Lost Chapters in CHL Black Holes: Untwisted $\frac{1}{4}$-BPS Dyons in the $\mathbb{Z}_2$ Model

Workshop on Black Holes: BPS, BMS and Integrability
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based on [1]=arXiv:2005.07712

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Introduction:

The purpose of this talk is to keep a Geometry-Physics dictionary alive that started with the seminal work of Yau and Zaslow (95) :

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \,.$$

By work of Göttsche (90), $\Delta(q)^{-1}$ was already known to arise as the generating series of the Euler characteristics of Hilbert schemes of points of $K3$ surfaces S,

$$\sum_{n \geq 0} \chi\big(\mathsf{Hilb}^n(S)\big) q^{n-1} = \frac{1}{\Delta(q)} \,.$$

An argument by Beauville in (97) provided a geometric link between curve counting in primitive classes and the Euler characteristics of $\mathsf{Hilb}^n(S)$. The Yau Zaslow conjecture for non primitive classes was finally proven in AK Maulik, Pandharipande and Scheidegger (95)

The next step was to refine this result to the χ_y genus of $\mathrm{Hilb}^n(S)$ proposed in Katz AK and Vafa (99):

$$\sum_{n\geq 0} \chi_y\big(\mathrm{Hilb}^n(S)\big) q^{n-1} = \frac{1}{q\prod_{n=1}^{\infty}(1-yq^n)^2(1-q^n)^{20}(1-y^{-1}q^n)^2} \, ,$$

and finally to the Hodge of $\mathrm{Hilb}^n(S)$ Katz AK and Pandharipande (14)

$$\sum_{n\geq 0} \chi_{\mathrm{Hodge}}\big(\mathrm{Hilb}^n(S)\big) q^{n-1} =$$

$$\frac{1}{q\prod_{n=1}^{\infty}(1-u^{-1}y^{-1}q^n)(1-u^{-1}yq^n)(1-q^n)^{20}(1-uy^{-1}q^n)(1-uyq^n)} \, .$$

An important step in this dictionary was the proof of the matching of the generating series of the Donaldson Thomas invariants on $X = K3\times E$ with the inverse of the Igusa Cusp form χ_{10}

$$Z^X(q,t,p) = \sum_{h=0}^{\infty}\sum_{d=1}^{\infty}\sum_{n\in\mathbb{Z}} \mathrm{DT}^X_{n,(\beta_h,d)} q^{d-1} t^{\frac{1}{2}\langle\beta_h,\beta_h\rangle}(-p)^n$$

Igusa cusp form conjecture[1]

$$Z^X(q,t,p) = -\frac{1}{\chi_{10}}$$

was recently proven[2]

The main motivation to keep this dictionary alive is that the mathematician work with techniques that are to large extend independent of the amount of supersymmetry which can eventually provide us a guide to the more interesting $\mathcal{N}=2$ cases.

[1] Oberdieck Pandharipande (2014), Bryan (2015)
[2] Oberdieck Shen (2016), Oberdieck Pixton (2017)

The main aim of this paper is to extend this dictionary to the CHL threefold $(K3 \times T^2)/\mathbb{Z}_N$ as orbifold by (g, δ)

- g: order N automorphism of $K3$ preserving holomorphic $(2,0)$-form, non-trivial action on middle cohom. $H^2(K3,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$
- δ: order N translation in elliptic curve T^2

We focus on $N = 2$ and g induces an exchange of $E_8(-1)$'s

Invariant piece: $H^2(K3,\mathbb{Z})^g \cong U^{\oplus 3} \oplus E_8(-2)$ [3]

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[3] with hyperbolic lattice U and diagonal $E_8(-2) \subset E_8(-1)^{\oplus 2}$

- Bryan Oberdieck 2018 conjecture two primitive DT partition functions (PF) for $X = (K3 \times T^2)/\mathbb{Z}_2$ in terms of weight -6 Siegel modular forms (SMF) for level 2 congruence subgroups of $\mathrm{Sp}_4(\mathbb{Z})$.

  DT-invariant $\mathrm{DT}^X_{n,(\gamma,d)}$ depends on whether curve class $\gamma \in \frac{1}{2}H_2(K3,\mathbb{Z})$ is contained in original homology $H_2(K3,\mathbb{Z})$ or not (case B/A resp.):

A: weight -6 SMF $\mathsf{Z}^A(Z) = -\frac{1}{\widetilde{\Phi}_2(Z)} \sim$ Borcherds lift of $K3$ twining genera.
Matches SMF found by David Jatkar Sen as PF of quarter-BPS indices (a.k.a. sixth helicity supertraces) in $\mathcal{N} = 4$ $\mathbb{Z}_2$-CHL model.

B: weight -6 SMF Z^B with no prior appearance in physics:

$$\mathsf{Z}^B(Z) = \frac{-8F_4(Z) + 8G_4(Z) - \frac{7}{30}E_4^{(2)}(2Z)}{\chi_{10}}$$

Here: $G_4(Z), E_4^{(2)}(2Z)$ weight 4 SMF for $\Gamma_0^{(2)}(2)$, while $F_4(Z)$ weight 4 SMF for the paramodular group $K(2) \subset \mathrm{Sp}_4(\mathbb{Q})$ of degree 2. Thus, numerator is a SMF for the intersection $B(2) = K(2) \cap \Gamma_0^{(2)}(2)$, the level 2 Iwahori subgroup.

- **Important clue for physics:** Near diagonal divisor $\left(\begin{smallmatrix}\tau & 0\\ 0 & \sigma\end{smallmatrix}\right)$ SMF Z^B exhibits quadratic pole

$$\mathsf{Z}^B\left(\left(\begin{smallmatrix}\tau & z\\ z & \sigma\end{smallmatrix}\right)\right) \propto \frac{1}{z^2}\,\frac{E_4(2\sigma)}{2\Delta(\sigma)}\,\frac{1}{\eta^8(\tau)\eta^8(2\tau)} + \mathcal{O}(z^0)\ .$$

$\rightarrow$ Admits wall-crossing interpretation:

Decay of 1/4-BPS states of charge (Q,P) (that are counted by the SMF) into two 1/2-BPS states of charge $(Q,0)$ and $(0,P)$ (counted by resp. modular forms in σ and τ).

– $\eta^{-8}(\tau)\eta^{-8}(2\tau)$ is the same as for Z^A, i.e. for Sen's SMF.
– $E_4(2\sigma)/(2\Delta(\sigma))$ different
– Suggestion: 1/4-BPS states with same magnetic P, but different electric Q

Compatible with known limitation of Sen's result:

Electric Q of respective dyons belong to twisted orbifold sector in heterotic frame, i.e. 1/2-BPS states of charge $(Q,0)$ have half-integral winding on S^1_{CHL}.

Four-dim. $\mathcal{N}=4$ models and BPS-index

Recall duality: IIA$[K3\times T^2]\leftrightarrow$ Het$[T^6]$.

Moduli space of 4D $\mathcal{N}=4$ theory:

$$[\mathrm{O}(22,6;\mathbb{Z})\backslash\mathrm{O}(22,6)/(\mathrm{O}(6)\times\mathrm{O}(22))]\ \times\ [\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{U}(1)]$$

- 1st factor: Heterotic Narain moduli space
  Quotient by $\mathcal{T}$-duality group (automorph. $\mathrm{O}(\Lambda_{22,6})\cong\mathrm{O}(22,6;\mathbb{Z})$ of N. lattice)

- 2nd factor: Heterotic axio-dilation modulus
  Quotient by $\mathcal{S}$-duality group $\mathrm{SL}_2(\mathbb{Z})$

Narain lattice vs. IIA charges:

$$\Lambda_{22,6}\ \cong\ \Lambda_{20,4}\oplus\Lambda_{2,2}\ \cong\ H^*(\mathsf{K3},\mathbb{Z})\oplus(\text{momentum-winding on }T^2)$$

Heterotic $\mathbb{Z}_2$ CHL model as asymmetric orbifold:

- g acts only on left-moving bosonic string by swapping E_8's
- δ acts by $\mathbb{Z}_2$-shift on $S^1 \subset T^6$

Moduli of CHL orbifold as invariant moduli of parent theory:

$$G_4(\mathbb{Z})\backslash \left([\mathrm{O}(14,6)/(\mathrm{O}(14)\times \mathrm{O}(6))] \;\times\; [\mathrm{SL}_2(\mathbb{R})/\mathrm{U}(1)] \right)$$

for some discrete U-duality group in four dimensions $G_4(\mathbb{Z}) \supset \mathcal{T} \times \mathcal{S}$

with $\mathcal{T}$ acting (only) on 1st factor

and $\mathcal{S} = \Gamma_1(2)$ acting on 2nd factor (Möbius transformation on het. axio-dilaton)

I.e. we only consider $\mathcal{T}$- and $\mathcal{S}$-transformations here.

Charges. Gauge group $U(1)^r$ of rank r at generic moduli (smooth $K3$).

- Electric charges Q quantized in rank r lattice Λ_e
- Magnetic charges P quantized in dual lattice $\Lambda_m \simeq \Lambda_e^*$

In non-orbifold case: rank $r = 28$ and $\Lambda_e \cong \Lambda_{22,6} \cong \Lambda_m$ unimodular (self-dual).

In $\mathbb{Z}_2$ orbifoldcase: rank $r = 20$ and lattices are

$$\Lambda_e = E_8\left(-\tfrac{1}{2}\right) \ \oplus U^{\oplus 5} \oplus U\left(\tfrac{1}{2}\right)$$
$$\Lambda_m = E_8\left(-2\right) \ \oplus U^{\oplus 5} \oplus U\left(2\right)$$

Note $E_8(-2) \oplus U^{\oplus 4} \cong H^*(K3, \mathbb{Z})^g$ invariant piece in Λ_m.

Now $\Lambda_{e,m}$ are 2-modular, i.e. $\Lambda_m^* \cong \Lambda_m(\frac{1}{2})$ or $\Lambda_m^*(2) \cong \Lambda_m$, not self-dual (can think: rescale quadratic form by 2 or vectors by $\sqrt{2}$).

Duality action. $\mathcal{T} \times \mathcal{S}$ act on charges (Q, P) in $\Lambda_e \oplus \Lambda_m$:

$$\begin{pmatrix} Q \\ P \end{pmatrix} \mapsto \begin{pmatrix} OQ \\ OP \end{pmatrix} \qquad\qquad \begin{pmatrix} Q \\ P \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

In $\mathbb{Z}_2$ case one expects T-duality

$$\mathrm{O}(\Lambda_e) \supset \mathcal{T} \supset C_{(g,\delta)} := \{h \in \mathrm{O}(\Lambda_{22,6}) \,|\, h(\delta) = \delta,\ hg = gh\}$$

Non-trivial problem: characterize (Q, P) orbits by (complete) set of invariants, e.g.

i) Quadratic $\mathcal{T}$-invariants: $T(Q, P) := (Q^2\,,\, P^2\,,\, Q \cdot P)$

ii) Discrete "Torsion invariant"[4]: $I(Q, P) := \gcd(Q \wedge P) \in \mathbb{Z}$

[4] Dabholkar, Gaiotto, Nampuri 2007

Proven[5] for non-orbifold theory where $\Lambda_{em} \cong \Lambda_{22,6}$:
(T, I) uniquely characterizes charge orbit under $\mathcal{T} \times \mathcal{S} = \mathrm{O}(22,6;\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$.

Does not hold for $\mathbb{Z}_2$ case. Can define further invariants, e.g.

iii) $\quad R(Q,P) := [Q] \in \Lambda_e/\Lambda_e^* \cong \mathbb{Z}_2^{2+8}$

[5] Banerjee Sen 2007-2008

BPS-indices. Will be interested in dyonic states of charge (Q, P) that are

$$1/2\text{-BPS}\,(\text{need } Q \parallel P) \qquad \text{or} \qquad 1/4\text{-BPS}\,(\text{need } Q \nparallel P)$$

They contribute to the nth helicity supertrace (BPS-index)

$$\Omega_n(Q, P; \cdot) = \frac{1}{n!}\,\mathrm{Tr}_{(Q,P)}\left[(-1)^F (2h)^n\right]$$

with F fermion parity and h spacetime helicity[6] for $n \geq 4$ and $n \geq 6$ respectively.

The dot denotes potential moduli dependence: for $n = 4$ none but for $n \geq 6$ Ω_6 is only piecewise continuous, i.e. have wall-crossing.

BPS-index expected to be duality invariant, i.e., a function of charge orbits. Can consider $\mathcal{Q} \subset \Lambda_e \oplus \Lambda_m$ consisting of complete orbits, characterized by invariants (T, I, R) — once (I, R) are fixed, the index becomes a function of the quadratic invariants T (and moduli chamber) only:

[6] Third component of angular momentum in rest frame of massive state.

For instance $\mathcal{Q}$ set of unit-torsion dyons ($I = 1$) in Het[T^6] (R trivial), write

$$\Omega_6(Q, P; \cdot) = f_{\mathcal{Q}}(Q^2, P^2, Q \cdot P; \cdot)$$

Form partition function

$$\mathsf{Z}_{\mathcal{Q}}(\tau, z, \sigma) = \sum_{Q^2, P^2, Q\cdot P} (-1)^{Q\cdot P+1} f_{\mathcal{Q}}(Q^2, P^2, Q \cdot P; \cdot)\; e^{2\pi i\left(\sigma\frac{Q^2}{2}+\tau\frac{P^2}{2}+z\,Q\cdot P\right)}$$

Also write as $\mathsf{Z}_{\mathcal{Q}}(\tau, z, \sigma) =: \frac{1}{\Phi_{\mathcal{Q}}(\tau,z,\sigma)}$. In the example above have DVV-result[7]

$$\mathsf{Z}_{\mathcal{Q}}(\tau, z, \sigma) = \frac{1}{\chi_{10}}, \qquad \chi_{10} : \text{weight 10 Igusa cusp form of } \mathrm{Sp}_4(\mathbb{Z})$$

[7] Dijkgraaf Verlinde Verlinde 1996

BPS-index now obtained as Fourier coefficient via contour integral:

$$f_{\mathcal{Q}}(Q^2, P^2, Q\cdot P; \cdot\,) = \frac{(-1)^{Q\cdot P+1}}{(\mathsf{q}_1\mathsf{q}_2\mathsf{q}_3)^{-1}} \oint_{\mathcal{C}} \frac{e^{-2\pi i\left(\sigma\frac{Q^2}{2}+\tau\frac{P^2}{2}+z\,Q\cdot P\right)}}{\Phi_{\mathcal{Q}}(\tau,\sigma,z)}\,\mathrm{d}\tau\wedge\mathrm{d}\sigma\wedge\mathrm{d}z$$

Contour $\mathcal{C} \simeq T^3$: deformations pick up residues from quadratic zeroes of $\Phi_{\mathcal{Q}}$ (e.g. $z=0$). Choose moduli dependence[8] in $\Im(\mathcal{C})$ s.t. poles are picked up precisely when decay to two $1/2$-BPS states possible, e.g. at $z=0$

$$(Q,P) \longrightarrow (Q,0)\; + (0,P)$$

or generic decay parametrized in the form (with $a_0d_0 - b_0c_0 = 1$)

$$(Q,P) \to (a_0d_0Q - a_0b_0P, c_0d_0Q - c_0b_0P) + (-b_0c_0Q + a_0b_0P, -c_0d_0Q + a_0d_0P)$$

[8] Cheng Verlinde, Sen 2007

Such a decay demands 2nd order pole in $\Phi_{\mathcal{Q}}^{-1}$ at

$$z' := c_0 d_0\, \tau + a_0 b_0\, \sigma + (a_0 d_0 + b_0 c_0)\, z = 0$$

and the change in the index matches the (primitive/$I = 1$) wall-crossing formula[9]

$$(-1)^{Q' \cdot P' + 1} \;\; Q' \cdot P' \;\; d_h(a_0 Q', c_0 Q')\; d_h(b_0 P', d_0 P')$$

with half-BPS indices $d_h \sim \Omega_4$ of decay products $(a_0 Q', c_0 Q')$, $(b_0 P', d_0 P')$.

Think of the d_h as Fourier coeff. of genus-1 modular forms $\phi_e(\sigma'; a_0, c_0) {-1}$ and $\phi_m(\tau'; b_0, d_0)^{-1}$ that appear near quadratic divisor $z' = 0$:

$$\Phi_{\mathcal{Q}}^{-1}(\tau, \sigma, z) \propto \Big(\phi_e(\sigma'; a_0, c_0)^{-1}\; \phi_m(\tau'; b_0, d_0)^{-1}\;\; z'^{-2} + \mathcal{O}(z'^0) \Big)$$

$$\sigma' := c_0^2\, \tau + a_0^2\, \sigma + 2 a_0 c_0\, z \qquad\qquad \tau' := d_0^2\, \tau + b_0^2\, \sigma + 2 b_0 d_0\, z$$

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[9] Denef Moore, ...

Their $\mathrm{SL}_2(\mathbb{Z})$ symmetries (might) lift to symmetries of $\Phi_{\mathcal{Q}}(\tau, z, \sigma)$, e.g. for $\phi_m(\tau'; b_0, d_0)$ can expect

$$\begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 & 0 & \beta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_1 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix} \qquad (\star)$$

These act on $Z = \left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)$ as $Z \mapsto (AZ + B)(CZ + D)^{-1}$ when written in usual block form. Can argue: $\phi_{e/m}$ weight $k + 2$ mod. form $\Rightarrow$ $\Phi_{\mathcal{Q}}$ weight k SMF.

Constraints from $\mathcal{S}$-duality: For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{S} \subset \mathrm{SL}_2(\mathbb{Z})$ index should satisfy $\Omega_6(Q, P; \cdot) = \Omega_6(aQ + bP, cQ + dP; \cdot')$. Implies $\Phi_{\mathcal{Q}}(Z) = \Phi_{\mathcal{Q}}(Z')$ where Z' is obtained by acting with the 4x4-matrix on the right of $(\star)$ (with a, b, c, d taking role of a_0, b_0, c_0, d_0).

Constraint from charge quantization: Shift symmetry $\Phi_{\mathcal{Q}}(Z) = \Phi_{\mathcal{Q}}(Z + B)$ for B subject to the values $T = (Q^2, P^2, Q \cdot P)$ takes on $\mathcal{Q}$. Consider as 4x4 matrix $\left(\begin{smallmatrix} 1_2 & B \\ 0_2 & 1_2 \end{smallmatrix}\right)$ acting on Z.

Conclusion so far: Assemble quarter-BPS indices for charge orbits $\mathcal{Q} \subset \Lambda_{em}$ into partition function $\Phi_{\mathcal{Q}}(Z)^{-1}$. Extraction as Fourier coeff. via contour integral.

- Quantization laws of $T = (Q^2, P^2, Q \cdot P)$: Shift symmetries $\left(\begin{smallmatrix} 1_2 & B \\ 0_2 & 1_2 \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z})$
- S-duality invariance: $\mathrm{SL}_2(\mathbb{Z}) \supset \mathcal{S} \hookrightarrow \mathrm{Sp}_4(\mathbb{Z})$
- Wall-crossing:
  - Quadratic divisors for allowed decays $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$.
  - Embedded modular symmetries $\subset \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{Sp}_4(\mathbb{Z})$ of half-BPS counting functions for decay products.

$\Rightarrow \Phi_{\mathcal{Q}}$ must be weight k SMF for group that embedds the above symmetries and has the right divisors. Strategy to explain new DT formula Z^B:

Identify the charges $\mathcal{Q} \subset \Lambda_{em}$ whose $1/4$-BPS PF $\mathsf{Z}_{\mathcal{Q}}$ gives rise to former. Check the above constraints. For this need to understand $1/2$-BPS counting functions. Also crucial for a physics derivation of $\mathsf{Z}_{\mathcal{Q}}$ via chiral genus-two partition function.

Dabholkar-Harvey $1/2$-BPS states in the $\mathbb{Z}_2$ model

Purely electric $(Q, 0)$ half-BPS states can be realized as DH states in the perturbative heterotic string. Keep right-movers in ground state, allow arbitrary left-movers (bosonic). Charge Q from momentum-winding in Narain lattice ($\mathbb{Z}_2$ invariant part).

Half-BPS indices Ω_4 via heterotic one-loop (orbifold) partition function with extra fugacities for left- and right- helicities. Take appropriate derivatives; then set chem. pot. to zero. Result of Dabholkar Denef Moore Pioline 2005:

$$B_4(q,\bar{q}) = \frac{3}{2\tau_2}\,\frac{1}{2}\left[\frac{\theta^2_{E_8(1)}(\tau)}{\eta^{24}(\tau)}\mathcal{Z}_{6,6}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \frac{\theta_{E_8(1)}(2\tau)}{\eta^8(\tau)\eta^8(2\tau)}\mathcal{Z}_{6,6}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right.$$
$$\left. + \frac{\theta_{E_8(1)}(\frac{\tau}{2})}{\eta^8(\tau)\eta^8(\frac{\tau}{2})}\mathcal{Z}_{6,6}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + e^{-2\pi i/3}\frac{\theta_{E_8(1)}(\frac{\tau+1}{2})}{\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\mathcal{Z}_{6,6}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right].$$

with Narain lattice (shifted in twisted sector $h=1$) $\Lambda_{6,6}^{[h]} = \left(\Lambda_{6,6} + \frac{h}{2}\delta\right)$ and

$$\mathcal{Z}_{6,6}[{}^h_g] = \sum_{Q \in \Lambda_{6,6}^{[h]}} (-1)^{g\,\delta\cdot Q}\ e^{i\pi Q_L \tau Q_L - i\pi Q_R \bar\tau Q_R}$$

How to read off the index: *In unorbifolded model* would only have term

$$B_4^{\rm unorb}(q,\bar q) = \frac{1}{\tau_2} \times \mathcal{Z}_{6,6}[{}^0_0](q,\bar q)\ \theta^2_{E_8(1)}(\tau) \times \frac{3}{2}\frac{1}{\eta^{24}(\tau)}\ .$$

For DH state with charge $(Q,0)$ and $Q \in \Lambda_{6,6} \oplus E_8^{\oplus 2}$ read off

$$d_h(Q,0) \equiv \Omega_4(Q,0) = \frac{3}{2} p_{24}(N)\,, \qquad N-1 = \frac{Q^2}{2}$$

Apply to $\mathbb{Z}_2$: *In $\mathbb{Z}_2$ model* consider $P_1 \pm P_2 = 2P_\pm \pm \mathcal{P}$ for root lattice vectors $P_1, P_2, P_+, P_- \in E_8$ and $\mathcal{P} \in E_8/(2E_8) \cong E_8(2)^*/E_8(2) \cong \mathbb{Z}_2^8$. Only P_1+P_2 physical electric charge, want to seperate this as above.

Rewrite theta function for $E_8^{\oplus 2}$ as

$$\theta^2_{E_8(1)} = \theta^2_{E_8(2),1} + 120\ \theta^2_{E_8(2),248} + 135\ \theta^2_{E_8(2),3875}\ .$$

with $E_8(2)$ theta functions with characteristics $\mathcal{P}$ (dep. only on orbit $[\mathcal{P}]$ under Weyl group)

$$\theta_{E_8(2),\mathcal{P}}(\tau) := \sum_{\Delta \in E_8(1)} \exp\left[2\pi i \tau \left(\Delta - \frac{1}{2}\mathcal{P}\right)^2\right]$$

From the untwisted sector B_4^{untw} of $B_4 = B_4^{\mathrm{untw}} + B_4^{\mathrm{tw}}$ get

$$B_4^{\mathrm{untw}}(q,\bar{q}) = \frac{3}{2\tau_2} \times \sum_{\epsilon \in \{+1,-1\}} \frac{\mathcal{Z}_{6,6}[{}^0_0] + \epsilon \mathcal{Z}_{6,6}[{}^0_1]}{2} \left[\theta_{E_8(2)} \times \frac{1}{2}\left(\frac{\theta_{E_8(2)}}{\eta^{24}} + \epsilon \frac{1}{\eta^8(\tau)\eta^8(2\tau)}\right)\right.$$
$$\left. + 120\ \theta_{E_8(2),248} \times \left(\frac{\theta_{E_8(2),248}}{2\eta^{24}}\right) + 135\ \theta_{E_8(2),3875} \times \left(\frac{\theta_{E_8(2),3875}}{2\eta^{24}}\right)\right].$$

Here $(\ldots)$ takes the role $1/\Delta$ plays in unorbifolded case, specific to the charge

subsector of untwisted charges $\Lambda_e^U = U \oplus U^{\oplus 5} \oplus E_8(-\frac{1}{2})$ [10].

The sign $\epsilon = \pm 1$ corresponds to momentum along S^1_{CHL} being even $(+)$ or odd (-) — this parity is conserved under $\mathcal{T}$.

Consider untwisted sector $Q \in U \oplus U^{\oplus 5} \oplus E_8(-2)$, i.e. with $\mathcal{P} = 0$, where the CHL momentum takes arbitrary values. Summing over $\epsilon = \pm 1$ gives a partition function for half-BPS states for with such $(Q, 0)$:

$$\frac{1}{\tau_2} \times \ \mathcal{Z}_{6,6}[{}^0_0](q,\bar{q})\ \theta_{E_8(2),1} \times \frac{3}{2} \frac{\theta_{E_8(2),1}}{2\,\eta^{24}(\tau)} \quad \longrightarrow \quad \phi_e^{-1}(\sigma) = \frac{E_4(2\sigma)}{2\Delta(\sigma)}$$

This is the precisely the modular form on the diagonal divisor $z = 0$ as seen in the DT partition function Z^B.

[10](recall $h = 0$: Narain lattice not shifted by $\delta/2$)

1/4-BPS partition function from heterotic genus-two amplitude

Dabholkar Gaiotto (2007) suggested to look at the chiral (bosonic) genus-two partition function of the heterotic $\mathbb{Z}_2$ orbifold in order to identify the quarter-BPS partition function discovered by Sen et al. Their motivation was that a 1/4-BPS dyon can be represented as a string web on a torus, which in M-theory becomes a genus-two Riemann surface. In fact (Dabholkar, Gaiotto, Nampuri (2007)) the genus of this surface satifies $g = I + 1 = 2$ and the derivation only captures $I = 1$ dyons.

We follow this ansatz to identify 1/4-BPS PFs for charges with untwisted sector electric charge Q, complementary to Dabholkar Gaiotto (2007). This should especially yield the DT partition function Z^B proposed by Bryan Oberdieck 2018.

Start: sum over orbifold blocks (at Narain moduli locus admitting factoriz'n):

$$\mathcal{Z}(\Omega) = \frac{1}{2^2} \sum_{\substack{h_1, h_2 \in \{0,1\} \\ g_1, g_2 \in \{0,1\}}} \mathcal{Z}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] = \frac{1}{2^2} \sum_{\substack{h_1, h_2 \in \{0,1\} \\ g_1, g_2 \in \{0,1\}}} \mathcal{Z}_8\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] \mathcal{Z}_{6,6}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right]$$

The $\mathcal{Z}_{6,6}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right]$ are Siegel-Narain theta functions for T^6:

$$\Lambda_{6,6}^{[h_1,h_2]} = \left(\Lambda_{6,6} + \frac{h_1}{2}\delta\right) \oplus \left(\Lambda_{6,6} + \frac{h_2}{2}\delta\right)$$

$$\mathcal{Z}_{6,6}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] = \sum_{(Q_1,Q_2)\in\Lambda_{6,6}^{[h_1,h_2]}} (-1)^{\delta\cdot(g_1 Q_1 + g_2 Q_2)} \; e^{i\pi Q_L^r \Omega_{rs} Q_L^s - i\pi Q_R^r \bar{\Omega}_{rs} Q_R^s}$$

Identify the period matrix Ω with chem. potentials conjugate to quadratic charge invariants:

$$\Omega \stackrel{!}{=} Z \quad \left(= \left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)\right) \qquad \leftrightarrow \qquad \begin{pmatrix} P^2/2 & Q\cdot P \\ Q\cdot P & Q^2/2 \end{pmatrix}$$

As in the genus-one (half-BPS) case, need to identify contributions in $\mathcal{Z}$ that arise from appropriate charges in the lattice sums.

First look at "toroidal" charge components as in $\mathcal{Z}_{6,6}$. Will identify dyon charge $(Q,P)|_{\text{toroid.}} \leftrightarrow (Q_2, Q_1)$ therein.

- Recall $\Lambda_m = U(2) \oplus U^5 \oplus E_8(2)$. If Q_1 should be magnetic, need to look at blocks with $h_1 = 0$ (no shifted U-lattice). Also: summing $g_1 = 0, 1$ projects to charges Q_1 with $(-1)^{g \cdot Q_1} = 1$, i.e. along a sublattice $U(2) \subset U$.

- Similarly: Want untwisted electric charges Q_2 in $\mathcal{Z}_{6,6}$, i.e. look at $h_2 = 0$.

Next will look at E_8 charge component. Use observation: Without a $\mathbb{Z}_2$-shift along S^1_{CHL} (just the $\mathbb{Z}_2$ action on the E_8's) one would obtain an equivalent theory, hence:

$$\mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \frac{\left[\Theta^{(2)}_{E_8}(\Omega)\right]^2}{\chi_{10}} = \sideset{}{'}\sum_{\substack{h_1, h_2 \in \{0,1\} \\ g_1, g_2 \in \{0,1\}}} \mathcal{Z}_8\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] .$$

Hence the full orbifold sum becomes

$$\mathcal{Z}(\Omega) = \frac{1}{2^2} \sideset{}{'}\sum_{\substack{h_1,h_2 \in \{0,1\} \\ g_1,g_2 \in \{0,1\}}} \mathcal{Z}_8\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] \left(\mathcal{Z}_{6,6}\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \mathcal{Z}_{6,6}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right]\right)$$

and from the above $\mathcal{Z}_{6,6}$ consideration focus on

$$\frac{1}{2^2} \sideset{}{'}\sum_{g_1,g_2 \in \{0,1\}} \mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ g_1 & g_2 \end{smallmatrix}\right] \left(\mathcal{Z}_{6,6}\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \mathcal{Z}_{6,6}\left[\begin{smallmatrix} 0 & 0 \\ g_1 & g_2 \end{smallmatrix}\right]\right)$$

Dabholkar Gaiotto (2007) used results of Dijkgraaf Verlinde[2] (1988) to compute the block[11]

$$\mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]\left(\left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)\right) = \frac{\Theta^{(2)}_{E_8}(2\tau, 2z, 2\sigma)}{\Phi_{6,0}} + \frac{\Theta^{(2)}_{E_8}(2\tau, z, \frac{\sigma}{2})}{16\,\Phi_{6,3}} + \frac{\Theta^{(2)}_{E_8}(2\tau, z, \frac{\sigma+1}{2})}{16\,\Phi_{6,4}}$$

[11] Their roles of (τ, σ) are swapped w.r.t. the ones here, formally $\mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ here.

where $\Phi_{6,0}$ is a weight 6 $\Gamma_0^{(2)}(2)$ SMF and $\Phi_{6,i}$ are modular images of it under the index 3 subgroup of group $\Gamma_0^{(2)}(2)$ that keeps the characteristic $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ mod 2. Can write them as

$$\Phi_{6,0} = \frac{Y_0}{\chi_{10}}, \quad \Phi_{6,3} = \frac{Y_3}{\chi_{10}}, \quad \Phi_{6,4} = \frac{Y_4}{\chi_{10}}$$

Recall from DH analysis that in the untwisted sector the charge components along the electric $E_8(\frac{1}{2}) \subset \Lambda_e$ came in three classes $P_+ = 2\Sigma + \mathcal{P}$, where $\mathcal{P} \in E_8/(2E_8)$ falls into one of three orbits $\mathcal{O}_1, \mathcal{O}_{248}, \mathcal{O}_{3875}$ under Weyl(E_8). Define for these $\mathcal{O}_x$

$$\Theta_x := \sum_{\mathcal{P} \in \mathcal{O}_x} \sum_{\substack{(Q_1, Q_2) \in \\ E_8(2) \oplus [E_8(2) + \mathcal{P}]}} e^{i\pi\, Q^r \Omega_{rs} Q^s}$$

Re-express the E_8 Siegel theta functions in numerator of $\mathcal{Z}_8[\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}]$:

$$\Theta^{(2)}_{E_8}(2\tau, 2z, 2\sigma) = \sum_{\substack{(Q_1,Q_2)\in\\ E_8(2)\oplus E_8(2)}} e^{i\pi\, Q^r \Omega_{rs} Q^s} = \Theta_1$$

$$\Theta^{(2)}_{E_8}(2\tau, 2z, \frac{\sigma}{2}) = 2^{-4} \sum_{\substack{(Q_1,Q_2)\in\\ E_8(2)\oplus E_8(\frac{1}{2})}} e^{i\pi\, Q^r \Omega_{rs} Q^s} = \Theta_1 + \Theta_{248} + \Theta_{3875}$$

$$\Theta^{(2)}_{E_8}(2\tau, 2z, \frac{\sigma+1}{2}) = 2^{-4} \sum_{\substack{(Q_1,Q_2)\in\\ E_8(2)\oplus E_8(\frac{1}{2})}} (-1)^{Q_2^2} e^{i\pi\, Q^r \Omega_{rs} Q^s} = \Theta_1 - \Theta_{248} + \Theta_{3875}$$

We are eventually interested in electric charges $Q = Q_2$ with components in $E_8(2) \subset E_8(\frac{1}{2})$, i.e. $\mathcal{O}_1 \leftrightarrow \Theta_1$.

$$\frac{1}{2^2}\left(\mathcal{Z}_{6,6}[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}] + \mathcal{Z}_{6,6}[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}]\right)\ \mathcal{Z}_8[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}] \qquad \leftarrow \text{Correct terms to look at.}$$

$$= \frac{1}{2}\sum_{\substack{Q_1\in U(2)\oplus U^{\oplus 5}\\ Q_2\in U^{\oplus 6}}} e_{Q_1,Q_2}(\Omega)\ \mathcal{Z}_8[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](\Omega) \qquad \leftarrow (e_{Q_1,Q_2}(\Omega) := \exp[\ldots])$$

$$= \sum_{\substack{Q_1\in U(2)\oplus U^{\oplus 5}\\ Q_2\in U^{\oplus 6}}} e_{Q_1,Q_2}(\Omega)\left(\Theta_1\frac{Y_0+\frac{1}{16}Y_3+\frac{1}{16}Y_4}{2\,\chi_{10}} + \Theta_{248}\frac{\frac{1}{16}Y_3+\frac{1}{16}Y_4}{2\,\chi_{10}} + \Theta_{3875}\frac{\frac{1}{16}Y_3-\frac{1}{16}Y_4}{2\,\chi_{10}}\right)$$

$$= \sum_{x\in\{1,248,3875\}}\left[\left(\sum_{\substack{Q_1\in U(2)\oplus U^{\oplus 5}\oplus E_8(2)\\ Q_2\in U^{\oplus 6}\oplus[E_8(2)+\mathcal{O}_x]}} e_{Q_1,Q_2}(\Omega)\right)\times \mathsf{Z}^{\text{untw.}}_{(x)}\right]$$

For the trivial shift orbit $\mathcal{O}_1 = \{0\}$:

✓: Summation $P \leftrightarrow Q_1$ over $U(2)\oplus U^{\oplus 5}\oplus E_8(2) = \Lambda_m$

✓: Summation $Q \leftrightarrow Q_2$ over $U \quad \oplus U^{\oplus 5}\oplus E_8(2) \subset \Lambda_e$

$$\Rightarrow \qquad \mathsf{Z}^{\text{untw.}}_{(1)} = \frac{Y_0 + \frac{1}{16}Y_3 + \frac{1}{16}Y_4}{2\,\chi_{10}}$$

should be the partition function of quarter-BPS dyons with indicated charge (Q, P) .

Matching DT: $\mathsf{Z}^{\text{untw.}}_{(1)}$ expected to match to new DT result:

$$\mathsf{Z}^B(Z) = \frac{-8F_4(Z) + 8G_4(Z) - \frac{7}{30}E_4^{(2)}(2Z)}{\chi_{10}}$$

Rewrite numerators as polyonomials in terms of genus-two theta constants, use relation $\theta^2_{0100}\theta^2_{0110} = \theta^2_{0000}\theta^2_{0010} - \theta^2_{0001}\theta^2_{0011}$. $\Rightarrow$ *perfect match!* ✓

Note: As a byproduct, have obtained two more $1/4$-BPS partition functions for unit-torsion dyons with untwisted sector electric charge Q (shifts by $\mathcal{O}_{248,3875}$).

Modular and polar constraints

Have also checked for $\mathsf{Z}^{\text{untw.}}_{(1)}$ constraints from $\Gamma_1(2)$ $\mathcal{S}$-duality, translation symmetries and wall-crossing. All checks passed! ✓

In fact, this analysis gives very stringent constraints:

- The weight of $\mathsf{Z}^{\text{untw.}}_{(1)}$ must be $-k = -6$
- $\mathsf{Z}^{\text{untw.}}_{(1)}$ should exhibit quadratic poles at all images of $z = 0$ under the group generated by integer translations and embedded $\mathcal{S} = \Gamma_1(2) \hookrightarrow \mathrm{Sp}_4(\mathbb{Z})$ symmetries.
- The coefficient of such a pole must be given by known half-BPS counting modular forms (inspect decay products at each pole)

- The (expected) Siegel symmetries of $\mathsf{Z}^{\text{untw.}}_{(1)}$ satisfy a common congruence relation

$$\begin{pmatrix} 2\mathbb{Z}+1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 & \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & \mathbb{Z} & 2\mathbb{Z}+1 \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & \mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z})$$

  The RHS is precisely the Iwahori subgroup $B(2)$!

Remarkable: Even we did not yet have a closed formula for $\mathsf{Z}^{\text{untw.}}_{(1)}$, the (simplest) ansatz $M_4(Z)/\chi_{10}$ with M_4 a weight 4 SMF for $B(2)$ would lead to the correct result, as the (even weight SMF) ring generators are known [Ibukiyama (1999)] and we only need to fix a finite number of coefficients.

Constraints from black hole entropy

Following DVV 1996, Cardoso de Wit Kappeli Mohaupt 2004, Sen 2005 for large $(Q^2, P^2, Q \cdot P)$ the log of the microscopic 1/4-BPS index should match the macroscopic entropy of an extremal black hole carrying the charge (Q, P) as computed in the supergravity approximation.

Bekenstein-Hawking area term $+$ leading correction in inverse powers of charges:

$$S_{BH} = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2} + 64\pi^2\,\phi\left(\frac{Q\cdot P}{P^2}, \frac{\sqrt{Q^2P^2-(Q\cdot P)^2}}{P^2}\right) + \cdots \ .$$

Correction worked out from the entropy function including Gauss-Bonnet term in effective sugra action (Gregori Kiritsis Kounnas Obers Petropoulos Pioline 1997, Sen 2005)

$$\int \mathrm{d}^4 x\ \sqrt{-\det g}\ \phi(a,S)\ \left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

where $\tau = a + iS$ axio-dilaton. Function ϕ for $\mathbb{Z}_2$ CHL model:

$$\begin{aligned} \phi(a,S) &= -\frac{1}{64\pi^2}\left[8\log S + \log g(a+iS) + \log g(-a+iS)\right] + \text{const.} \\ g(\tau) &:= \eta^8(\tau)\eta^8(2\tau) \end{aligned}$$

Following the analysis of previously known 1/4-BPS PF in CHL models, estimate asymptotic growth of Fourier coeff. by saddle-point approximation in (τ,σ) after picking up the dominant pole in the z-plane. This is the divisor $z' := \tau\sigma - z^2 + z = 0$, where $\mathsf{Z}^{\text{untw.}}_{(1)}$ behaves as

$$\mathsf{Z}^{\text{untw.}}_{(1)} \propto \frac{1}{(2z' - \tau' - \sigma')^6}\left(\frac{1}{z'^2}\frac{1}{g(\tau')}\frac{1}{g(\sigma')} + \mathcal{O}(z'^4)\right)$$

Same coefficients as for known (twisted sector) partition function! From here on rely on results given in literature.

$\Rightarrow$ Leading and subleading terms in entropy S_{BH} also reproduced microscopically from untwisted sector partition function$\mathsf{Z}^{\text{untw.}}_{(1)}$! ✓

Conclusions, outlook and questions

✓ Physical interpretation of DT invariants on CHL threefold $(K3 \times T^2)/\mathbb{Z}_2$ as quarter-BPS index.
Re-derived conjectural DT formula of [Bryan Oberdieck (2018)] using $\mathcal{N} = 4$ string duality from a heterotic genus two partition function.

✓ Checked physical constraints from charge quantization, S-duality and wall-crossing. Physical explanation for the congruence subgroup $B(2)$.

✓ Checked constraints on asymptotic growth of index by comparing to macroscopic BH entropy.

? Generalize to orbifolds by other symplectomorphism groups G on $K3$?

? ...