fundamental classes of Σ and M

Cohomological HQFTs

$\theta \in H^n(X, \Bbbk^*) \quad \rightsquigarrow \quad n$-dim HQFT τ^θ with target X

τ^θ is characterized by :

- M closed oriented n-manifold, $h \in [M, X]$

$$\tau^\theta(M, h) = \langle h^*(\theta), [M] \rangle \in \Bbbk$$

  where $[M] \in H_n(M, \mathbb{Z})$ is the fundamental class of M

- Σ closed oriented $(n-1)$-manifold, $f\colon \Sigma \to X$

$$\tau^\theta(\Sigma, f) \text{ is one-dimensional}$$

Rk: τ^θ can be explicitly defined using singular chains representing the fundamental classes of Σ and M

Cohomological HQFTs

$\theta \in H^n(X, \Bbbk^*) \quad \rightsquigarrow \quad n$-dim HQFT τ^θ with target X

τ^θ is characterized by :

- M closed oriented n-manifold, $h \in [M, X]$

  $$\tau^\theta(M, h) = \langle h^*(\theta), [M] \rangle \in \Bbbk$$

  where $[M] \in H_n(M, \mathbb{Z})$ is the fundamental class of M

- Σ closed oriented $(n-1)$-manifold, $f \colon \Sigma \to X$

  $\tau^\theta(\Sigma, f)$ is one-dimensional

Rk: τ^θ can be explicitly defined using singular chains representing the fundamental classes of Σ and M

Cohomological HQFTs

$\theta \in H^n(X, \Bbbk^*) \quad \rightsquigarrow \quad n$-dim HQFT τ^θ with target X

τ^θ is characterized by :

- M closed oriented n-manifold, $h \in [M, X]$

$$\tau^\theta(M, h) = \langle h^*(\theta), [M] \rangle \in \Bbbk$$

  where $[M] \in H_n(M, \mathbb{Z})$ is the fundamental class of M

- Σ closed oriented $(n-1)$-manifold, $f \colon \Sigma \to X$

  $\tau^\theta(\Sigma, f)$ is one-dimensional

Rk: τ^θ can be explicitly defined using singular chains representing the fundamental classes of Σ and M

Cohomological HQFTs

$\theta \in H^n(X, \mathbb{k}^*) \quad \rightsquigarrow \quad n$-dim HQFT τ^θ with target X

τ^θ is characterized by :

- M closed oriented n-manifold, $h \in [M, X]$

  $$\tau^\theta(M, h) = \langle h^*(\theta), [M] \rangle \in \mathbb{k}$$

  where $[M] \in H_n(M, \mathbb{Z})$ is the fundamental class of M

- Σ closed oriented $(n-1)$-manifold, $f \colon \Sigma \to X$

  $\tau^\theta(\Sigma, f)$ is one-dimensional

Rk: τ^θ can be explicitly defined using singular chains representing the fundamental classes of Σ and M

Cohomological HQFTs

$\theta \in H^n(X, \mathbb{k}^*)$ $\rightsquigarrow$ n-dim HQFT τ^θ with target X

τ^θ is characterized by :

- M closed oriented n-manifold, $h \in [M, X]$

  $$\tau^\theta(M, h) = \langle h^*(\theta), [M] \rangle \in \mathbb{k}$$

  where $[M] \in H_n(M, \mathbb{Z})$ is the fundamental class of M

- Σ closed oriented $(n-1)$-manifold, $f \colon \Sigma \to X$

  $\tau^\theta(\Sigma, f)$ is one-dimensional

Rk: τ^θ can be explicitly defined using singular chains representing the fundamental classes of Σ and M

Cohomological HQFTs

$\theta \in H^n(X, \Bbbk^*) \quad \rightsquigarrow \quad n$-dim HQFT τ^θ with target X

τ^θ is characterized by :

- M closed oriented n-manifold, $h \in [M, X]$

  $$\tau^\theta(M, h) = \langle h^*(\theta), [M] \rangle \in \Bbbk$$

  where $[M] \in H_n(M, \mathbb{Z})$ is the fundamental class of M

- Σ closed oriented $(n-1)$-manifold, $f \colon \Sigma \to X$

  $\tau^\theta(\Sigma, f)$ is one-dimensional

Rk: τ^θ can be explicitly defined using singular chains representing the fundamental classes of Σ and M

The case of aspherical targets

From now, assume that X is aspherical (i.e., $\pi_i(X) = 0$ for $i \geq 2$)

$\rightsquigarrow$ X is a $K(G, 1)$-space with $G = \pi_1(X)$

(Turaev, 2000)

2-dim HQFTs with target X $\Leftrightarrow$ G-graded Frobenius algebras

(Sozer, 2019)

Classification of 2-dim extended HQFTs with target X

The case of aspherical targets

From now, assume that X is aspherical (i.e., $\pi_i(X) = 0$ for $i \geq 2$)

$\rightsquigarrow$ X is a $K(G, 1)$-space with $G = \pi_1(X)$

(Turaev, 2000)

2-dim HQFTs with target X $\Leftrightarrow$ G-graded Frobenius algebras

(Sozer, 2019)

Classification of 2-dim extended HQFTs with target X

The case of aspherical targets

From now, assume that X is aspherical (i.e., $\pi_i(X) = 0$ for $i \geq 2$)

$\rightsquigarrow$ X is a $K(G, 1)$-space with $G = \pi_1(X)$

(Turaev, 2000)

2-dim HQFTs with target X $\Leftrightarrow$ G-graded Frobenius algebras

(Sozer, 2019)

Classification of 2-dim extended HQFTs with target X

The case of aspherical targets

From now, assume that X is aspherical (i.e., $\pi_i(X) = 0$ for $i \geq 2$)

$\rightsquigarrow$ X is a $K(G, 1)$-space with $G = \pi_1(X)$

(Turaev, 2000)

2-dim HQFTs with target X $\Leftrightarrow$ G-graded Frobenius algebras

(Sozer, 2019)

Classification of 2-dim extended HQFTs with target X

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

- **Turaev-Viro (92), Barret-Westburry (96)**

triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

- **Turaev-Viro (92), Barret-Westburry (96)**

triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

- **Turaev-Viro (92), Barret-Westburry (96)**

triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

- **Turaev-Viro (92), Barret-Westburry (96)**

triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

- **Turaev-Viro (92), Barret-Westburry (96)**

triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery  + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

- **Turaev-Viro (92), Barret-Westburry (96)**

triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery  + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

- **Turaev-Viro (92), Barret-Westburry (96)**

triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery  + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

- **Turaev-Viro (92), Barret-Westburry (96)**

triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery  + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

- **Turaev-Viro (92), Barret-Westburry (96)**

triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery  + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery + $\mathscr{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathscr{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

3-dimensional HQFTs with target $X = K(G,1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\mathbb{k}$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\left(\right.$  $\left. \right)$

Example: G- $\mathrm{vect}_{\mathbb{k}}^{\theta}$ with $\theta \in H^3(G, \mathbb{k}^*)$

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\mathbb{k}$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}} \left(\right.$  $\left. \right)$

Example: $G\text{-}\mathrm{vect}_{\mathbb{k}}^{\theta}$ with $\theta \in H^3(G, \mathbb{k}^*)$

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\mathbb{k}$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\left(\right.$  $\left.\right)$

Example: $G\text{-}\mathrm{vect}_{\mathbb{k}}^{\theta}$ with $\theta \in H^3(G, \mathbb{k}^*)$

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\Bbbk$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

$\rightsquigarrow$ $6j$-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\left(\right.$  $\left. \right)$

Example: $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ with $\theta \in H^3(G, \Bbbk^*)$

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\mathbb{k}$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}} \left(\right.$  $\left. \right)$

Example: $G\text{-}\mathrm{vect}_{\mathbb{k}}^{\theta}$ with $\theta \in H^3(G, \mathbb{k}^*)$

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\Bbbk$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\left(\right.$  $\left. \right)$

Example: $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ with $\theta \in H^3(G, \Bbbk^*)$

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\mathbb{k}$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\left(\right.$  $\left. \right)$

Example: $G\text{-}\mathrm{vect}_{\mathbb{k}}^{\theta}$ with $\theta \in H^3(G, \mathbb{k}^*)$

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\mathbb{k}$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

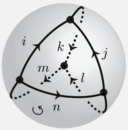
$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}} \left(\right.$  $\left. \right)$

Example: $G\text{-}\mathrm{vect}_{\mathbb{k}}^{\theta}$ with $\theta \in H^3(G, \mathbb{k}^*)$

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\mathbb{k}$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - ▸ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - ▸ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

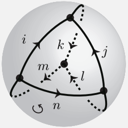
$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\left(\right.$  $\left. \right)$

Example: G- $\mathrm{vect}_{\mathbb{k}}^{\theta}$ with $\theta \in H^3(G, \mathbb{k}^*)$

Spherical fusion graded categories

$\mathcal{C}$= spherical fusion G-graded category:

- $\mathcal{C}$ is $\mathbb{k}$-linear monoidal
- each object X has a 2-sided dual X^* (+ sphericity condition)
- $\mathcal{C}$ has a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$:
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h \quad \Rightarrow \quad X \otimes Y \in \mathcal{C}_{gh}$
  - ▹ $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ with $g \neq h \Rightarrow \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$
- $\mathcal{C}$ is semisimple
- each $\mathcal{C}_g$ has finitely many simple objects

$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\left(\right.$  $\left. \right)$

Example: $G\text{-}\mathrm{vect}_{\mathbb{k}}^{\theta}$ with $\theta \in H^3(G, \mathbb{k}^*)$

State sum HQFT with target $X = K(G,1)$

$C = \bigoplus_{g \in G} C_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in C_{[g(e)]}$ simple object

$$\mathrm{HTV}_C(M,h) = \sum_c \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta \ |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta \ |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

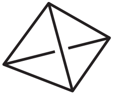
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta \, |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

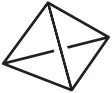
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object

 $\xrightarrow{g} X$

$$\mathrm{HTV}_{\mathcal{C}}(M, h) = \sum_{c} \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta \ |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object

$\xrightarrow{g} X$

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta |\Delta|) \in \mathbb{k}$$

State sum HQFT with target $X = K(G,1)$

$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object

$\xrightarrow{g} X$

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta \; |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object

$\xrightarrow{g} X$

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object

$\xrightarrow{g} X$

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \mathrm{coef}(c) \; \mathrm{ctr}_f(\otimes_\Delta |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G, 1)$

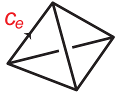
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$$\mathrm{HTV}_{\mathcal{C}}(M, h) = \sum_{c} \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

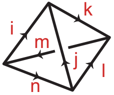
$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta \ |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

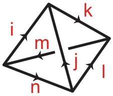
$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \mathrm{coef}(c) \; \mathrm{ctr}_f(\otimes_\Delta |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

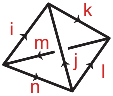
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M, h) = \sum_{c} \mathrm{coef}(c) \; \mathrm{ctr}_f\big(\otimes_\Delta |\Delta|\big) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

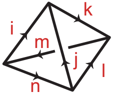
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

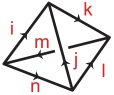
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta |\Delta|) \in \Bbbk$$

State sum HQFT with target $X = K(G,1)$

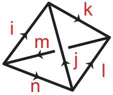
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

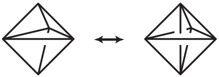
c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta \ |\Delta|) \in \Bbbk$$

Pachner moves



State sum HQFT with target $X = K(G, 1)$

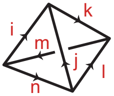
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

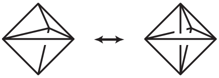
e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M, h) = \sum_{c} \left(\prod_{e} \dim_q(c_e) \right) \mathrm{ctr}_f\left(\otimes_{\Delta} |\Delta| \right) \in \Bbbk$$

Pachner moves 

State sum HQFT with target $X = K(G,1)$

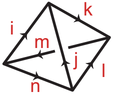
$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \left(\prod_{e} \dim_q(c_e) \right) \mathrm{ctr}_f\big(\otimes_\Delta \, |\Delta| \big) \in \Bbbk$$

Example: $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ $\leftsquigarrow$ $\theta \in H^3(G, \Bbbk^*) \cong H^3(X, \Bbbk^*)$ $\rightsquigarrow$ τ^θ

$$\mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}} \cong \tau^\theta$$

State sum HQFT with target $X = K(G,1)$

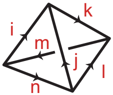
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M, h) = \sum_{c} \left(\prod_{e} \dim_q(c_e) \right) \mathrm{ctr}_f\left(\otimes_\Delta \ |\Delta| \right) \in \Bbbk$$

Example: $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta} \leftrightsquigarrow \theta \in H^3(G, \Bbbk^*) \cong H^3(X, \Bbbk^*) \rightsquigarrow \tau^{\theta}$

$$\mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}} \cong \tau^{\theta}$$

State sum HQFT with target $X = K(G,1)$

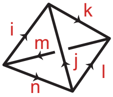
$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \left(\prod_{e} \dim_q(c_e) \right) \mathrm{ctr}_f\big(\otimes_{\Delta} \; |\Delta| \big) \in \Bbbk$$

Example: $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ $\leftsquigarrow$ $\theta \in H^3(G, \Bbbk^*) \cong H^3(X, \Bbbk^*)$ $\rightsquigarrow$ τ^{θ}

$$\mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}} \cong \tau^{\theta}$$

State sum HQFT with target $X = K(G,1)$

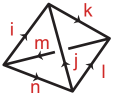
$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X,*) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M,h) = \sum_{c} \left(\prod_{e} \dim_q(c_e) \right) \mathrm{ctr}_f\left(\otimes_\Delta \; |\Delta| \right) \in \Bbbk$$

Example: $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta} \;\leftsquigarrow\; \theta \in H^3(G,\Bbbk^*) \cong H^3(X,\Bbbk^*) \;\rightsquigarrow\; \tau^\theta$

$\mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}} \cong \tau^\theta$

State sum HQFT with target $X = K(G, 1)$

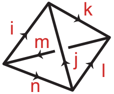
$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in \mathcal{C}_{[g(e)]}$ simple object



$\rightsquigarrow$ $|\Delta| = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 6j-symbol

$$\mathrm{HTV}_{\mathcal{C}}(M, h) = \sum_{c} \left(\prod_{e} \dim_q(c_e) \right) \mathrm{ctr}_f\left(\otimes_\Delta |\Delta| \right) \in \Bbbk$$

Example: $G\text{-}\mathrm{vect}^{\theta}_{\Bbbk}$ $\leftsquigarrow$ $\theta \in H^3(G, \Bbbk^*) \cong H^3(X, \Bbbk^*)$ $\rightsquigarrow$ τ^θ

$$\mathrm{HTV}_{G\text{-}\mathrm{vect}^{\theta}_{\Bbbk}} \cong \tau^\theta$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_\mathcal{C}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_\mathcal{C}$

$$\mathrm{TV}_\mathcal{C}(\Sigma) = \bigoplus_{f\in[\Sigma,B\Gamma]} \mathrm{HTV}_\mathcal{C}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_\mathcal{C}(M) = \sum_{h\in[M,B\Gamma]} \mathrm{HTV}_\mathcal{C}(M,h)$$

Example: $\theta \in H^3(G, \Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_\Bbbk^\theta$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M) = \sum_{h\in[M,BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M,h) = \sum_{h\in[M,BG]} \tau^\theta(M,h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h\colon \pi_1(M)\to G} \langle h^*(\theta), [M]\rangle$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

$$\mathrm{TV}_{\mathcal{C}}(\Sigma) = \bigoplus_{f \in [\Sigma, B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_{\mathcal{C}}(M) = \sum_{h \in [M, B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(M, h)$$

Example: $\theta \in H^3(G, \Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M) = \sum_{h \in [M, BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M, h) = \sum_{h \in [M, BG]} \tau^{\theta}(M, h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h\colon \pi_1(M) \to G} \langle h^*(\theta), [M] \rangle$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_\mathcal{C}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_\mathcal{C}$

$$\mathrm{TV}_\mathcal{C}(\Sigma) = \bigoplus_{f \in [\Sigma, B\Gamma]} \mathrm{HTV}_\mathcal{C}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_\mathcal{C}(M) = \sum_{h \in [M, B\Gamma]} \mathrm{HTV}_\mathcal{C}(M, h)$$

Example: $\theta \in H^3(G, \Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_\Bbbk^\theta$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M) = \sum_{h \in [M, BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M, h) = \sum_{h \in [M, BG]} \tau^\theta(M, h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h \colon \pi_1(M) \to G} \langle h^*(\theta), [M] \rangle$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

$$\mathrm{TV}_{\mathcal{C}}(\Sigma) = \bigoplus_{f \in [\Sigma, B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_{\mathcal{C}}(M) = \sum_{h \in [M, B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(M, h)$$

Example: $\theta \in H^3(G, \Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M) = \sum_{h \in [M, BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M, h) = \sum_{h \in [M, BG]} \tau^{\theta}(M, h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h\colon \pi_1(M) \to G} \langle h^*(\theta), [M] \rangle$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_\mathcal{C}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_\mathcal{C}$

$$\mathrm{TV}_\mathcal{C}(\Sigma) = \bigoplus_{f\in[\Sigma,B\Gamma]} \mathrm{HTV}_\mathcal{C}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_\mathcal{C}(M) = \sum_{h\in[M,B\Gamma]} \mathrm{HTV}_\mathcal{C}(M,h)$$

Example: $\theta \in H^3(G,\Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_\Bbbk^\theta$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M) = \sum_{h\in[M,BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M,h) = \sum_{h\in[M,BG]} \tau^\theta(M,h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h\colon \pi_1(M)\to G} \langle h^*(\theta), [M]\rangle$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_\mathcal{C}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_\mathcal{C}$

$$\mathrm{TV}_\mathcal{C}(\Sigma) = \bigoplus_{f \in [\Sigma, B\Gamma]} \mathrm{HTV}_\mathcal{C}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_\mathcal{C}(M) = \sum_{h \in [M, B\Gamma]} \mathrm{HTV}_\mathcal{C}(M, h)$$

Example: $\theta \in H^3(G, \Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_\Bbbk^\theta$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M) = \sum_{h \in [M, BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M, h) = \sum_{h \in [M, BG]} \tau^\theta(M, h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h\colon \pi_1(M) \to G} \langle h^*(\theta), [M] \rangle$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

$$\mathrm{TV}_{\mathcal{C}}(\Sigma) = \bigoplus_{f \in [\Sigma, B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_{\mathcal{C}}(M) = \sum_{h \in [M, B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(M, h)$$

Example: $\theta \in H^3(G, \Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M) = \sum_{h \in [M, BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M, h) = \sum_{h \in [M, BG]} \tau^{\theta}(M, h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h:\, \pi_1(M) \to G} \langle h^*(\theta), [M] \rangle$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

$$\mathrm{TV}_{\mathcal{C}}(\Sigma) = \bigoplus_{f\in[\Sigma,B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_{\mathcal{C}}(M) = \sum_{h\in[M,B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(M, h)$$

Example: $\theta \in H^3(G, \Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M) = \sum_{h\in[M,BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M, h) = \sum_{h\in[M,BG]} \tau^{\theta}(M, h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h\colon \pi_1(M)\to G} \langle h^*(\theta), [M]\rangle$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

$$\mathrm{TV}_{\mathcal{C}}(\Sigma) = \bigoplus_{f\in[\Sigma,B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_{\mathcal{C}}(M) = \sum_{h\in[M,B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(M, h)$$

Example: $\theta \in H^3(G, \Bbbk^*)$ $\rightsquigarrow$ | $G\text{-}\,\mathrm{vect}^{\theta}_{\Bbbk}$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\,\mathrm{vect}^{\theta}_{\Bbbk}}(M) = \sum_{h\in[M,BG]} \mathrm{HTV}_{G\text{-}\,\mathrm{vect}^{\theta}_{\Bbbk}}(M, h) = \sum_{h\in[M,BG]} \tau^{\theta}(M, h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h\colon \pi_1(M)\to G} \langle h^*(\theta), [M]\rangle$$

HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

$\mathcal{C}$ spherical fusion Γ-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

$$\mathrm{TV}_{\mathcal{C}}(\Sigma) = \bigoplus_{f\in[\Sigma,B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \quad \text{and} \quad \mathrm{TV}_{\mathcal{C}}(M) = \sum_{h\in[M,B\Gamma]} \mathrm{HTV}_{\mathcal{C}}(M, h)$$

Example: $\theta \in H^3(G, \Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M) = \sum_{h\in[M,BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}}(M, h) = \sum_{h\in[M,BG]} \tau^{\theta}(M, h)$$

$$\rightsquigarrow \quad \mathrm{DW}_{G,\theta}(M) = \sum_{h:\, \pi_1(M)\to G} \langle h^*(\theta), [M]\rangle$$

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

Modular fusion graded categories

$\mathcal{B}$= modular fusion G-graded category:

- $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is spherical fusion G-graded
- $\mathcal{B}$ has an action $\varphi \colon \underline{G} \to \mathrm{Aut}_\otimes(\mathcal{B})$ such that $\varphi_g(\mathcal{B}_h) \subset \mathcal{B}_{ghg^{-1}}$
- $\mathcal{B}$ has a G-braiding: for $X \in \mathcal{B}_g$ and $Y \in \mathcal{B}_h$,

$$\tau_{X,Y} \colon X \otimes Y \to \varphi_g(Y) \otimes X$$

- the S-matrix of fusion category $\mathcal{B}_1$ is invertible

$\rightsquigarrow$ Invariant $I_\mathcal{B}$ of $\mathcal{B}$-colored framed oriented G-links in S^3

$$(L, f \colon \pi_1(L) \to G)$$

whose longitudes are sent to 1 by f

Modular fusion graded categories

$\mathcal{B}$= modular fusion G-graded category:

- $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is spherical fusion G-graded
- $\mathcal{B}$ has an action $\varphi\colon \underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{B})$ such that $\varphi_g(\mathcal{B}_h) \subset \mathcal{B}_{ghg^{-1}}$
- $\mathcal{B}$ has a G-braiding: for $X \in \mathcal{B}_g$ and $Y \in \mathcal{B}_h$,

$$\tau_{X,Y}\colon X \otimes Y \to \varphi_g(Y) \otimes X$$

- the S-matrix of fusion category $\mathcal{B}_1$ is invertible

$\rightsquigarrow$ Invariant $I_{\mathcal{B}}$ of $\mathcal{B}$-colored framed oriented G-links in S^3

$$(L, f\colon \pi_1(L) \to G)$$

whose longitudes are sent to 1 by f

Modular fusion graded categories

$\mathcal{B}$= modular fusion G-graded category:

- $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is spherical fusion G-graded
- $\mathcal{B}$ has an action $\varphi \colon \underline{G} \to \mathrm{Aut}_\otimes(\mathcal{B})$ such that $\varphi_g(\mathcal{B}_h) \subset \mathcal{B}_{ghg^{-1}}$
- $\mathcal{B}$ has a G-braiding: for $X \in \mathcal{B}_g$ and $Y \in \mathcal{B}_h$,

$$\tau_{X,Y} \colon X \otimes Y \to \varphi_g(Y) \otimes X$$

- the S-matrix of fusion category $\mathcal{B}_1$ is invertible

$\rightsquigarrow$ Invariant $I_\mathcal{B}$ of $\mathcal{B}$-colored framed oriented G-links in S^3

$$(L, f \colon \pi_1(L) \to G)$$

whose longitudes are sent to 1 by f

Modular fusion graded categories

$\mathcal{B}$= modular fusion G-graded category:

- $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is spherical fusion G-graded
- $\mathcal{B}$ has an action $\varphi \colon \underline{G} \to \mathrm{Aut}_\otimes(\mathcal{B})$ such that $\varphi_g(\mathcal{B}_h) \subset \mathcal{B}_{ghg^{-1}}$
- $\mathcal{B}$ has a G-braiding: for $X \in \mathcal{B}_g$ and $Y \in \mathcal{B}_h$,

$$\tau_{X,Y} \colon X \otimes Y \to \varphi_g(Y) \otimes X$$

- the S-matrix of fusion category $\mathcal{B}_1$ is invertible

$\rightsquigarrow$ Invariant $I_\mathcal{B}$ of $\mathcal{B}$-colored framed oriented G-links in S^3

$$(L, f \colon \pi_1(L) \to G)$$

whose longitudes are sent to 1 by f

Modular fusion graded categories

$\mathcal{B}$= modular fusion G-graded category:

- $\mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$ is spherical fusion G-graded
- $\mathcal{B}$ has an action $\varphi\colon \underline{G} \to \mathrm{Aut}_\otimes(\mathcal{B})$ such that $\varphi_g(\mathcal{B}_h) \subset \mathcal{B}_{ghg^{-1}}$
- $\mathcal{B}$ has a G-braiding: for $X \in \mathcal{B}_g$ and $Y \in \mathcal{B}_h$,

$$\tau_{X,Y}\colon X \otimes Y \to \varphi_g(Y) \otimes X$$

- the S-matrix of fusion category $\mathcal{B}_1$ is invertible

$\rightsquigarrow$ Invariant $I_\mathcal{B}$ of $\mathcal{B}$-colored framed oriented G-links in S^3

$$(L, f\colon \pi_1(L) \to G)$$

whose longitudes are sent to 1 by f

Modular fusion graded categories

$\mathcal{B}$= modular fusion G-graded category:

- $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is spherical fusion G-graded
- $\mathcal{B}$ has an action $\varphi \colon \underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{B})$ such that $\varphi_g(\mathcal{B}_h) \subset \mathcal{B}_{ghg^{-1}}$
- $\mathcal{B}$ has a G-braiding: for $X \in \mathcal{B}_g$ and $Y \in \mathcal{B}_h$,

$$\tau_{X,Y} \colon X \otimes Y \to \varphi_g(Y) \otimes X$$

- the S-matrix of fusion category $\mathcal{B}_1$ is invertible

$\rightsquigarrow$ Invariant $I_{\mathcal{B}}$ of $\mathcal{B}$-colored framed oriented G-links in S^3

$$(L, f \colon \pi_1(L) \to G)$$

whose longitudes are sent to 1 by f

Modular fusion graded categories

$\mathcal{B}$= modular fusion G-graded category:

- $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is spherical fusion G-graded
- $\mathcal{B}$ has an action $\varphi\colon \underline{G} \to \mathrm{Aut}_\otimes(\mathcal{B})$ such that $\varphi_g(\mathcal{B}_h) \subset \mathcal{B}_{ghg^{-1}}$
- $\mathcal{B}$ has a G-braiding: for $X \in \mathcal{B}_g$ and $Y \in \mathcal{B}_h$,

$$\tau_{X,Y}\colon X \otimes Y \to \varphi_g(Y) \otimes X$$

- the S-matrix of fusion category $\mathcal{B}_1$ is invertible

$\rightsquigarrow$ Invariant $I_\mathcal{B}$ of $\mathcal{B}$-colored framed oriented G-links in S^3

$$(L, f\colon \pi_1(L) \to G)$$

whose longitudes are sent to 1 by f

Surgical HQFT with target $X = K(G,1)$

$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f\colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L

Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M,h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i) \right) \mathcal{I}_{\mathcal{B}}(L, f, \underline{V})$$

Surgical HQFT with target $X = K(G,1)$

$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f \colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L

Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M, h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i) \right) \mathcal{I}_{\mathcal{B}}(L, f, \underline{V})$$

Surgical HQFT with target $X = K(G, 1)$

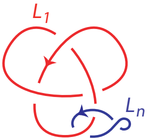
$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f\colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L



Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M, h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i) \right) \mathcal{I}_{\mathcal{B}}(L, f, \underline{V})$$

Surgical HQFT with target $X = K(G, 1)$

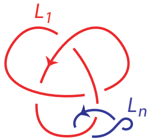
$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f \colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L



Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M, h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i) \right) \mathcal{I}_{\mathcal{B}}(L, f, \underline{V})$$

Surgical HQFT with target $X = K(G, 1)$

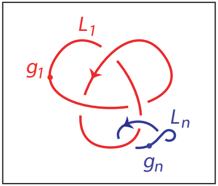
$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f\colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L



Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M, h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i) \right) \mathcal{T}_{\mathcal{B}}(L, f, \underline{V})$$

Surgical HQFT with target $X = K(G, 1)$

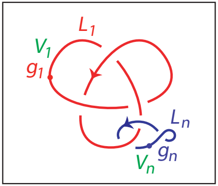
$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f\colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L



Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M, h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i) \right) \mathcal{I}_{\mathcal{B}}(L, f, \underline{V})$$

Surgical HQFT with target $X = K(G,1)$

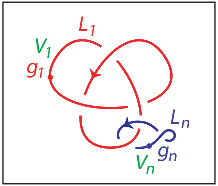
$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f \colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L



Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M, h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i) \right) \mathcal{I}_{\mathcal{B}}(L, f, \underline{V})$$

Surgical HQFT with target $X = K(G,1)$

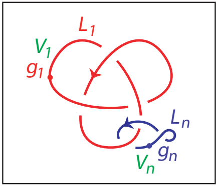
$\mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f\colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L



Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M,h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i)\right) \mathcal{I}_{\mathcal{B}}(L, f, \underline{V})$$

Surgical HQFT with target $X = K(G,1)$

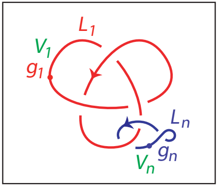
$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f\colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L



Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M,h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i) \right) \mathcal{I}_{\mathcal{B}}(L, f, \underline{V})$$

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

The graded center

$\mathcal{C}$ monoidal category, $\mathcal{D}$ monoidal subcategory of $\mathcal{C}$

The **center of $\mathcal{C}$ relative to $\mathcal{D}$** is the monoidal category $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

- objects of $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

$$X \in \mathcal{C} \quad \text{such that} \quad X \otimes Y = Y \otimes X \quad \forall\, Y \in \mathcal{D}$$

- $(X,\sigma) \otimes (X',\sigma') = \big(X \otimes X', (\sigma \otimes \mathrm{id}_{X'})(\mathrm{id}_X \otimes \sigma')\big)$

$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ spherical fusion G-graded category

The **G-center of $\mathcal{C}$** is $\mathcal{Z}_G(\mathcal{C}) = \mathcal{Z}(\mathcal{C},\mathcal{C}_1)$

$\mathcal{Z}_G(\mathcal{C})$ is G-graded with $\big(\mathcal{Z}_G(\mathcal{C})\big)_g = \big\{(X,\sigma) \in \mathcal{Z}_G(\mathcal{C}) \,\big|\, X \in \mathcal{C}_g\big\}$

Theorem (Gelaki-Naidu-Nikshych, 2009)

$\mathcal{Z}_G(\mathcal{C})$ has a G-action and a G-braiding

The graded center

$\mathcal{C}$ monoidal category, $\mathcal{D}$ monoidal subcategory of $\mathcal{C}$

The **center of $\mathcal{C}$ relative to $\mathcal{D}$** is the monoidal category $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

- objects of $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

$$X \in \mathcal{C} \quad \text{such that} \quad X \otimes Y = Y \otimes X \quad \forall\, Y \in \mathcal{D}$$

- $(X,\sigma) \otimes (X',\sigma') = \big(X \otimes X', (\sigma \otimes \mathrm{id}_{X'})(\mathrm{id}_X \otimes \sigma')\big)$

$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

The **G-center of $\mathcal{C}$** is $\mathcal{Z}_G(\mathcal{C}) = \mathcal{Z}(\mathcal{C},\mathcal{C}_1)$

$\mathcal{Z}_G(\mathcal{C})$ is G-graded with $\big(\mathcal{Z}_G(\mathcal{C})\big)_g = \big\{(X,\sigma) \in \mathcal{Z}_G(\mathcal{C}) \,\big|\, X \in \mathcal{C}_g\big\}$

Theorem (Gelaki-Naidu-Nikshych, 2009)

$\mathcal{Z}_G(\mathcal{C})$ has a G-action and a G-braiding

The graded center

$\mathcal{C}$ monoidal category, $\mathcal{D}$ monoidal subcategory of $\mathcal{C}$

The **center of $\mathcal{C}$ relative to $\mathcal{D}$** is the monoidal category $\mathcal{Z}(\mathcal{C}, \mathcal{D})$:

- objects of $\mathcal{Z}(\mathcal{C}, \mathcal{D})$:

$$X \in \mathcal{C} \quad \text{such that} \quad X \otimes Y = Y \otimes X \quad \forall\, Y \in \mathcal{D}$$

- $(X, \sigma) \otimes (X', \sigma') = \big(X \otimes X', (\sigma \otimes \mathrm{id}_{X'})(\mathrm{id}_X \otimes \sigma')\big)$

$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

The **G-center of $\mathcal{C}$** is $\mathcal{Z}_G(\mathcal{C}) = \mathcal{Z}(\mathcal{C}, \mathcal{C}_1)$

$\mathcal{Z}_G(\mathcal{C})$ is G-graded with $\big(\mathcal{Z}_G(\mathcal{C})\big)_g = \big\{(X, \sigma) \in \mathcal{Z}_G(\mathcal{C}) \,\big|\, X \in \mathcal{C}_g\big\}$

Theorem (Gelaki-Naidu-Nikshych, 2009)

$\mathcal{Z}_G(\mathcal{C})$ has a G-action and a G-braiding

The graded center

$\mathcal{C}$ monoidal category, $\mathcal{D}$ monoidal subcategory of $\mathcal{C}$

The **center of $\mathcal{C}$ relative to $\mathcal{D}$** is the monoidal category $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

- objects of $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

$$X \in \mathcal{C} \quad \text{with} \quad \sigma = \left\{\sigma_Y \colon\ X \otimes Y \xrightarrow{\cong} Y \otimes X\ \right\}_{Y \in \mathcal{D}}$$

- $(X,\sigma)\otimes(X',\sigma') = \big(X \otimes X', (\sigma \otimes \mathrm{id}_{X'})(\mathrm{id}_X \otimes \sigma')\big)$

$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

The **G-center of $\mathcal{C}$** is $\mathcal{Z}_G(\mathcal{C}) = \mathcal{Z}(\mathcal{C},\mathcal{C}_1)$

$\mathcal{Z}_G(\mathcal{C})$ is G-graded with $\big(\mathcal{Z}_G(\mathcal{C})\big)_g = \left\{(X,\sigma) \in \mathcal{Z}_G(\mathcal{C}) \,\middle|\, X \in \mathcal{C}_g\right\}$

Theorem (Gelaki-Naidu-Nikshych, 2009)

$\mathcal{Z}_G(\mathcal{C})$ has a G-action and a G-braiding

The graded center

$\mathcal{C}$ monoidal category, $\mathcal{D}$ monoidal subcategory of $\mathcal{C}$

The **center of $\mathcal{C}$ relative to $\mathcal{D}$** is the monoidal category $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

- objects of $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

$$X \in \mathcal{C} \quad \text{with} \quad \sigma = \left\{\sigma_Y\colon\ X \otimes Y \xrightarrow{\cong} Y \otimes X\ \right\}_{Y \in \mathcal{D}}$$

- $(X,\sigma) \otimes (X',\sigma') = \big(X \otimes X', (\sigma \otimes \mathrm{id}_{X'})(\mathrm{id}_X \otimes \sigma')\big)$

$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

The **G-center of $\mathcal{C}$** is $\mathcal{Z}_G(\mathcal{C}) = \mathcal{Z}(\mathcal{C},\mathcal{C}_1)$

$\mathcal{Z}_G(\mathcal{C})$ is G-graded with $\big(\mathcal{Z}_G(\mathcal{C})\big)_g = \big\{(X,\sigma) \in \mathcal{Z}_G(\mathcal{C}) \,\big|\, X \in \mathcal{C}_g\big\}$

Theorem (Gelaki-Naidu-Nikshych, 2009)

$\mathcal{Z}_G(\mathcal{C})$ has a G-action and a G-braiding

The graded center

$\mathcal{C}$ monoidal category, $\mathcal{D}$ monoidal subcategory of $\mathcal{C}$

The **center of $\mathcal{C}$ relative to $\mathcal{D}$** is the monoidal category $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

- objects of $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

$$X \in \mathcal{C} \quad \text{with} \quad \sigma = \left\{\sigma_Y \colon\ X \otimes Y \xrightarrow{\cong} Y \otimes X \right\}_{Y \in \mathcal{D}}$$

- $(X,\sigma) \otimes (X',\sigma') = \big(X \otimes X', (\sigma \otimes \mathrm{id}_{X'})(\mathrm{id}_X \otimes \sigma')\big)$

$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

The **G-center of $\mathcal{C}$** is $\mathcal{Z}_G(\mathcal{C}) = \mathcal{Z}(\mathcal{C},\mathcal{C}_1)$

$\mathcal{Z}_G(\mathcal{C})$ is G-graded with $\big(\mathcal{Z}_G(\mathcal{C})\big)_g = \left\{(X,\sigma) \in \mathcal{Z}_G(\mathcal{C}) \,\middle|\, X \in \mathcal{C}_g\right\}$

Theorem (Gelaki-Naidu-Nikshych, 2009)

$\mathcal{Z}_G(\mathcal{C})$ has a G-action and a G-braiding

The graded center

$\mathcal{C}$ monoidal category, $\mathcal{D}$ monoidal subcategory of $\mathcal{C}$

The **center of $\mathcal{C}$ relative to $\mathcal{D}$** is the monoidal category $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

- objects of $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

$$X \in \mathcal{C} \quad \text{with} \quad \sigma = \left\{\sigma_Y \colon\ X \otimes Y \xrightarrow{\cong} Y \otimes X \right\}_{Y \in \mathcal{D}}$$

- $(X,\sigma) \otimes (X',\sigma') = \big(X \otimes X', (\sigma \otimes \mathrm{id}_{X'})(\mathrm{id}_X \otimes \sigma')\big)$

$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ spherical fusion G-graded category

The **G-center of $\mathcal{C}$** is $\mathcal{Z}_G(\mathcal{C}) = \mathcal{Z}(\mathcal{C},\mathcal{C}_1)$

$\mathcal{Z}_G(\mathcal{C})$ is G-graded with $\big(\mathcal{Z}_G(\mathcal{C})\big)_g = \left\{(X,\sigma) \in \mathcal{Z}_G(\mathcal{C}) \,\middle|\, X \in \mathcal{C}_g\right\}$

Theorem (Gelaki-Naidu-Nikshych, 2009)

$\mathcal{Z}_G(\mathcal{C})$ has a G-action and a G-braiding

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

Steps of the proof of $\mathrm{HTV}_{\mathcal{C}} \simeq \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

1. Extend $\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ to graph HQFTs
   - provides basis of $\mathrm{TV}_{\mathcal{C}}(S^1 \times S^1, f_\alpha)$
2. $\mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \cong \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}(\Sigma, f)$ for G-surfaces (Σ, f)
   - via a description of $\mathcal{Z}_G(\mathcal{C})$ by *graded Hopf monad*
3. $\mathrm{HTV}_{\mathcal{C}}(M, h) = \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}(M, h)$ for closed G-manifolds (M, h)
   - via surgical TQFT techniques

Steps of the proof of $\mathrm{HTV}_{\mathcal{C}} \simeq \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

1. Extend $\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ to graph HQFTs
   - ▹ provides basis of $\mathrm{TV}_{\mathcal{C}}(S^1 \times S^1, f_\alpha)$
2. $\mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \cong \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}(\Sigma, f)$ for G-surfaces (Σ, f)
   - ▹ via a description of $\mathcal{Z}_G(\mathcal{C})$ by *graded Hopf monad*
3. $\mathrm{HTV}_{\mathcal{C}}(M, h) = \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}(M, h)$ for closed G-manifolds (M, h)
   - ▹ via surgical TQFT techniques

Steps of the proof of $\mathrm{HTV}_{\mathcal{C}} \simeq \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

1. Extend $\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ to graph HQFTs
   - ▹ provides basis of $\mathrm{TV}_{\mathcal{C}}(S^1 \times S^1, f_\alpha)$
2. $\mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \cong \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}(\Sigma, f)$ for G-surfaces (Σ, f)
   - ▹ via a description of $\mathcal{Z}_G(\mathcal{C})$ by *graded Hopf monad*
3. $\mathrm{HTV}_{\mathcal{C}}(M, h) = \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}(M, h)$ for closed G-manifolds (M, h)
   - ▹ via surgical TQFT techniques

3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs