Workshop on Black Holes *September 2020*

Three Avatars of Mock Modularity

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Mock Modularity

Holography **Topology** **Duality**

- *Dabholkar, Murthy, Zagier* ***(arXiv:1208.4074)***
- *Dabholkar, Jain, Rudra* ***(arXiv:1905.05207)***
- *Dabholkar, Putrov, Witten* ***(arXiv:2004.14387)***






Mock Theta Functions

- In Ramanujan's famous last letter to Hardy in 1920, he gave 17 examples (without any definition) of mock theta functions that he found very interesting with hints of modularity. For example,

$$f(\tau) = -q^{-25/168}\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q^n)\ldots(1-q^{2n-1})}$$

- Despite much work by many eminent mathematicians, this fascinating `mock' or `hidden' modular symmetry remained mysterious for a century until the thesis of Zwegers in 2002.

A mock theta function is `*almost modular*'

As difficult to characterize as a figure that is `*almost circular*'

Modular Forms

A modular form $f(\tau)$ of weight k on $SL(2,\mathbb{Z})$ is a holomorphic function on the upper half plane $\mathbb{H}$ that transforms as

$$(c\tau + d)^{-k} f(\frac{a\tau + b}{c\tau + d}) = f(\tau) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

(a, b, c, d, k integers and $ad\text{-}bc = 1$)

In particular, $f(\tau + 1) = f(\tau)$. Hence,

$$f(\tau) = \sum_n a_n q^n \qquad \big(q := \exp(2\pi i\tau)\big)$$

Modularity in Physics

- **Holography**: In AdS_3 quantum gravity the modular group $SL(2,\mathbb{Z})$ is the group of ***Boundary Global Diffeomorphisms***
- **Duality:** For a four-dimensional quantum gauge theory, a congruence subgroup of $SL(2,\mathbb{Z})$ is the ***S-duality Group***
- **Topology:** For a two-dimensional conformal field theory $SL(2,\mathbb{Z})$ is the ***Mapping Class Group***
- Physical observables in all these three physical contexts are expected to exhibit modular behavior.

Power of Modularity

Modular forms are highly symmetric functions.

- Ring of modular forms generated by the two Eisenstein Series $E_4(\tau)$ and $E_6(\tau)$. For example, for a weight 12 form we have

$$f_{12}(\tau) = aE_4^3(\tau) + bE_6^2(\tau) = \sum_n a_n q^n$$

- It's enough to know the first few Fourier coefficients to determine a modular form completely.
- Modular symmetry relates $f(\tau)$ to $f(-1/\tau)$. High temperature to low temperature or Strong coupling to Weak coupling. Cardy formula or Hardy-Ramanujan-Rademacher expansion.

Mock Modular Forms

The definition of a mock modular form involves *a pair* (f, g) where $f(\tau)$ is a holomorphic function on $\mathbb{H}$ with at most exponential growth at all cusps and $g(\tau)$ is a holomorphic modular form of weight $2-k$ such that the sum

$$\hat{f}(\tau,\bar{\tau}) := f(\tau) + g^*(\tau,\bar{\tau})$$

transforms like a holomorphic modular form of half-integer weight k of some congruence subgroup of $SL(2,\mathbb{Z})$ *(Zwegers, Zagier)*

- f is a *mock modular form* holomorphic ✓ modular ×
- g is its *shadow* holomorphic ✓ modular ✓
- $\hat{f}$ is its *modular completion* holomorphic × modular ✓

Holomorphic Xor Modular

- The function f is holomorphic but not modular.
- The function $\hat{f}$ is modular but not holomorphic

This incompatibility between holomorphy and modularity is the essence of mock modularity.

Holomorphic Anomaly

The ***modular completion*** of a ***mock modular form*** is related to the ***shadow*** by a holomorphic anomaly equation:

$$(4\pi\tau_2)^k \frac{\partial \hat{f}(\tau,\bar{\tau})}{\partial \bar{\tau}} = -\,2\pi\overline{g(\tau)}$$

- The shadow is both holomorphic and modular but by itself does not determine the modular completion.
- The non-holomorphic correction $g^*(\tau,\bar{\tau})$ by itself is not modular.
- The holomorphic mock modular form by itself is not modular.
- The completion contains all the information but is not holomorphic.

Incomplete Gamma Function

Given the q-expansion $g(\tau) = \sum_{n\geq 0} b_n q^n$ one can obtain

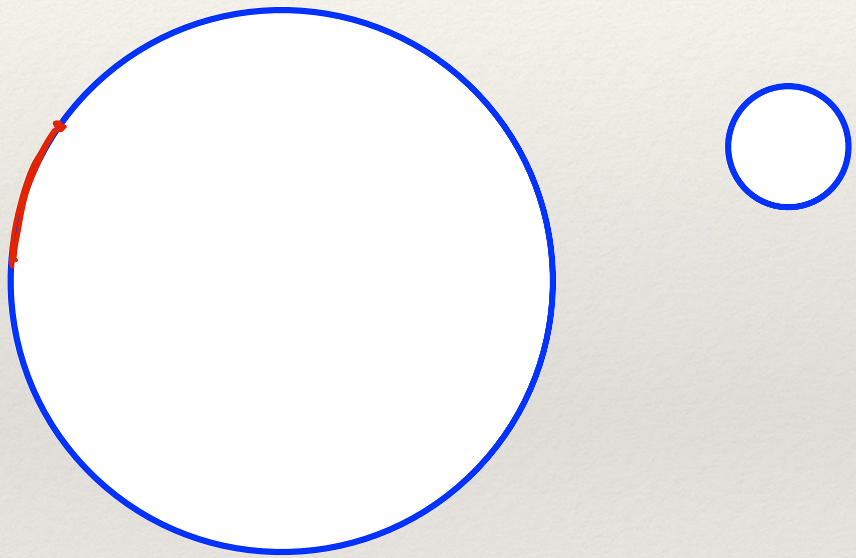
$$g^*(\tau, \bar{\tau}) = \bar{b}_0 \frac{(4\pi\tau_2)^{-k+1}}{k-1} + \sum_{n>0} n^{k-1}\, \bar{b}_n\, \Gamma(1-k, 4\pi n \tau_2)\, q^{-n}$$

where the incomplete Gamma function is defined by

$$\Gamma(1-k, x) = \int_x^\infty t^{-k}\, e^{-t}\, dt$$

The incomplete gamma function will make its appearance again

Holomorphic Modular ~Blue Circular



Jacobi Forms

A holomorphic function $\varphi(\tau, z)$ of two variables on $\mathbb{H} \times \mathbb{C}$ is a Jacobi form of weight k and index m if it is ***modular*** in τ and ***elliptic*** (doubly periodic up to phase) in z :

- $\varphi\Big(\dfrac{a\tau+b}{c\tau+d}, \dfrac{z}{c\tau+d}\Big) = (c\tau+d)^k \, e^{\frac{2\pi i m c z^2}{c\tau+d}} \, \varphi(\tau, z)$
- $\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\varphi(\tau, z) \qquad \forall \quad \lambda, \mu \in \mathbb{Z}$

with Fourier expansion

$$\varphi(\tau, z) = \sum_{n,r} c(n,r) q^n y^r \qquad \big(y := \exp(2\pi i z)\big)$$

Elliptic Genera and Jacobi Forms

Consider a conformally invariant nonlinear sigma model with target space X with $(2,2)$ superconformal symmetry with left and right super Virasoro algebra. One can define

$$\chi(\tau, z) = \mathrm{Tr}\,(-1)^{F_R+F_L}\, e^{2\pi i\tau H_L}\, e^{-2\pi i\bar{\tau} H_R}\, e^{2\pi i z F_L}\,.$$

Elliptic genus thus defined is a Jacobi form.

- By ***conformal symmetry***, the path integral is ***modular***.
- By ***spectral flow symmetry***, the path integral is ***elliptic.***

Right-moving Witten index that counts right-moving ground states.

Examples of Jacobi forms

- Modified Elliptic genus of T^4 weight -2 and index 1

$$A(\tau, z) = \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)} \qquad (k = -2\,, m = 1)$$

Since $\vartheta_1(\tau, z)$ has a simple zero, $A(\tau, z)$ has a **double zero at $z = 0$.**

- Elliptic genus of K_3 weight 0 and index 1

$$B(\tau, z) = 8\left(\frac{\vartheta_2^2(\tau, z)}{\vartheta_2^2(\tau)} + \frac{\vartheta_3^2(\tau, z)}{\vartheta_3^2(\tau)} + \frac{\vartheta_4^2(\tau, z)}{\vartheta_4^2(\tau)}\right) \qquad (k = 0\,, m = 1)$$

where $\eta(\tau)$ is the Dedekind eta and $\vartheta_i(\tau, z)$ are the Jacobi theta function.

Ring of Jacobi forms

- Ring of Jacobi forms is generated by $A(\tau, z)$ and $B(\tau, z)$ with ordinary modular forms as coefficients. Index and weight both add when you multiply Jacobi forms.
- For example, a $k = 4\,, m = 2$ Jacobi form is given by

$$a\, E_4^2(\tau)\, A^2(\tau, z) + b\, E_6(\tau)\, AB(\tau, z) + c\, E_4(\tau)\, B^2(\tau, z)$$

- Jacobi forms in are equivalent to vector valued modular forms in a single variable τ

Theta Decomposition

Using elliptic property, the z dependence goes into theta functions

$$\varphi(\tau, z) = \sum_{\ell=0}^{2m-1} \vartheta_{m,\ell}\, h_\ell(\tau)$$

where $\vartheta_{m,\ell}$ are the standard level m theta functions.

$$\vartheta_{m,\ell}(\tau, z) := \sum_{n\in\mathbb{Z}} q^{(\ell+2mn)^2/4m}\, y^{\ell+2mn}$$

A ***Jacobi form*** $\varphi(\tau, z)$ *of weight* k *and index* m *is equivalent to a* ***vector-valued modular form*** $\{h_\ell(\tau)\}$ *of weight* $(k - \frac{1}{2})$

“ My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but ***mock theta-functions*** *. . . But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal further.”*

Freeman Dyson

(1987 Ramanujan Centenary Conference)

I. A Puzzle With Black Holes

The counting function for the degeneracies of quarter-BPS black holes is given in Type-II string compactified on in $K_3 \times T^2$ is a ***meromorphic*** Jacobi form (weight -10 , index m)

$$\psi_m(\tau, z) = \frac{B^{m+1}(\tau, z) + \ldots}{\eta^{24}(\tau)\, A(\tau, z)}$$

The quantum degeneracies are given by the Fourier coefficients of this counting function. However, given the ***double pole in*** z, the Fourier coefficients depend on the choice of the contour and are not uniquely defined. ***What's wrong with the counting?***

II. A Puzzle With Instantons

- Consider the Vafa-Witten partition function $Z(\tau)$ of $\mathcal{N} = 4$ four-dimensional supersymmetric gauge theory on $\mathbb{CP}^2$ for gauge group $SO(3)$ with coupling constant $\tau = \theta + i\dfrac{4\pi}{g^2}$.
- The twisted supercharge Q squares to zero $Q^2 = 0$. By a general argument, the partition function is holomorphic because

$$\frac{d}{d\bar{\tau}} Z(\tau) = \langle \{Q, \Lambda\} \rangle = 0$$

- The holomorphic partition function receives contributions only from instantons but does not have any modular properties as expected from S-duality. ***What's wrong with duality?***

III. A Puzzle With Elliptic Genus

- The elliptic genus of a SCFT like a sigma model is again expected to be holomorphic by a general argument. But for a noncompact SCFT (for example, a sigma model with infinite cigar as the target space) the holomorphic function is not modular as expected by conformal invariance.
- *What's wrong with conformal invariance?*

In all three cases the answer involves ***mock modularity*** *from a mathematical point of view and a* ***noncompact path integral*** *from a physical point of view.*

Poles and Walls

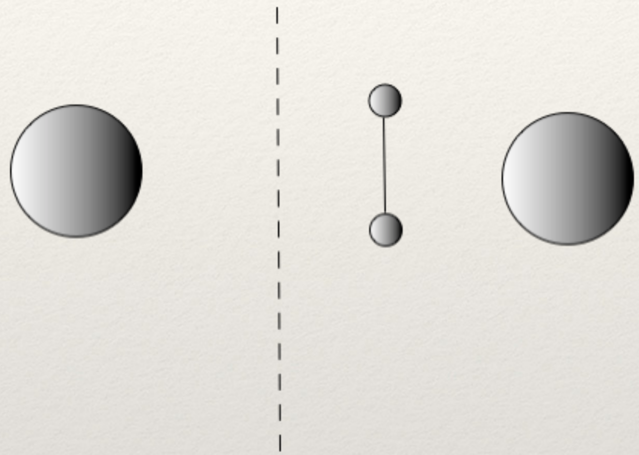
- The poles in the counting function is a reflection of the ***wall-crossing phenomenon***. The moduli space of the theory is divided into chambers separated by walls. The quantum degeneracies jump upon crossing walls.

pole $\longleftrightarrow$ wall

residue at the pole $\longleftrightarrow$ jump in the degeneracy

How come? *The quantum degeneracy of a black hole is a property of its horizon and not of asymptotic moduli.*

Multi-centered Black Holes



What counts the horizon degrees of freedom relevant for quantum gravity?

Decomposition Theorem

- There is a unique decomposition *(DMZ 2012)*

$$\psi_m(\tau, z) = \psi_m^P(\tau, z) + \psi_m^F(\tau, z)$$

- The polar part $\psi_m^P(\tau, z)$ called the Appel-Lerch sum has exactly the same poles in z as the poles of $\psi_m(\tau, z)$.
- The finite part $\psi_m^F(\tau, z)$ is free of poles and it's Fourier coefficients are uniquely defined without ambiguity.

Black Hole Degeneracies

The decomposition has a nice physical interpretation:

- The polar part $\psi_m^P(\tau, z)$ counts the microstates of multi-centered black holes which are moduli-dependent.
- The finite part $\psi_m^F(\tau, z)$ counts the microstates of single-centered black hole which is a property of the horizon.

The nontrivial part of the theorem is that both these terms admit a ***modular completion****.*

Appel-Lerch Sum

- The polar term is proportional to the Appel-Lerch sum which is an `***elliptic average***' of the double pole at the origin (obtained by summing over its images under translations with phases)

$$\psi^{\mathrm{P}}_m(\tau, z) \ := \ \frac{p_{24}(m+1)}{\eta^{24}(\tau)} \mathscr{A}_{2,m}(\tau, z)$$

$$\mathscr{A}_{2,m}(\tau, z) \ := \ \sum_{s \in \mathbb{Z}} \frac{q^{ms^2+s} y^{2ms+1}}{(1 - q^s y)^2}$$

- The Appel-Lerch sum has `elliptic symmetry' like a Jacobi form of weight $2$ index $m$ but no `modular symmetry'

Modular Completions

- Similarly it's convenient to define

$$\psi^F_m(\tau, z) := \frac{p_{24}(m+1)}{\eta^{24}(\tau)}\varphi^F_{2,m}(\tau, z)$$

$\varphi^F_{2,m}(\tau, z)$ is a *mock Jacobi form* of weight 2 index m and admits a *modular completion* $\widehat{\varphi}^F_{2,m}(\tau, z)$.

- One can similarly define the *modular completion* $\widehat{\mathscr{A}}_{2,m}(\tau, z)$ for the Appel-Lerch sum which has good modular properties.
- Both completions are nonholomorphic in τ but transform like a proper Jacobi form of weight 2 index m.

Mock Jacobi Forms

- Using the elliptic property one can still obtain a theta decomposition, but the theta coefficients are (vector-valued) mock modular forms. In our case

$$\varphi_{2,m}(\tau, z) = \sum_{\ell=1}^{2m} \vartheta_{m,\ell}\, h_\ell(\tau)$$

- The modular completions $\{\widehat{h}_\ell(\tau) := h_\ell(\tau) + g_\ell^*(\tau)\}$ transform as weight $3/2$ modular forms but have a holomorphic anomaly.

Holomorphy Xor Modularity

	Holmorph ic	Modular	Pole-free	Simple
Meromorphic Jacobi ψ	✓	✓	✗	✗
Mock Jacobi φ	✓	✗	✓	✗
Appel Lerch A	✓	✗	✗	✓
Completion $\widehat{\varphi}$	✗	✓	✓	✗

Holomorphic Anomaly

The shadows in the black hole problem (and in all of Ramanujan's examples) are particularly simple.

- For vector-valued modular forms:

$$\tau_2^{3/2}\,\frac{\partial}{\partial\bar{\tau}}\,\widehat{h_\ell}(\tau)\;\doteq\;\sqrt{m}\;\overline{\vartheta_{m,\ell}(\tau)}$$

- For the Jacobi form:

$$\tau_2^{3/2}\,\frac{\partial}{\partial\bar{\tau}}\,\widehat{\phi_m^F}(\tau,z)\;\doteq\;\sqrt{m}\;\sum_{\ell=0}^{2m-1}\overline{\vartheta_{m,\ell}(\tau)}\;\vartheta_{m,\ell}(\tau,z)$$

Given the shadow what is the `optimal' mock modular form?

Zagier Mock Modular Forms

Consider the special case of $m = 1$. Then the following objects define mock modular forms of weight $3/2$ with the desired shadow expected from the holomorphic anomaly.

$$\widehat{h}_0(\tau) = \sum_{n\geq 0} H(4n)q^n + \frac{32\pi}{\sqrt{\tau_2}} \sum_{n\in Z} \Gamma\left[-\frac{1}{2}, 4\pi n^2\tau_2\right] q^{-n^2}$$

$$\widehat{h}_1(\tau) = \sum_{n>0} H(4n-1)q^{n-\frac{1}{4}} + \frac{32\pi}{\sqrt{\tau_2}} \sum_{n\in Z} \Gamma\left[-\frac{1}{2}, 4\pi (n+\frac{1}{2})^2\tau_2\right] q^{-(n+\frac{1}{2})^2}$$

Hurwitz-Kronecker Class Numbers

- These class numbers $H(N)$ are defined for $N > 0$ as the number of $PSL(2,\mathbb{Z})$ equivalence classes of integral binary quadratic forms of discriminant $-N$, weighted by the reciprocal of the number of their automorphisms, and for other values of N by $H(0) = -1/12$ and $H(N) = 0$ for $N < 0$.
- $H(N) = 0$ unless N equals 0 or -1 modulo 4.

Holography and Mock Modularity

The Fourier coefficients of the mock Jacobi form are unambiguously defined and count the horizon degrees of freedom of a single-centered black hole.

- The black hole can be thought of as a left-moving excitation of an effective black string (MSW string)
- From the near horizon AdS_3/CFT_2 holography one expects $SL(2,\mathbb{Z})$ modular symmetry for the boundary of solid torus.
- It is natural to identify $\widehat{\psi}^F_m(\tau, z)$ with the elliptic genus of the boundary that exhibits this symmetry. ***Why nonholomorphic?***

Holomorphy of Elliptic Genus

- For a compact SCFT, the elliptic genus $\chi(\tau, \bar{\tau})$ is independent of $\bar{\tau}$ by the standard argument that right-moving Bose and Fermi states with *nonzero* right-moving energy are paired by supersymmetry.
- The nonholomorphy is consequence of the failure of this naïve argument. In a noncompact theory one can have a continuum of states.

The density of states of bosons and fermions need not be exactly equal.

Witten Index

- Consider a simpler problem of supersymmetric quantum mechanics of a particle on a ($2n$-dimensional) Riemannian manifold X with Hamiltonian $H = \{Q, Q\}$ and supercharge $Q = \psi^i p_i \sim \gamma^i D_i = D$ = a Dirac operator.
- The Witten index is defined by

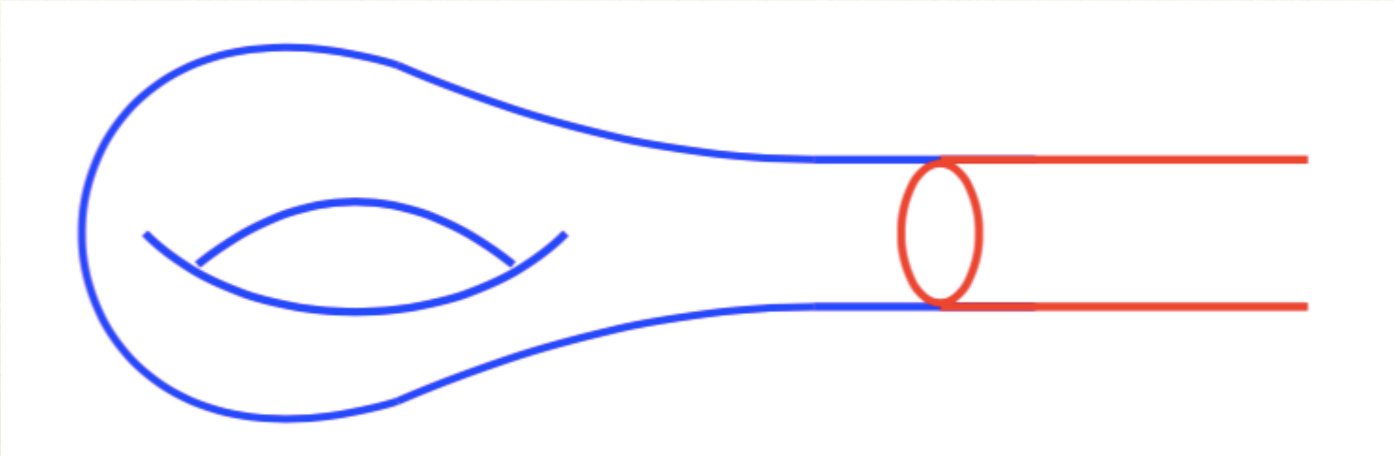
$$W(\beta) := \mathrm{Tr}\left[(-1)^F e^{-\beta H}\right]$$

- If X is compact, then the spectrum H is discrete. It's nonzero eigenvalues are paired, $Q|B\rangle = |F\rangle$. Only ground states contribute, and the Witten index is independent of β :

$$W(\beta) = n_+ - n_-$$

Non-compact Witten Index

- Consider now a noncompact manifold X with a cylindrical product form at infinity $X \sim \mathbb{R}^+ \times Y$.
- The spectrum of the Hamiltonian is no longer discrete. The Witten index needs to be more carefully defined to include the continuum of scattering states.
- The Bose-Fermi cancellation from the continuum may not be exact. This explains why noncompact Witten index is in general β-dependent.



The Dirac operator near the boundary takes the form $D = \gamma^u(\partial_u + \bar{\gamma}\mathscr{B})$ and the eigenvalue equation is

$$\begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix} \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix} = \sqrt{E} \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}$$

$$\Psi_\pm(u, y) = \sum_\lambda \Psi^\lambda_\pm(u)\, e_\lambda(y)$$

where $\{e_\lambda(y)\}$ is the complete set of eigenfunctions of $\mathscr{B}$ with eigenvalues $\{\lambda\}$

Phase Shifts and Density of States

Scattering states with energy $E(k) = k^2$ labeled by momentum k

$$\psi_{\pm}^{\lambda k}(u) \sim c_{\pm}^{\lambda}\left[e^{iku} + e^{i\delta_{\pm}^{\lambda}(k) - iku}\right]$$

Substituting in eigenvalue equation, we obtain

$$\delta_{+}^{\lambda}(k) - \delta_{-}^{\lambda}(k) = -i\,\ln\left(\frac{ik+\lambda}{ik-\lambda}\right) + \pi$$

This gives the *difference* in the density of states using

$$\rho_{+}^{\lambda}(k) - \rho_{-}^{\lambda}(k) = \frac{1}{\pi}\frac{d}{dk}\left[\delta_{+}^{\lambda}(k) - \delta_{-}^{\lambda}(k)\right]$$

β-dependence from the Continuum

$$\rho_+^\lambda(k) - \rho_-^\lambda(k) = -\frac{2\lambda}{\pi(k^2+\lambda^2)}$$

- The temperature-dependent contribution to the Witten index coming from the continuum states is

$$W(\beta) - W(\infty) = \sum_\lambda \int dk \left[\rho_+^\lambda(k) - \rho_-^\lambda(k)\right] e^{-\beta(k^2+\lambda^2)}$$

$$= \frac{1}{2}\sum_\lambda \mathrm{sgn}(\lambda)\,\mathrm{erfc}\left(|\lambda|\sqrt{\beta/2}\right)$$

- Semi-topological: depends on only the asymptotic data.

Localization and Path integral

Consider a world-point *`path integral'* for Witten index

$$W(\beta) = -i\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dF \int d\psi_-\, d\psi_+ \, \exp\left[-\beta S(U)\right]$$

$$S(u, F, \psi_-, \psi_+) = \frac{1}{2}F^2 + iF\,h'(u) + ih''(u)\,\psi_-\psi_+$$

Choose $h'(u) = -\lambda\tanh(u)$ to again find $W(\beta) - W(\infty)$ equal to

$$1 - \frac{1}{\sqrt{\pi}}\int_{-\sqrt{\frac{\beta}{2}}\lambda}^{\sqrt{\frac{\beta}{2}}\lambda} dy\, e^{-y^2} = \frac{1}{2}\mathrm{sgn}(\lambda)\,\mathrm{erfc}\left(\sqrt{\frac{\beta}{2}}\,|\,\lambda\,|\right)$$

Atiyah-Patodi-Singer η-invariant

The contribution from the continuum states equals half the (regularized) eta-invariant of the boundary manifold

$$\hat{\eta}(\beta) := \sum_{\lambda} \operatorname{sgn}(\lambda)\, \operatorname{erfc}\left(|\lambda| \sqrt{\beta} \right) \;\to\; \sum_{\lambda} \operatorname{sgn}(\lambda)$$

This can be written in terms of an incomplete gamma Function using

$$\Gamma(\frac{1}{2}, x) \;=\; \sqrt{\pi}\, \operatorname{erfc}(\sqrt{x})$$

Appearance of *incomplete gamma function* is related to the one in the modular completion. Consequence of the non-compact field space.

(APS theorem)

- For a target manifold $\hat{X}$ with a boundary Y, it is difficult to define a path integral to compute the index. One can map the problem to the problem on a cigar-like noncompact manifold X (introduced already in the original work of APS) that asymptotes to $\mathbb{R}^+ \times Y$.
- The above reasoning yields a new proof of APS theorem using supersymmetric quantum mechanics. The contribution from the continuum gives rise to the eta-invariant that must be included in the index theorem.

A puzzle with S-duality

- The anti-holomorphic dependence of the effective action on the coupling parameter τ is Q-exact. Therefore,

$$\frac{\partial Z(\tau,\bar{\tau})}{\partial\bar{\tau}} = \langle\frac{\partial S_{4\mathrm{d}}}{\partial\bar{\tau}}\rangle = \langle\{Q,\Lambda\}\rangle = 0$$

For example for $X = K_3$ and $G = SU(2)$

$$Z(\tau) = \frac{1}{\eta(\tau)^{24}}$$

(Vafa-Witten 94)

A striking strong-coupling test of S-duality and a precursor of string-duality. For $X = \mathbb{CP}^2$ and $G = SO(3)$ naively it fails.

Vafa-Witten Partition Functions

- Path integral localizes to Yang-Mills instantons. The instanton number has a fractional part because of possible discrete magnetic flux or the second Stieffel-Whitney class.

$$\frac{h_0(\tau)}{\eta^3(\tau)} = \sum a_n q^n \qquad \frac{h_1(\tau)}{\eta^3(\tau)} = \sum b_n q^{(n+\frac{1}{2})}$$

- Sum over Euler characteristics of instanton moduli spaces. Holomorphic but not modular. Admit completions that satisfy an anomaly equation:

$$\frac{\partial Z_0}{\partial \bar{\tau}} \doteq \frac{3}{\tau^{3/2}\eta^3(\tau)} \sum \bar{q}^{n^2} \qquad \frac{\partial Z_1}{\partial \bar{\tau}} \doteq \frac{3}{\tau^{3/2}\eta^3(\tau)} \sum \bar{q}^{(n+\frac{1}{2})^2}$$

What is the physical origin of this mock modularity?

Topological Twist

- Rotation group $SU(2)_l \times SU(2)_r$ R-symmetry $Spin(6)$
- Bosons $F^+_{\mu\nu}$ (3,1,1) ϕ^I (1,1,6) Supercharge Q^a_α (2,1,4)
- The twisted rotation group $SU(2)'_l \times SU(2)'_r$ such that one of the supercharges is a scalar Q (1,1) with $Q^2 = 0$
- Three scalars transform as (3,1) and are sections of bundle of self-dual two forms $\Omega^{2+}(X)$

M-Theory Realization

- Consider M-theory on $X \times T^2 \times \mathbb{R}^5$ with two M5-branes wrapping $X \times T^2$ with $(0,2)$ theory on the world-volume where $SO(5)$ R-symmetry is geometric rotations of $\mathbb{R}^5$.

Gauge theory region when T^2 is small.

Sigma model region with X is small.

- For $X = K3$ one obtains the M5-brane/Heterotic string duality.
- Taking M-theory on $\Omega^{2+}X \times T^2 \times \mathbb{R}^2$ gives the twisted version of the Yang-Mills theory. For Kähler X, consider $KX \times T^2 \times \mathbb{R}^3$

Sigma Model Field Content

- Consider well-separated M5-branes. For the relative motion, we can obtain the sigma model fields by the usual KK reduction of the tensor multiplet.
- Bosonic fields: Self-dual 2-form b^+ and five scalars ϕ^I
- The bosonic fields are three bosons X^m corresponding to transverse motion in $\mathbb{R}^3$ with $m = 1,2,3$
- One right-moving compact X^4 boson from the zero mode of the B-field or radius $\sqrt{2}$ determined by A_1 root lattice.

Sigma Model Calculation

- Target space $\frac{\mathbb{R}^3 \times S^1_R}{\mathbb{Z}_2}$ with $(0,1)$ supersymmetry.
- Free action with a worldsheet B-field corresponding to a Wess-Zumino term with H-quantum -3 determined by first chern class of the canonical bundle of $\mathbb{CP}^2$

$$\frac{\partial Z(\tau, \bar{\tau})}{\partial \bar{\tau}} = \frac{1}{\sqrt{-8\tau_2}\eta(\tau)} \langle G_+ \rangle_Y$$

(Gaiotto, Johnson-Freyd 19)

- Naively this vanishes as in the gauge theory case.

Wess-Zumino Term

- There are four right-moving fermions and three in the boundary theory.
- The sigma model theory is not totally free but must have nontrivial Kalb-Ramond field corresponding to a Wess-Zumino term. As a result

$$\langle G_+ \rangle\rangle_Y = \langle \psi^i \partial X^i - i H_{ijk} \psi^i \psi^j \psi^k \rangle$$

The second term soaks up the zero modes. Originates in a 6d topological term *Ganor Motl (1998) Intrilligator (2000)*

Sigma Model Regime

- The factor of 3 comes from the quantum of H-flux.
- The factor of $\eta(\tau)^{-3}$ is the contribution of three left-moving bosonic oscillators.
- The anti-holomorphic theta-function is the contribution of right-moving momenta of a compact chiral boson.
- The factor of $\tau_2^{-3/2}$ comes from the integral over the three noncompact bosonic zero modes.

Gauge Theory Calculation

- On the Coulomb branch when the two M5-branes are far separated, there are four bosonic and four fermionic zero modes. In the low energy $U(1)$ theory, there must exist a Wess-Zumino term to match the 't Hooft anomaly of the UV $SU(2)$ theory corresponding to the $\mathrm{Tr}\,(F_R^3)$ term in the anomaly polynomial. Related by supersymmetry to the Dine-Seiberg term and ultimately to the 6d term.
- Once again supersymmetrization gives rise to terms with the right number of fermion zero modes to give a nonzero answer for the boundary contribution.

Gauge Theory Regime

- The factor of 3 comes from the the first Chern class of the canonical line bundle of $\mathbb{CP}^2$.
- The factor of $\eta(\tau)^{-3} = \eta(\tau)^{-\chi(\mathbb{CP}^2)}$ is the contribution of point-like instantons.
- The anti-holomorphic theta-function is the contribution of abelian anti-instantons.
- The factor of $\tau_2^{-3/2}$ comes from the integral over the constant mode of an auxiliary field in the off-shell multiplet.

Mock Modularity

Holography **Topology** **Duality**