The growth of the $\frac{1}{16}$-BPS index in 4d N=4 SYM

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Black Holes: BPS, BMS and Integrability
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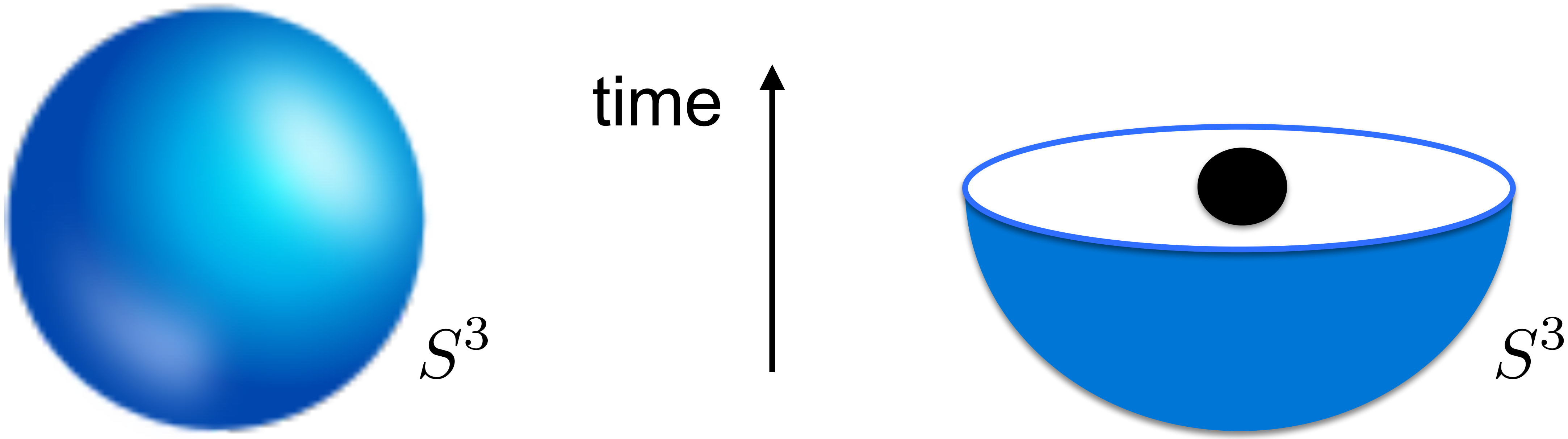
Does the most general index in $\mathcal{N}=4$ SYM capture the susy BH entropy in $\mathrm{AdS}_5\times\mathrm{S}^5$?



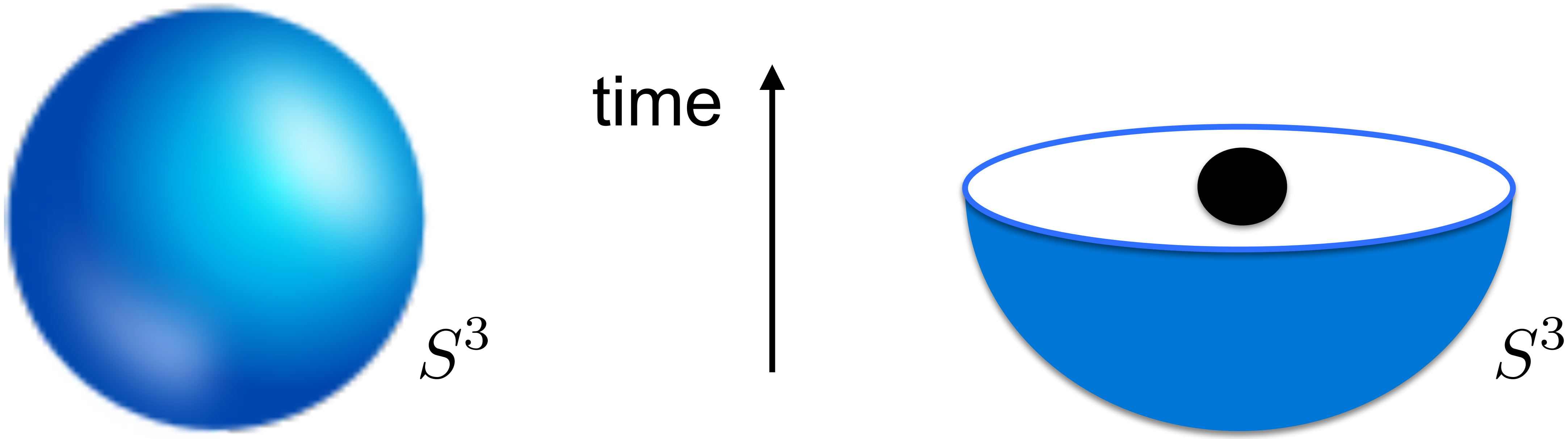
$\mathcal{N}=4\ \ U(N)$ SYM	$\mathrm{AdS}_5\times S^5$
$\frac{1}{16}$-BPS states	$\frac{1}{16}$-BPS BH

$\mathcal{I}_N(\tau) \ = \ \mathrm{Tr}(-1)^F e^{2\pi \mathrm{i}\tau_i q_i}$ $\log \mathcal{I}_N \ \stackrel{?}{=} \ O(N^2)$	$S_{\mathrm{BH}} \ = \ \dfrac{A_H}{4G_N} \ = \ O(N^2)$

[Sundborg '99; Aharony, Marsano, Minwalla, Papadodimas, van Raamsdonk '03; Kinney, Maldacena, Minwalla, Raju '05]

[Gutowski, Reall '04; Chong, Cvetic, Lu, Pope '05; Kunduri, Lucietti, Reall '06]

Does the most general index in $\mathcal{N}=1$ SCFT4 capture the susy BH entropy in dual AdS5 ?



$\mathcal{N}=1$ SCFT

$\frac{1}{4}$-BPS states

$\mathrm{AdS}_5 \times M_5$

$\frac{1}{4}$-BPS BH

$\mathcal{I}_N(\tau) = \mathrm{Tr}(-1)^F e^{2\pi \mathrm{i} \tau q}$ $\log \mathcal{I}_N \stackrel{?}{=} O(N^2)$	$S_{\mathrm{BH}} = \frac{A_H}{4G_N} = O(N^2)$

Index counts BPS states labelled by (R-charge + spin)

$\mathcal{N}=4\,\mathrm{SYM}$ → $\mathcal{N}=1\,\mathrm{SCFT}$ on S^3

$SU(4)_R$ (Q_1, Q_2, Q_3)

$U(1)_R\ Q$ + Flavors F_a

$SO(4)$ (J_1, J_2)

Susy: $\{\mathcal{Q}, \overline{\mathcal{Q}}\} = E - J_1 - J_2 - \frac{3}{2}Q$

Commutant: $\left[J_i + \frac{1}{2}Q\,,\, \mathcal{Q}\right] = 0$, $\left[F_a\,,\, \mathcal{Q}\right] = 0$

Most general $\frac{1}{4}$- BPS index:

$$\mathrm{Tr}\,(-1)^F\, e^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}}\, e^{2\pi i\sigma(J_1+\frac{1}{2}Q)+2\pi i\tau(J_2+\frac{1}{2}Q)}\, e^{2\pi i\mu_a F_a}$$

Index counts BPS states labelled by (R-charge + spin)

$\mathcal{N}=4$ SYM → $\mathcal{N}=1$ SCFT on S^3

$SU(4)_R$ (Q_1, Q_2, Q_3)

$U(1)_R$ Q + Flavors F_a

$SO(4)$ (J_1, J_2)

Susy: $\{\mathcal{Q}, \overline{\mathcal{Q}}\} = E - J_1 - J_2 - \frac{3}{2}Q$

Commutant: $[J_i + \frac{1}{2}Q \,,\, \mathcal{Q}] = 0$, $[F_a \,,\, \mathcal{Q}] = 0$

Simplest $\frac{1}{4}$-BPS index $(2J = J_1 + J_2)$

$$\mathcal{I}_N(\tau) = \mathrm{Tr}\,(-1)^F\, \mathrm{e}^{2\pi \mathrm{i} \tau (2J+Q)} = \sum_{n \geq 0} d_N(n)\, \mathrm{e}^{2\pi \mathrm{i}\, \tau\, n/3}$$

Any statement about d_n can be translated to I_N(t) and vice versa.

Susy BH has $O(N^2)$ semiclassical entropy

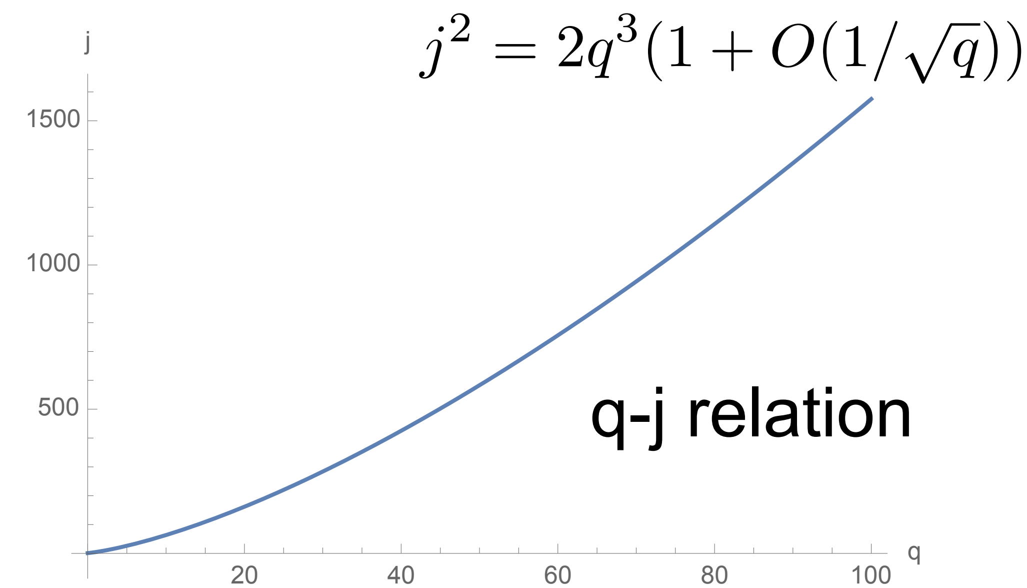
IIB on $\mathrm{AdS}_5 \times S^5$
(IIB sugra in 10d)

→

IIB on $\mathrm{AdS}_5 \times M^5$
Minimal AdS_5 sugra
$g_{\mu\nu},\, A_\mu \qquad G_N = 1/N^2$

$\frac{1}{4}$-BPS BH charges

$(J_1, J_2, Q) = N^2 (j_1, j_2, q)$ obeying



Simplest: $j_1 = j_2 = j \neq 0$

$$S_{\mathrm{BH}}(N, j) \;=\; N^2\, s(j)$$

cf $\quad d_N(n)$

What is the meaning of BH/exponential growth of states in the index?

$$\log d_N(n) \xrightarrow{?} N^2\, s(n/N^2)$$

Large-N limit

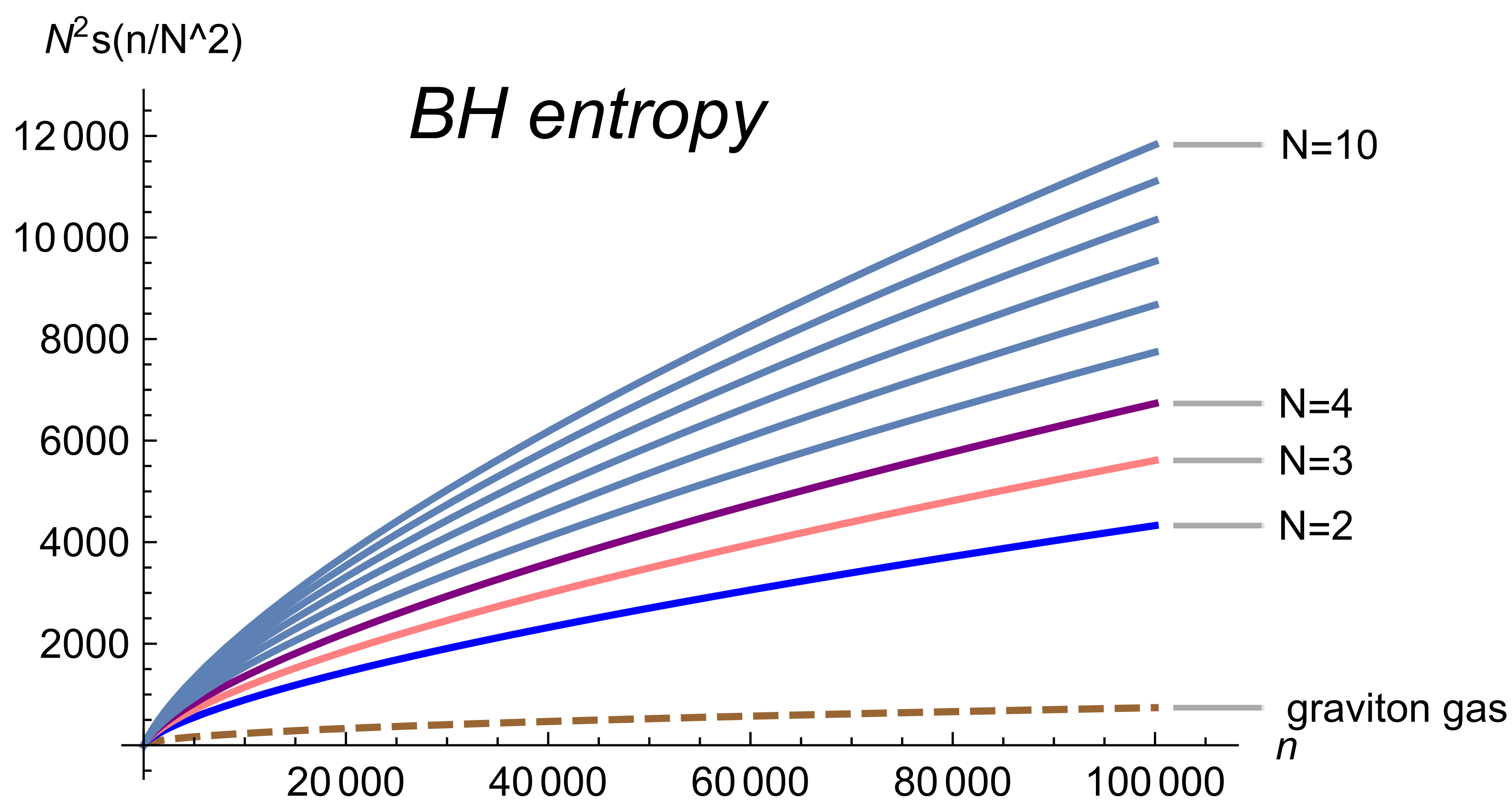
$n, N \to \infty$

fixed $j = n/N^2$

Cardy-like limit

$n/N^2 \to \infty$

fixed N



$$s(j) \;=\; \pi\sqrt{3q^2 - 4j}$$

$$\xrightarrow[j\to\infty]{} \sqrt{3}\,\pi\left(\frac{j^2}{2}\right)^{1/3}$$

The N=4 superconformal index is an integral over unitary matrices

[Romelsberger '05; Kinney, Maldacena, Minwalla, Raju '05]

$$\mathcal{I}_N(\tau) = \int [DU] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\, i_{\rm s}(e^{2\pi i n\tau})\, \mathrm{tr}\, U^n\, \mathrm{tr}\,(U^\dagger)^n \right)$$

$U(N)$ matrix

"Single-letter index"

$$i_{\rm s}(\mathrm{x}) = \frac{3\,\mathrm{x}^2 - 3\,\mathrm{x}^4 - 2\,\mathrm{x}^3 + 2\,\mathrm{x}^6}{(1-\mathrm{x}^3)^2}$$

- Purely double trace, i.e. purely two-particle interaction among eigenvalues.

Recent progress has established BH growth of states in SYM index

- Gravitational thermodynamic analysis
  [Cabo-Bizet, Cassani, Martelli, S.M. 1810]

- Cardy-like limit = infinite charge limit of matrix model (BHs much larger than AdS scale) [Choi, Kim, Kim 1810;
  Kim, Kim, Song 1904;
  Cabo-Bizet, Cassani, Martelli, S.M. 1904]

- Bethe-ansatz-type approach at large N.
  (Different contour integral representation of $\mathcal{I}$)
  [Benini, Milan 1811, 1812]

- Direct large-N analysis of matrix model. [Cabo-Bizet, S.M. 1909;
  Cabo-Bizet, Cassani, Martelli, S.M. 2005.]
  (= CCMM)

- Numerical results [S.M. 2005.]
  [Agarwal, Choi, Kim, Kim, Nahmgoong 2005.]

Related recent work on this subject

Cabo-Bizet, Cassani, Martelli, S.M. 1810

Choi, Kim, Kim, Nahmgoong 1810

Choi, Hwang, Kim, Nahmgoong 1811

Benini, Milan 1811, 1812

Suh 1812

Honda 1901

Ardehali 1902

Kim, Kim, Song 1902

Zaffaroni 1902

Cabo-Bizet, Cassani, Martelli, S.M. 1904

Amariti, Garozzo, Lo Monaco 1904

Cassani, Papini 1906

Larsen, Nian, Zeng 1907

Kantor, Papageorgakis, Richmond 1907

Nahmgoong 1907

Lezcano, Pando-Zayas 1907

Lanir, Nedelin, Sela 1908

Cabo-Bizet, S.M. 1909

Goldstein, Jejjala, Lei, van Leuven, Li 1910

Ardehali, Hong, Liu 1912

Gadde 2004

Nian, Pando-Zayas 2003

David, Nian, Pando-Zayas 2005

S.M. 2005

Agarwal, Choi, Kim, Kim, Nahmgoong 2005

Lee, Nahmgoong 2006

Lezcano, Hong, Liu, Pando-Zayas 2007

…

Questions addressed in today's talk

(How) do susy BHs contribute to (gravitational) path integral for index?

→ Regularized saddles, subtle BPS-limit

[CCMM. 1810.]

(How) can BHs be realized as large-N saddle-points of matrix integral? Other phases?

→ Help from number theory: Bloch-Wigner elliptic dilogarithm

[Cabo-Bizet, S.M. 1909.]

What about lore: "*gauge-invariant operators in N=4 SYM correspond to gravitons*"?

→ Representation theory of U(N), random partitions

[S.M. 2005.]

Is $S_{\rm BH} = \log d_N$ a consequence of AdS/CFT?

- BH has thermodynamic entropy.

  Boltzmann relation $\Longrightarrow$ large degeneracy of states.

  Should be reflected in dual SYM.

- Potential holes in argument:
  - Presupposes Hilbert space of quantum gravity.
  - Total number of states not preserved in moduli space.
  - Index $d_N(n)$ is only a lower bound on the total number of states with charge n .

Euclidean functional integral viewpoint

- Start with microscopic SYM index.
  $\mathcal{I}(\tau)$ = functional integral on $S^3 \times S^1$.
  [up to Casimir energy terms, cf CCMM 1810.]

- Index protected. [N*ot* dimension of $\frac{1}{16}$-BPS operators cf Beisert '03]
  $\Longrightarrow$ $\mathcal{I}(\tau)$ = functional integral in AdS gravity.

- Question: does susy BH contribute to the gravitational functional integral with susy boundary conditions?

  Subtle: needs regularization of infinite throat.

The entropy of these BHs can be recast as an extremization problem [Hosseini, Hristov, Zaffaroni 1705.]

$$S_{\rm BH}(J_1, J_2, Q) \; = \; -{\rm ext}_{\varphi,\omega_i}\big(I(\omega_1,\omega_2,\varphi) - \omega_1 J_1 - \omega_2 J_2 - \varphi\, Q\big)$$

with $I(\omega_1,\omega_2,\varphi) \; = \; \dfrac{4N^2}{27}\dfrac{\varphi^3}{\omega_1\omega_2}$

under the constraint $\omega_1 + \omega_2 - 2\varphi \; = \; -2\pi{\rm i}$

Note similarity to “Quantum statistical relation” [Gibbons, Hawking ’77]
I is like the on-shell action of the AdS5 BH (canonical ensemble).

Does extremization principle follow from semiclassical BH thermodynamics in AdS5?

BH thermodynamics

Temperature	=	surface gravity β
Chemical potential for angular momentum	=	angular velocity $\Omega_{1,2}$
Chemical potential for electric charge	=	gauge field value Φ

Susy BH values: $\beta \to \infty \ , \ \Omega^*_{1,2} = 1 \ , \ \Phi^* = \frac{3}{2}$

- Frozen: nothing left to vary!
- Wrong relation! $\Omega^*_1 + \Omega^*_2 - 2\Phi^* = -1$

Treat supersymmetric solutions as a limit of non-susy BHs

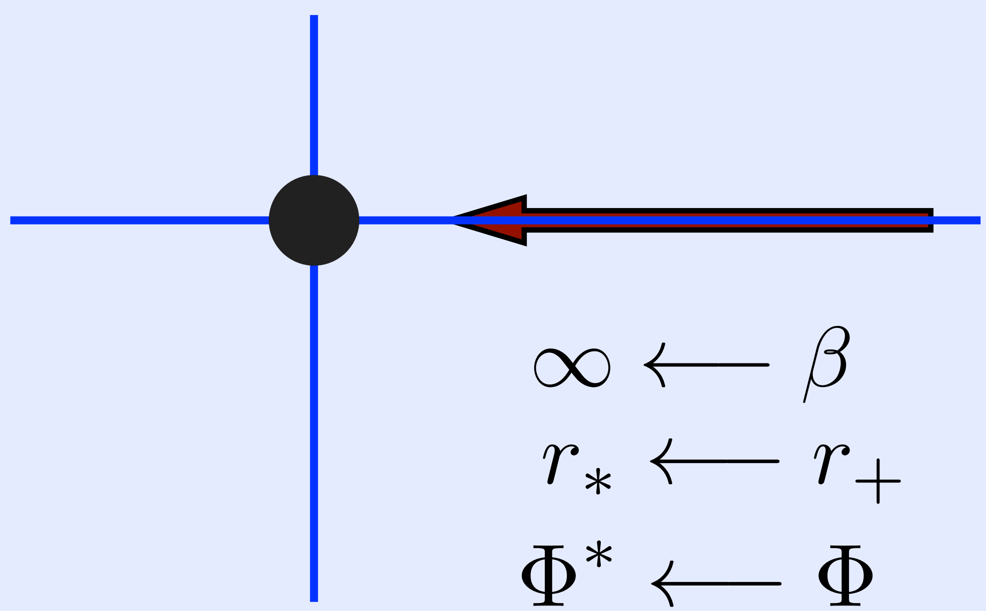
[CCMM 1810.]

- Minimal 5d sugra

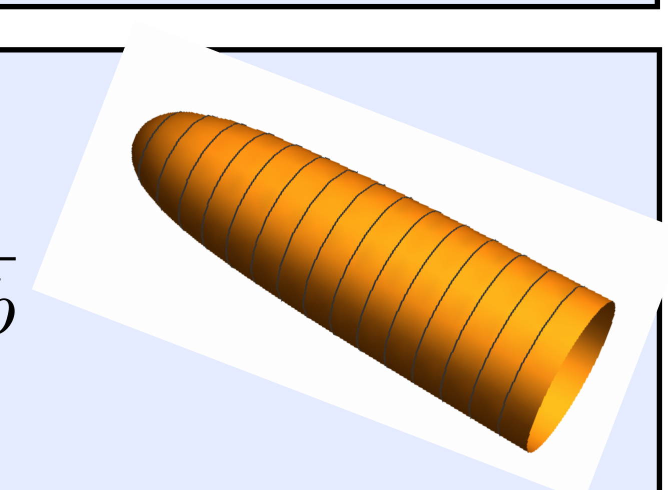
$$\mathcal{L} = (R + 12g^2) *1 - \tfrac{2}{3} F \wedge *F + \tfrac{8}{27} F \wedge F \wedge A$$

- Four-parameter family of BH solutions: (a, b, m, q)

[Gutowski, Reall, '04] [Chong, Cvetic, Lu, Pope, '05; Kunduri, Lucietti, Reall, '06]



Susy $\quad q = \dfrac{m}{1+a+b}$



$$\infty \longleftarrow \beta$$

$$r_* \longleftarrow r_+$$

$$\Phi^* \longleftarrow \Phi$$

$$\Omega^*_{1,2} \longleftarrow \Omega_{1,2}$$

Extremal

$$\omega_{1,2} := \beta(\Omega_{1,2} - \Omega^*_{1,2})$$

$$\varphi := \beta(\Phi - \Phi^*)$$

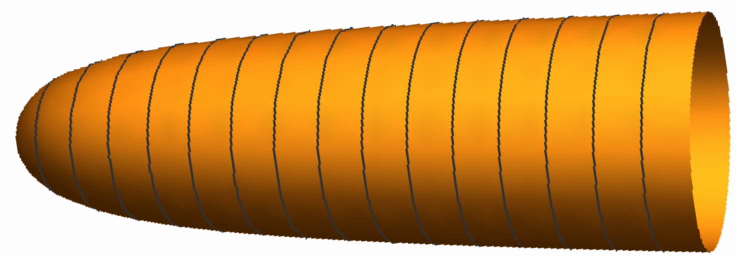
[Idea first presented in Silva '06. General formalism in CCMM 1810.]

Extremal non-susy analysis in Larsen, Nian, Zeng 1907., David Nian, Pando-Zayas 2005.]

Extremization principle follows from regularized Euclidean AdS5 path integral

[CCMM 1810.]

- Susy solutions



- On-shell action $$I = \frac{4N^2}{27}\frac{\varphi^3}{\omega_1\omega_2}$$

- Global regularity of Killing spinor $$\omega_1 + \omega_2 - 2\varphi = -2\pi\mathrm{i}$$

Wilson line R-symmetry backgnd

- Solve for φ $\omega_1 = \omega_2 = 2\pi\mathrm{i}\tau$ $\Longrightarrow$ $$I = \frac{\pi\mathrm{i}\,N^2}{27}\frac{(2\tau+1)^3}{\tau^2}$$

Identified with the parameter of the CFT background

AdS/CFT prediction for matrix model: large-N free energy = on-shell action of BH

$\mathcal{I}_N(\tau) \Longleftrightarrow d_N(n)$

Change of ensemble

$$\mathcal{I}_N(\tau) = \sum_{n\geq 0} d_N(n)\, \mathrm{e}^{2\pi \mathrm{i}\, \tau\, n/3}$$

$$\log d_N(n) \xrightarrow{?} N^2\, s(n/N^2)$$

$n \to \infty$, fixed N

$n, N \to \infty$, fixed $j = \frac{n}{N^2}$

$\Updownarrow$

$$\log \mathcal{I}_N(\tau) \xrightarrow{?} \frac{\pi \mathrm{i}\, N^2}{27} \frac{(2\tau+1)^3}{\tau^2}$$

$\tau \to 0$, fixed N

$N \to \infty$, fixed τ

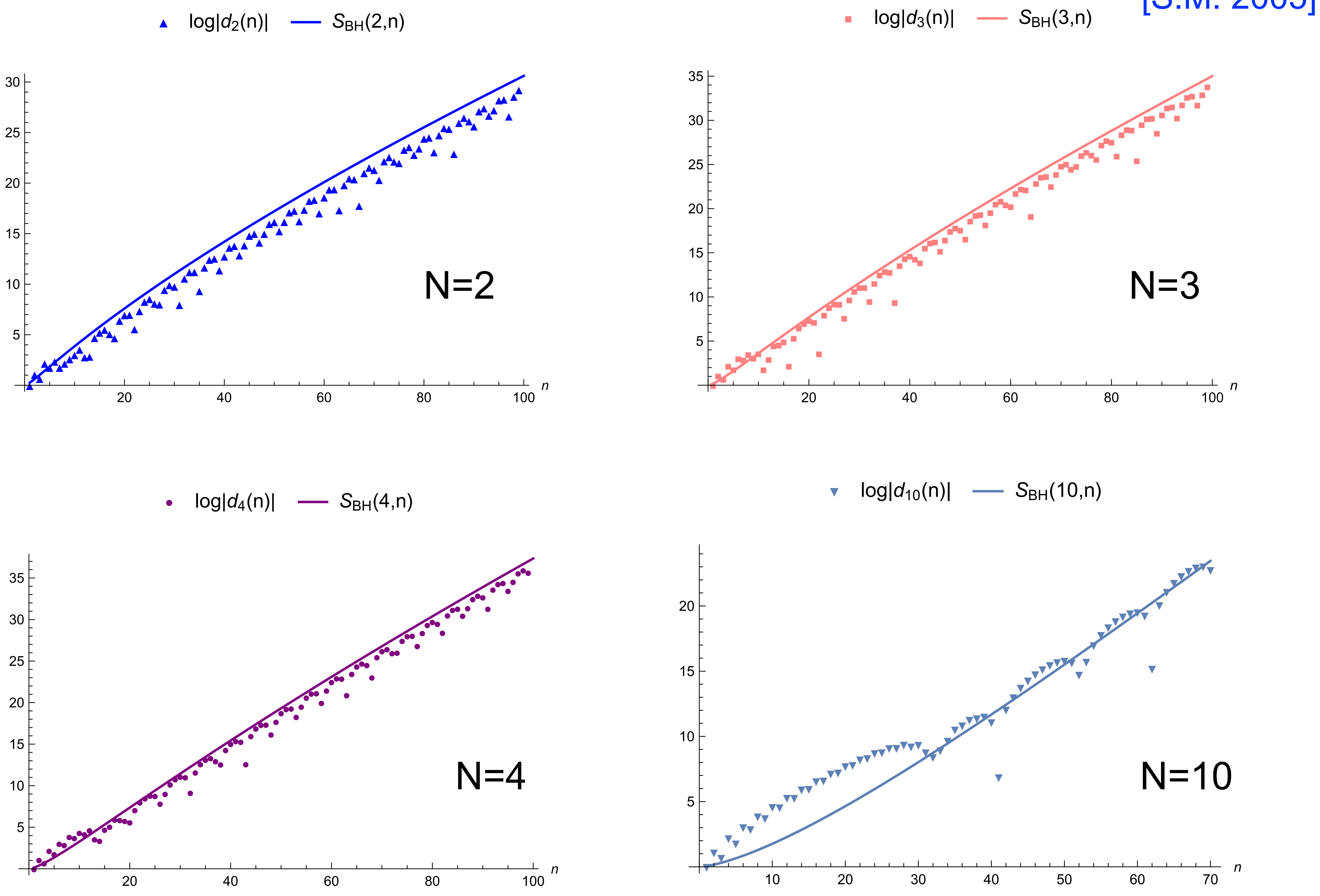
Not $SL_2(\mathbb{Z})$ modular form!

Cardy-like limit

Large-N limit

Comparison for fixed N (Cardy-like limit)

[S.M. 2005]



How to analyze matrix model large-N limit?

$$\mathcal{I}_N(\tau) \;=\; \int [DU]\, \exp\Bigg(\sum_{n=1}^{\infty} \frac{1}{n}\, i_{\mathrm{s}}(\mathrm{e}^{2\pi \mathrm{i} n\tau})\, \mathrm{tr}\, U^n\, \mathrm{tr}\, (U^\dagger)^n \Bigg)$$

Eigenvalues of U $\quad \{\mathrm{e}^{2\pi \mathrm{i} u_i}\}_{i=1,\ldots,N}$

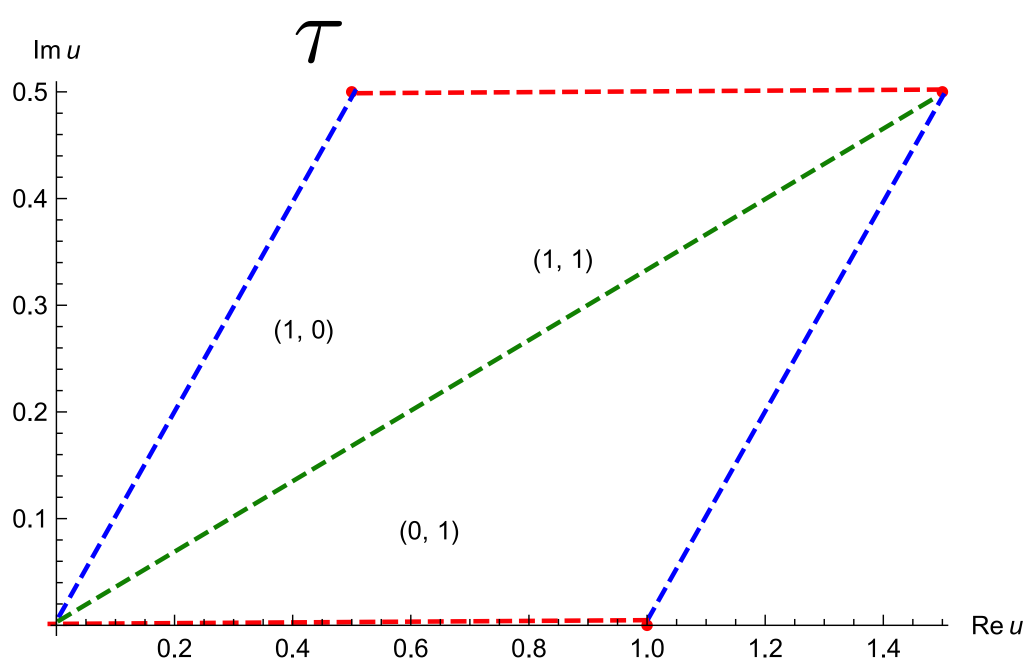
$$\underline{u} \equiv \{u_i\}_{i=1,\ldots,N} \qquad\qquad [D\underline{u}] = \frac{1}{N!} \prod_{i=1}^{N} \mathrm{d}u_i$$

$$\mathcal{I}_N(\tau) \;=\; \int [D\underline{u}]\, \exp\big(-S(\underline{u})\big) \qquad S(\underline{u}) \;=\; \sum_{i,j=1}^{N} V(u_{ij})$$

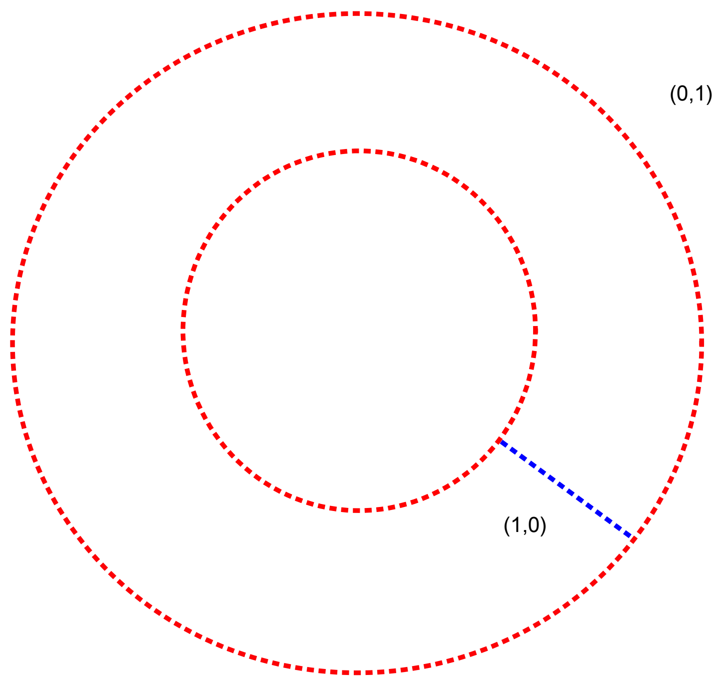
Pure two-particle interaction

Matrix integral has complex saddles

[Cabo-Bizet, S.M. 1909.]



$$u \mapsto e^{2\pi \mathrm{i} u}$$

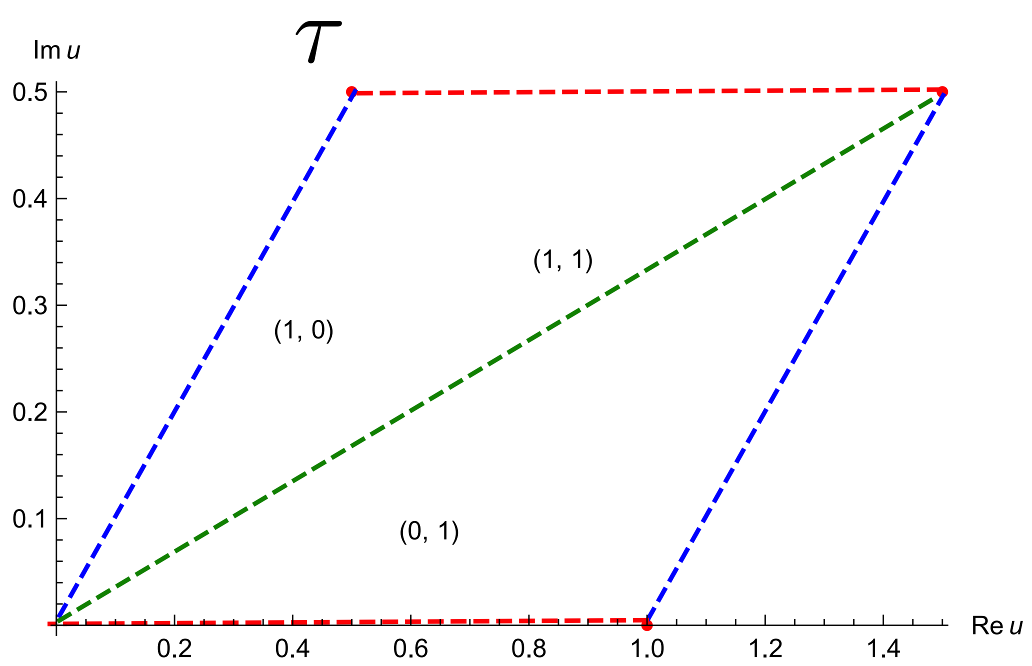


Family of saddles labelled by lattice points (m, n)

$$m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \qquad \gcd(m,n) = 1$$

Matrix integral has complex saddles with macroscopic entropy

[Cabo-Bizet, S.M. 1909.]



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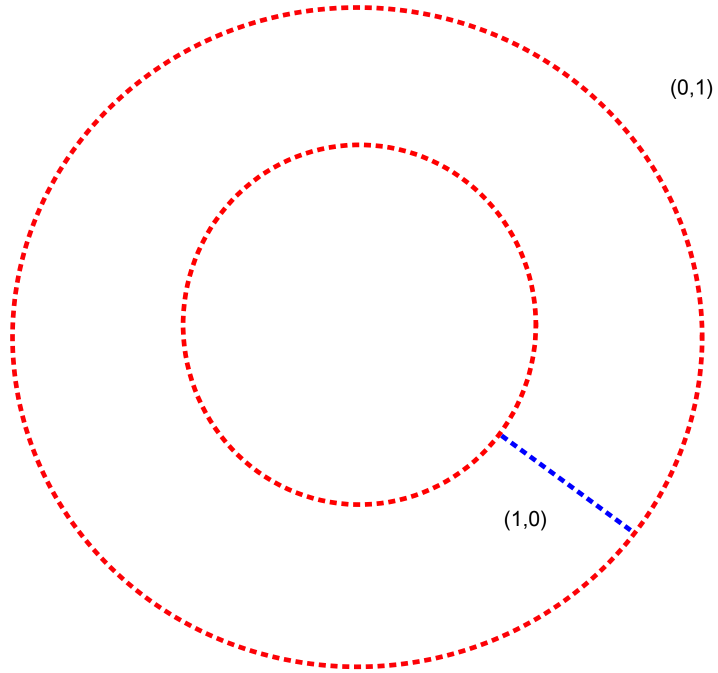


Family of saddles labelled by lattice points (m, n)

$m\tau + n \in \mathbb{Z}\tau + \mathbb{Z}$ $\quad \gcd(m,n) = 1$

$$\mathcal{S}_{(0,1)} = \text{entropy of pure AdS}_5 = O(1)$$

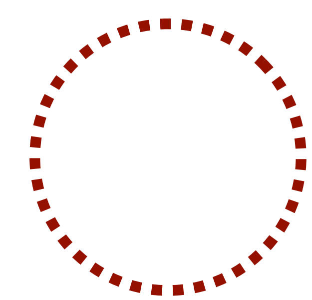
$$\mathcal{S}_{(1,0)} = \text{entropy of AdS}_5 \text{ BH} = O(N^2)$$

Lemma: Periodic eigenvalue distributions are saddle points of periodic potentials

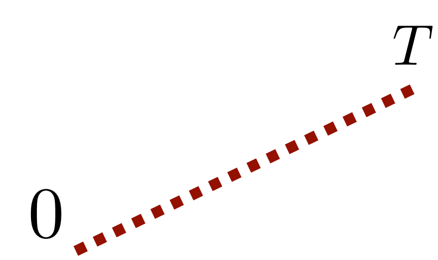
$$\partial_{u_i} S(\underline{u}) = \partial_{\overline{u}_i} S(\underline{u}) = 0 \ , \qquad S(\underline{u}) = \sum_{i,j=1}^{N} V(u_{ij})$$

$$V(-z) = V(z) \qquad V(z+T) = V(z)$$

1. Uniform distribution along [0,T] $u_i = \frac{i}{N} T + u_0$ is a solution of the variational equations.



2. Action of this distribution = average value of V



$$S_{\rm eff}(T) = \sum_{i,j=1}^{N} V\left(\frac{i}{N}T\right) = N^2 \int_0^1 dx \, V(xT)$$

3. Need separate argument for contour deformation.

The potential of our index is quasi-periodic under lattice translations

$$\mathcal{I}(\tau) \;=\; \eta(\tau)^{2N} \int [D\underline{u}] \prod_{i\neq j=1}^{N} \Gamma_{\rm e}(u_{ij}+2\tau;\tau,\tau)\times$$

$$\times \prod_{i,j=1}^{N} \Gamma_{\rm e}\Big(u_{ij}+\frac{1}{3}(2\tau+1);\tau,\tau\Big)^{3}$$

[Dolan-Osborn ’08]

$$\Gamma_{\rm e}\big(z;\sigma,\tau\big) \;=\; \prod_{j,k=0}^{\infty} \frac{1-{\rm e}^{-2\pi {\rm i} z}\,{\rm e}^{2\pi{\rm i}\sigma({\rm j}+1)}\,{\rm e}^{2\pi{\rm i}\tau({\rm k}+1)}}{1-{\rm e}^{2\pi{\rm i}z}\,{\rm e}^{2\pi{\rm i}\sigma j}\,{\rm e}^{2\pi{\rm i}\tau k}}$$

[Felder-Varchenko ’99; Spiridonov ’10; Spiridonov-Vartanov ’12]

$$\Gamma_{\rm e}\big(z;\tau,\tau\big) \;=\; \Gamma_{\rm e}\big(z+1;\tau,\tau\big) \;=\; \theta_0\big(z;\tau\big)^{-1}\,\Gamma_{\rm e}\big(z+\tau;\tau,\tau\big)$$

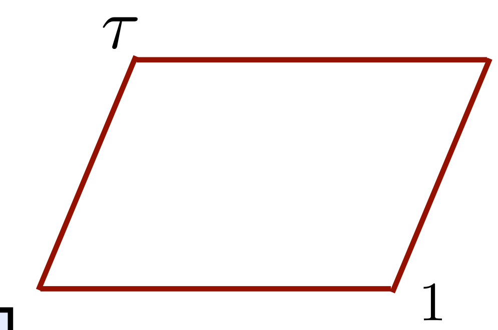
From quasi-elliptic to elliptic functions

$$\theta_0(z;\tau) = (1-\zeta)\prod_{n=1}^{\infty}(1-q^n\zeta)(1-q^n\zeta^{-1})$$

[Jacobi]

$$\theta_0(z) = \theta_0(z+1) = -e^{2\pi i z}\,\theta_0(z+\tau)$$

Lattice $\mathbb{Z}\tau + \mathbb{Z}$



$$q = e^{2\pi i \tau}$$
$$\zeta = e^{2\pi i z}$$

$$z = z_1 + \tau z_2$$

Construct

$$P(z;\tau) := q^{\frac{1}{2}B_2(z_2)}\,\theta_0(z;\tau) \qquad B_2(x) = x^2 - x + \frac{1}{6}$$

1. $q^{-\frac{1}{12}}P(z;\tau) = \theta_0(z;\tau) \quad \text{when } z_2 = 0$

2. $|P(z)|$ is elliptic 3. $|P(z)|$ is modular!

$$-\log|P(z;\tau)| = \lim_{s\to 1}\frac{\tau_2^s}{2\pi}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{e^{2\pi i(nz_2 - mz_1)}}{|m\tau+n|^{2s}}$$

[Kronecker's second limit formula]

The elliptic gamma function can be deformed to an elliptic function

Construct $Q(z;\tau) = q^{\frac{1}{3}B_3(z_2)-\frac{1}{2}z_2B_2(z_2)}\,\dfrac{P(z;\tau)^{z_2}}{\Gamma_{\mathrm{e}}(z+\tau;\tau,\tau)}$

1. $Q(z;\tau) = \Gamma_{\mathrm{e}}(z+\tau;\tau,\tau)^{-1} \quad \text{when } z_2 = 0$

2. $|Q(z)|$ is elliptic → Elliptic dilogarithm [S.Blo

Kronecker-Eisenstein series

Period functions

3. $$\log\left|Q\left(\frac{z}{\tau};-\frac{1}{\tau}\right)\right| - \tau\log|Q(z;\tau)| = F(z;\tau)$$

$$F(z;\tau) = \frac{\tau_2^2}{2\pi^2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{e^{2\pi\mathrm{i}(nz_2-mz_1)}}{(m\tau+n)(m\bar{\tau}+n)^2}$$

Bloch formula

[Ramakrishnan '86] [Zagier '90] [Duke-Imamoglu '07] [Pasol-Zudilin '18]

We can thus extend the SYM action to an elliptic function (the elliptic SYM action)

$$
\begin{aligned}
S(\underline{u}) \;=\; & -2N\log\big(q^{-1/24}\,\eta(\tau)\big) - \frac{1}{6}N\pi\mathrm{i}\tau + \frac{8}{27}\pi\mathrm{i}\tau N^2 \\
& + \sum_{i\neq j}\log Q\big(u_{ij}+\tau\big) + 3\sum_{i,j}\log Q\big(u_{ij}-\tfrac{1}{3}\tau+\tfrac{1}{3}\big) \\
& - \sum_{i\neq j}\log P\big(u_{ij}+\tau\big) + \sum_{i,j}\log P\big(u_{ij}-\tfrac{1}{3}\tau+\tfrac{1}{3}\big)
\end{aligned}
$$

- Well-defined on the torus $\Longrightarrow$ saddles labelled by

  $(m,n) \quad m\tau+n\in\mathbb{Z}\tau+\mathbb{Z} \quad \mathrm{gcd}(m,n)=1$

- Action calculated as the integral along a given direction — use double Fourier expansion!

The action of the (m,n) saddle is a simple rational function of τ

$$S_{\text{eff}}(m,n;\tau) \;=\; \frac{N^2\pi \mathrm{i}}{27\,m}\,\frac{\big(2(m\tau+n)+\chi_1(m+n)\big)^3}{(m\tau+n)^2} \qquad m\neq 0$$

$$S_{\text{eff}}(0,1;\tau) \;=\; 0$$

ℓ	0	1	2
χ_1	0	1	-1

Dirichlet character mod 3.

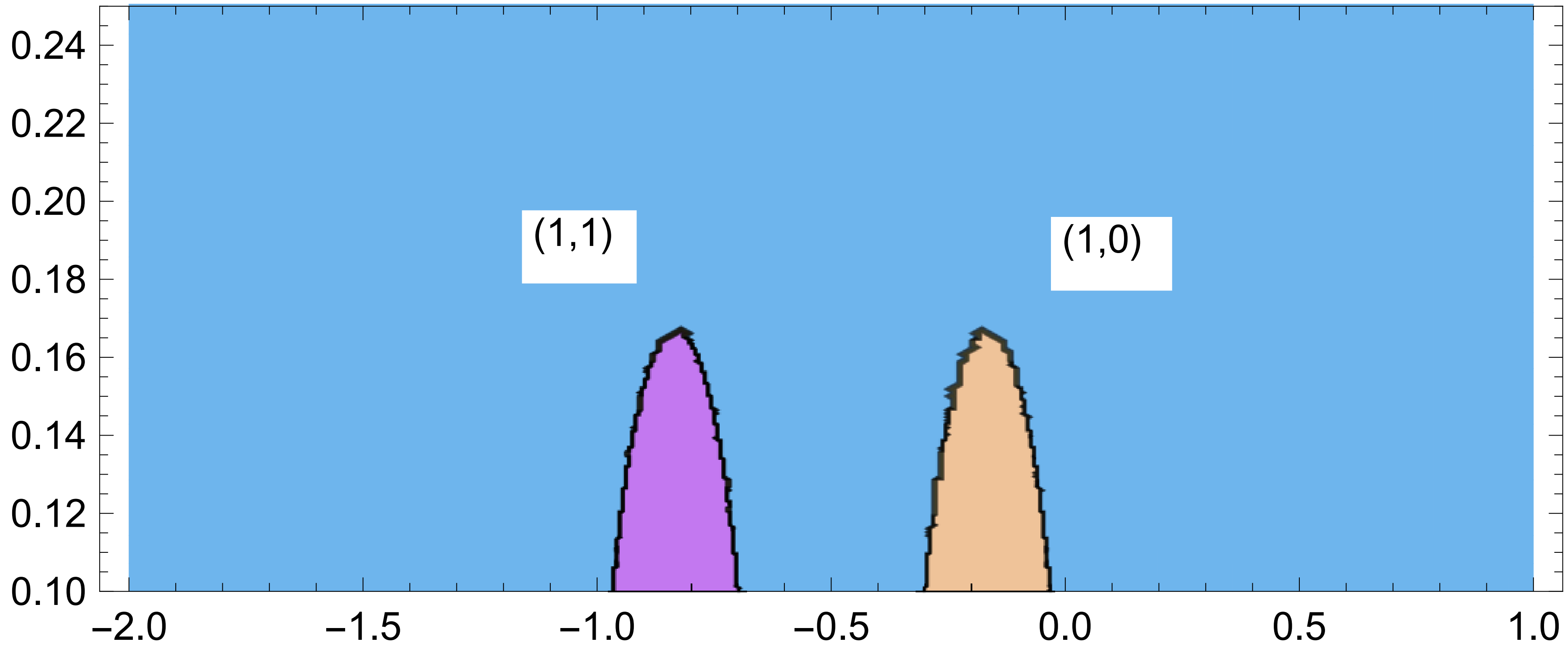
In particular,

$$S_{\text{eff}}(1,0;\tau) \;=\; \frac{N^2\pi \mathrm{i}}{27}\,\frac{\big(2\tau+1\big)^3}{\tau^2}$$

= the on-shell sugra action of AdS5 BH

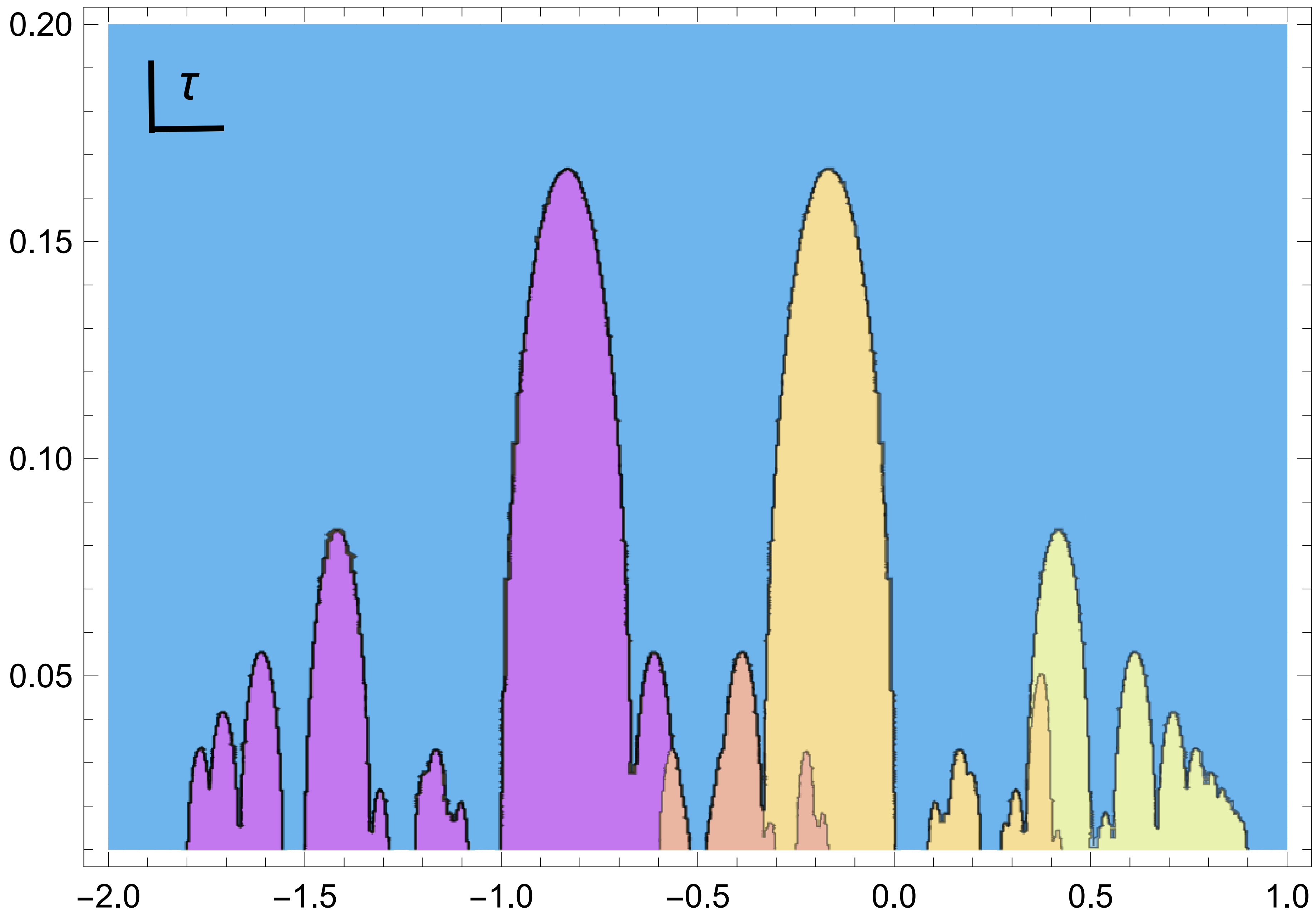
Phase structure: black hole dominates in a region surrounding $\tau = 0$

[Cabo-Bizet, S.M. 1909.]



Phase structure: new phases appear near rational points

[Cabo-Bizet, S.M. 1909.]



Method of elliptic extension is very general

Generic N=1 SCFTs, including flavor chemical potentials

[Cabo-Bizet, Cassani, Martelli, S.M. 2005]

- Universal family of saddles with large entropy. Should have gravitational duals.
- Action controlled by anomalies

$$S_{\text{eff}}(m,n) \;=\; \frac{\pi \mathrm{i}}{3mT^2}\, C_{IJK}\, [\Delta^I]^m_T\, [\Delta^J]^m_T\, [\Delta^K]^m_T$$

- Large family of saddles, classified by finite Abelian groups of order N.

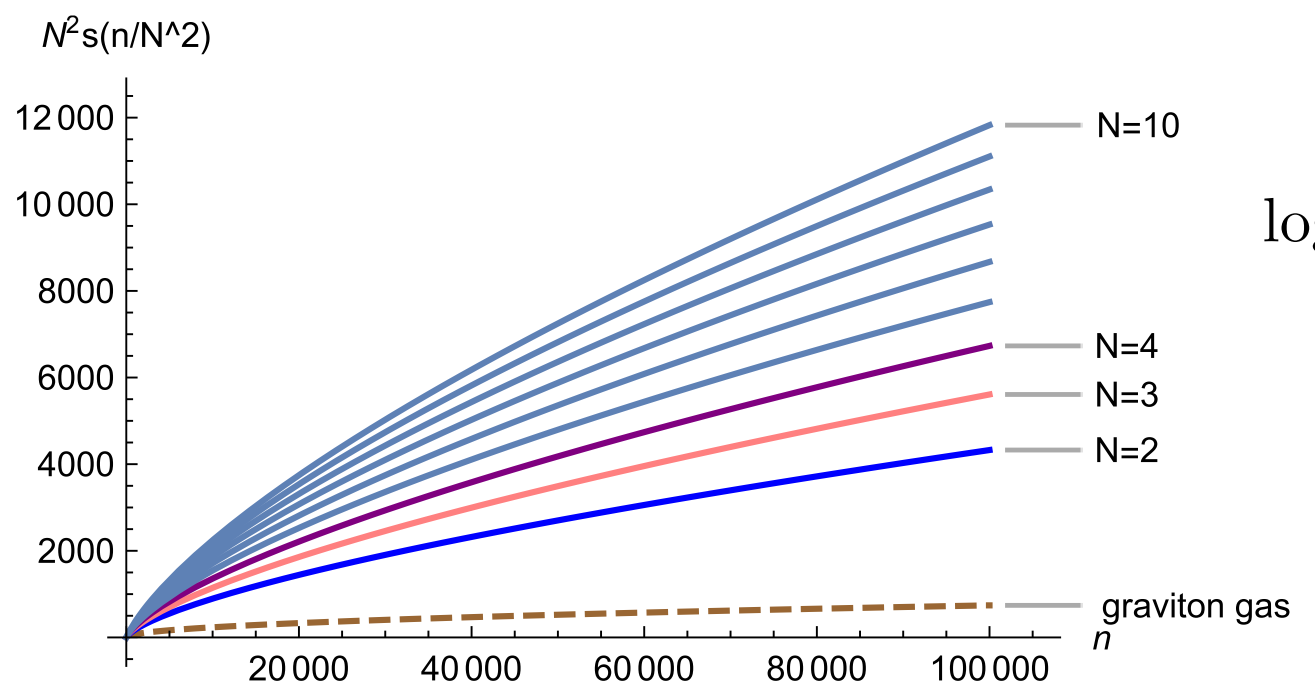
Are SYM operators BHs or gravitons?

- At $N = \infty$ single-trace operators = single gravitons.

  At $N = \infty$ multi-trace BPS operators = multi-gravitons.

  Sharpened and checked to first few orders in [Chang, Yin '13]

  [cf Janik, Trzetrzelewski, '07; Grant, Grassi, Kim, Minwalla '08]



$$\log d_N(n) \xrightarrow{n\to\infty} \frac{\sqrt{3}\pi}{2^{1/3}} N^{2/3} n^{2/3}$$

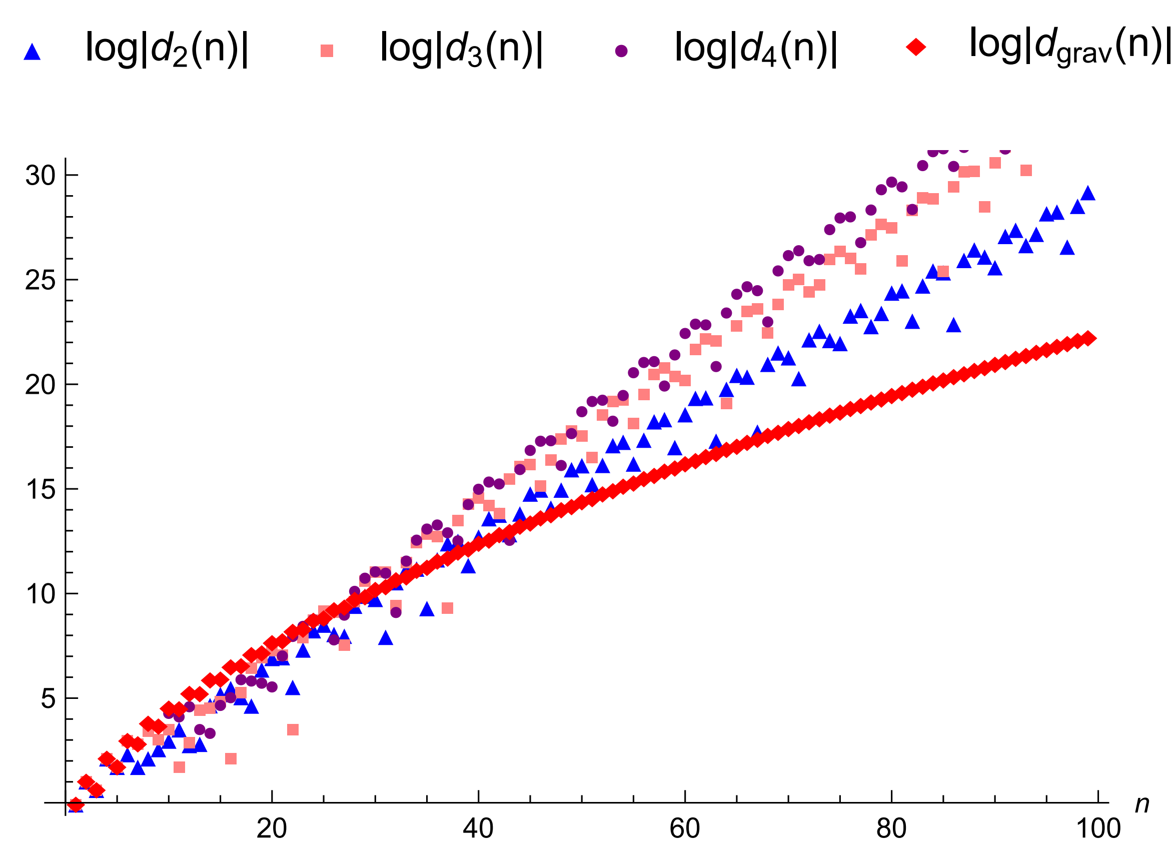
$$\log d_{\rm grav}(n) \xrightarrow{n\to\infty} \frac{\pi}{3}\sqrt{5n}$$

- On the other hand, we clearly see much larger growth of SYM operators controlled by N^2, as consistent with BH.

Both! The index *interpolates* between multi-gravitons and BHs

[S.M. 2005]

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
d_2	3	-2	9	-6	11	-6	9	14	-21	36	-17	-18	114	-194	258	-168
d_3	3	-2	9	-6	21	-18	33	-22	36	6	-19	90	-99	138	-9	-210
d_4	3	-2	9	-6	21	-18	48	-42	78	-66	107	-36	30	114	-165	390
d_5	3	-2	9	-6	21	-18	48	-42	99	-96	172	-156	252	-160	195	48
d_6	3	-2	9	-6	21	-18	48	-42	99	-96	200	-198	345	-340	540	-426
d_{grav}	3	-2	9	-6	21	-18	48	-42	99	-96	200	-198	381	-396	711	-750



This can be proved using ideas from representation theory

[S.M. 2005] [cf Dolan '07; Dutta, Gopakumar '07]

- Expand exponential in terms of U(N) characters.
- Frobenius: $\prod_{j=1}^{m} \left(\mathrm{Tr}\, U^j\right)^{k_j} = \sum_{\ell(\lambda)\leq N} \widetilde{\chi}_\lambda(U)\, \chi^\lambda(P)$ $\left(P \equiv \prod_{j=1}^{m} (j)^{k_j}\right)$

$$\mathcal{I}_N(\mathrm{x}) = \sum_P i_{\mathrm{s}}(\mathrm{x})_P \, \frac{1}{z_P} \sum_{\ell(\lambda)\leq N} \chi^\lambda(P)^2$$

$$z_P = \prod_{j=1}^{m} k_j!\, j^{k_j}$$

$$f(\mathrm{x})_P := \prod_{j=1}^{m} f(\mathrm{x}^j)^{k_j}$$

- At $N = \infty$, we sum over all partitions, and =1

$$\Rightarrow \quad \mathcal{I}_{N=\infty}(\mathrm{x}) = \prod_{k=1}^{\infty} \frac{1}{1 - i_{\mathrm{s}}(\mathrm{x}^k)} = \prod_{n=1}^{\infty} \frac{(1-\mathrm{x}^{3n})^2}{(1-\mathrm{x}^{2n})^3} = \sum_n d_{\mathrm{grav}}(n)\, \mathrm{x}^n$$

Outlook and Questions

Faith in AdS/CFT reconfirmed (in its simplest form)

- Complex saddles of matrix model important.
  BHs, many other phases (what are they in gravity?)

- Importance of lattice $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ and torus $\mathbb{C}/\Lambda_\tau$
  Role of modular symmetry: τ^{-1} vs τ^{-2} period functions

- Elliptic deformation of SYM action — Q-exact term?

- Sum over random partitions (cf 2d gauge theory).

- Deeper role of number theory? Relation to L-functions/ Mahler measure/ Algebraic K-theory.

Thank you very much!