BPS indices, Vafa-Witten invariants and quivers

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Workshop on "Black Holes: BPS, BMS and Integrability"
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based on arXiv:2004.14466 with Guillaume Beaujard and Jan Manschot

and earlier work with Sergei Alexandrov and Ashoke Sen

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- While the net number of BPS microstates with fixed charge γ (known as the BPS index $\Omega(\gamma)$) is known exactly in all string backgrounds with $\mathcal{N} \geq 4$ supersymmetry, this is not so in $\mathcal{N} = 2$ string vacua, except for very special charges.
- Part of the reason is that $\Omega(\gamma, t)$ depends on the moduli t in a very intricate way, due to wall-crossing phenomena associated to BPS bound states with arbitrary number of constituents. The moduli space itself receives quantum corrections, unlike in $\mathcal{N} \geq 4$.

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Maldacena Strominger Witten 1998; Gaiotto Strominger Yin 2006; Denef Moore 2007; Manschot 2009; Alexandrov Banerjee Manschot BP 2016-2019

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- On the math side, $\Omega(\gamma, t)$ are generalized Donaldson-Thomas invariants of the Calabi-Yau three-fold X. Morally, the Euler number of the moduli space of stable coherent sheaves on X with Chern character γ. They are subtle to define and hard to compute. The mathematical origin of (mock) modularity is still mysterious.

Introduction

- In this talk, I will consider D4-D2-D0 bound states in type II string compactified on a local (non-compact) Calabi-Yau manifold K_S, the total space of the canonical bundle over a complex Fano surface S. D4-D2-D0 branes supported on S are then described by stable coherent sheaves on S (or derived category thereof).

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- For $[D4] = N[S]$, $\Omega(\gamma, t)$ coincides with the Vafa-Witten invariants of S, computed by topologically twisted $\mathcal{N} = 4$ SYM with gauge group $U(N)$. S-duality implies that generating functions should be (mock) modular.

Vafa Witten 1994; Minahan Nemeschansky Vafa Warner 1998;
Gholampour Sheshmani Yau 2017

Introduction

- For Fano surfaces S, the derived category of coherent sheaves is known to be isomorphic to the derived category of representations of a certain quiver (Q, W). The nodes of the quiver correspond to certain rigid sheaves E_i on S forming an exceptional collection.

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- The BPS index $\Omega(\gamma, t)$ is equal to the Euler number $\Omega(\vec{N}, \vec{\zeta})$ of the moduli space of semi-stable quiver representations with dimension vector $\vec{N}$ and FI parameters $\vec{\zeta}$ determined from (γ, t).

Introduction

- Unless Q has no loops, the BPS index $\Omega(\vec{N},\vec{\zeta})$ is in general difficult to compute. However, quivers coming from exceptional collections on Fano surfaces are special: the 'attractor index'

$$\Omega_*(\vec{N}) = \Omega(\vec{N},\vec{\zeta}_*(\vec{N}))$$

vanishes unless $\vec{N}$ is supported on a single node. Here $\vec{\zeta}_*(\vec{N})$ is the 'attractor' or 'self-stability condition'.

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Beaujard Manschot BP 2020

- The BPS index elsewhere can be computed by performing a sequence of wall-crossings, or more directly by using the flow tree formula, which expresses $\Omega(\vec{N}, \vec{\zeta})$ in terms of $\Omega_*(\vec{N}_i)$ for all decompositions $\vec{N} = \sum_i \vec{N}_i$.

Denef Green Raugas 2001; Alexandrov Pioline 2018

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- The (mock) modular properties of generating functions should have a natural explanation from the quiver description.
- In the rest of this talk, I will explain some background about exceptional collections, toric surfaces, quivers, etc, and demonstrate how the method works in simple examples.

Outline

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D-branes and coherent sheaves

- At large volume, D-branes on a Calabi-Yau threefold X are described by coherent sheaves E on X: morally, a vector bundle whose fiber dimension may jump. A D6-brane is supported on all of X, a D4-brane on a divisor, a D2-brane on a curve and a D0-brane on a point.

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- The D-brane charge can be read off from the Chern character $\mathrm{ch}(E) = [\mathrm{rk}, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3] \in H^{\rm even}(X, \mathbb{Q})$.

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- The spectrum of open strings between D-branes associated to coherent sheaves E, E' is determined from the extension groups $\mathrm{Ext}^k_X(E, E')$. Ext^0_X corresponds to tachyons (projected out when $E = E'$), Ext^1_X to nearly massless states, $\mathrm{Ext}^{k\geq 2}_X$ to massive strings irrelevant at low energy.

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- The Ext groups on X are related to those on S by

$$\mathrm{Ext}^k_X(i_*E, i_*E') = \mathrm{Ext}^k_S(E, E') \oplus \mathrm{Ext}^{3-k}_S(E, E')$$

Thus, light open strings originate both from Ext^1_S and Ext^2_S, while Ext^0_S and Ext^3_S lead to tachyons.

D-branes and coherent sheaves

- The dimension of Ext groups can be inferred from the Euler form

$$\chi(E,E') := \sum_{k\geq 0} (-1)^k \dim \mathrm{Ext}^k_S(E,E')$$

By the Riemann-Roch formula, it depends only on the Chern characters $\gamma(E) = [\mathrm{rk}(E), c_1(E), \mathrm{ch}_2(E)]$,

$$\begin{aligned}\chi(E,E') = & \,\mathrm{rk}(E)\,\mathrm{rk}(E') + \mathrm{rk}(E)\;\mathrm{ch}_2(E') + \mathrm{rk}(E')\;\mathrm{ch}_2(E) \\ & - c_1(E)\cdot c_1(E') + \frac{1}{2}\left[\mathrm{rk}(E)\;\deg(E') - \mathrm{rk}(E')\;\deg(E)\right]\end{aligned}$$

where $\deg(E) = c_1(E)\cdot c_1(S)$.

D-branes and coherent sheaves

- Stable D-branes correspond to Gieseker-stable sheaves on S. The sheaf E is stable if all proper subsheaves E' have

$$\begin{cases} \nu_J(E') < \nu_J(E) \\ \nu_J(E') = \nu_J(E) \quad \text{and} \quad \frac{\mathrm{ch}_2(E')}{\mathrm{rk}(E')} < \frac{\mathrm{ch}_2(E)}{\mathrm{rk}(E)} \end{cases}$$

where $\nu_J(E) = \frac{c_1(E)\cdot J}{\mathrm{rk}(E)}$ is the slope and J the Kähler form.

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where $\nu_J(E) = \frac{c_1(E)\cdot J}{\mathrm{rk}(E)}$ is the slope and J the Kähler form.

- The moduli space of stable sheaves of Chern vector γ has dimension

$$d_{\mathbb{C}}(\mathcal{M}^S_{\gamma,J}) = 1 - \chi(E,E)\ .$$

and is invariant under tensoring with a line bundle $\mathcal{L}$,

$$c_1 \to c_1 + Nc_1(\mathcal{L})\ , \quad \mathrm{ch}_2 \to \mathrm{ch}_2 - Nc_1(\mathcal{L})\cdot c_1 + \frac{1}{2}N\,c_1(\mathcal{L})^2$$

D-branes and coherent sheaves

- An exceptional sheaf is one such that

$$\mathrm{Ext}^0_S(E,E) \simeq \mathbb{C}, \quad \mathrm{Ext}^k_S(E,E) = 0 \quad \forall k > 0$$

Since $\chi(E,E) = 1$ it is necessarily rigid.

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- An exceptional collection is an ordered set $\mathcal{C} = (E_1, \dots, E_r)$ of exceptional sheaves such that

$$\mathrm{Ext}^k_S(E_i,E_j) = 0 \quad \forall k \geq 0, \; 1 \leq j < i \leq r$$

The matrix $S_{ij} = \chi(E_j, E_i)$ is then upper triangular with 1's on the diagonal.

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- A full exceptional collection collection is one such that the Chern characters $\{\mathrm{ch}\, E_i, i=1\dots r\}$ span the lattice $K(S)$. For a simply connected surface S, $r=\chi(S)$.

D-branes and coherent sheaves

- Full exceptional collections satisfying the no tachyon condition

$$\mathrm{Ext}^0_S(E_i, E_j) = \mathrm{Ext}^3_S(E_i, E_j) = 0 \quad \forall i \neq j$$

can be constructed from a strongly cyclic exceptional collection $\mathcal{C}^\vee = (E^1_\vee, \dots, E^r_\vee)$, such that $\chi(E_i, E^j_\vee) = \delta^i{}_j$.

Aspinwall Melnikov 2004; Herzog Karp 2006

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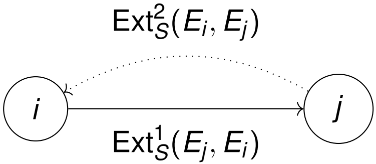
- The dual $E^i_\vee$ can be bone fide coherent sheaves, while E_i necessarily live in the derived category of coherent sheaves.
- Note that $E_i, E^i_\vee$ are denoted $E_i^\vee, E^i$ in our paper !

Exceptional collections and quivers

- To any such collection one associates a quiver Q with nodes $i \in Q_0$ corresponding to E_i. Arrows come from $\mathrm{Ext}^1_S(E_j, E_i)$ (morphisms $\Phi_{ij\alpha}$) and $\mathrm{Ext}^2_S(E_j, E_i)$ (constraints $C_{ij\alpha}$)



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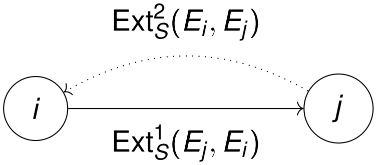

- The constraints can be implemented by introducing morphisms $\Phi_{ij\alpha}$ for $\mathrm{Ext}^2_S(E_j, E_i)$ such that $C_{ij\alpha} = \partial W / \partial \Phi_{ji\alpha} = 0$, where W is a gauge-invariant superpotential.

Coherent sheaves and quiver representations

- The net number of arrows is then

$$\kappa_{ij} = S_{ji} - S_{ij} = \langle E_i, E_j \rangle$$

where

$$\begin{aligned} \langle E, E' \rangle &= \chi(E, E') - \chi(E', E) \\ &= \mathrm{rk}(E)\deg(E') - \mathrm{rk}(E')\deg(E) \end{aligned}$$

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- Different exceptional collections lead to different quivers, typically related by Seiberg duality.

Herzog 2004

Coherent sheaves and quiver representations

- By the Baer-Bondal-Rickard theorem, given a (full,cyclic, strong) exceptional collection on S, the derived category of coherent sheaves $\mathcal{D}(S)$ is isomorphic to the derived category of quiver representations $\mathcal{D}(Q)$:

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- $\mathcal{D}(S)$ is graded by the Chern vector $\mathrm{ch}(E) \in K(S)$ while $\mathcal{D}(Q)$ is graded by the dimension vector $\vec{N} \in \mathbb{Z}^{Q_0}$. The two are related by

$$ch(E) = -\sum_i N_i \, \mathrm{ch}(E_i^\vee)$$

with overall minus sign such that $N_i > 0$ for large D0-brane charge.

Coherent sheaves and quiver representations

- The Gieseker stability condition on $\mathcal{D}(S)$ translates into a stability condition $\vec{\zeta}$ on Q,

$$\zeta_i = \lambda \, \mathrm{Im}(Z_{\gamma_i} \overline{Z_\gamma}) \ , \quad \lambda \in \mathbb{R}^+$$

where $Z_\gamma = -\frac{N}{2} J^2 + J \cdot c_1 - \mathrm{ch}_2$ is the central charge in the large volume limit.

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- This automatically satisfies $\sum_i N_i \zeta_i = 0$, and yields, for subrepresentations with dimension vector $\vec{N}' \leq \vec{N}$,

$$\sum_i N'_i \zeta_i = \rho \left[N \int_S J \cdot c_1(E') - N' \int_S J \cdot c_1(E) \right] + N'\, \mathrm{ch}_2(E) - N\, \mathrm{ch}_2(E')$$

where $\rho \gg 1$. The first term is the standard difference of slopes.

Coherent sheaves and quiver representations

- Under the assignment $(\mathrm{ch}\, E, J) \to (\vec{N}, \vec{\zeta})$, the moduli spaces of semi-stable objects are expected to be isomorphic. In particular, their dimension should match:

$$
\begin{aligned}
d_{\mathbb{C}}(\mathcal{M}^S_{\gamma,J}) =& 1 - \chi(E,E) = 1 - \sum_{i,j} N_i\, S_{ij}\, N_j \\
=& \sum_{S_{ij}<0} |S_{ij}| N_i N_j - \sum_{S_{ij}>0} S_{ij}\, N_i N_j - \sum_i N_i^2 + 1
\end{aligned}
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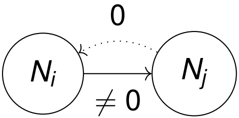
This matches the expected dimension of the quiver moduli space $\mathcal{M}^{Q}_{\vec{N},\vec{\zeta}}$ in the Beilinson branch where $\Phi_{ij\alpha} = 0$ whenever $S_{ij} > 0$.



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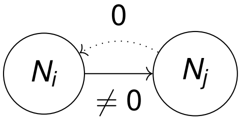

- The Beilinson branch is consistent with $\vec{\zeta}$ only when the slope $\nu_J(E)$ lies in a certain window.

DT invariants, VW invariants and modularity

- The DT invariants counting semi-stable coherent sheaves on S are then equal to the DT invariants counting semi-stable representations of (Q, W). When $J \cdot c_1(S) > 0$, by virtue of vanishing theorems they coincide with VW invariants.

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- The refined DT/VW invariants are given by the Poincaré polynomial of the moduli space $\mathcal{M} = \mathcal{M}^S_{\gamma,J} = \mathcal{M}^Q_{\vec{N},\vec{\zeta}}$ (for intersection homology)

$$\Omega(\vec{N}, \vec{\zeta}, y) = \sum_{p=0}^{d_{\mathbb{C}}(\mathcal{M})} (-y)^{2p-d_{\mathbb{C}}(\mathcal{M})} b_p(\mathcal{M})$$

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- The 'rational DT invariants' have simpler behavior under wall-crossing,

$$\bar{\Omega}(\vec{N},\vec{\zeta},y) = \sum_{m|\vec{N}} \frac{y-1/y}{m(y^m-1/y^m)}\,\Omega(\vec{N}/m,\vec{\zeta},y^m),$$

DT invariants and VW invariants

- In a sector with fixed ('t Hooft flux) c_1, the partition function

$$h^S_{N,c_1,J}(\tau,y) = \sum_n \frac{\bar{\Omega}([N, c_1, \frac12 c_1^2 - n], J, y)}{y - y^{-1}}\, q^{n - \frac{N-1}{2N}c_1^2 - \frac{N\chi(S)}{24}}$$

is expected to transform as a vector-valued Jacobi form of weight $-\frac12 b_2(S)$ and index $-\frac16 K_S^2(N^3 - N)$.

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  is expected to transform as a vector-valued Jacobi form of weight $-\frac12 b_2(S)$ and index $-\frac16 K_S^2 (N^3 - N)$.
- When $b_2^+(S) = 1$, additional non-holomorphic contributions from reducible connections at the boundary of moduli space $\mathcal{M}^S_{\gamma,J}$ are needed to restore modularity. In general $h^S_{N,c_1,J}(\tau,y)$ is a vector-valued *mock* Jacobi form of depth $N-1$, subject to wall-crossing in J.

Vafa Witten 1994; Alexandrov Manschot BP 2019; Dabholkar Putrov Witten 2020

DT invariants and VW invariants

- For $N=1$, there are no non-holomorphic contributions, nor any dependence on J, and h_1 is truly modular,

$$h_1^S(\tau,y) = \frac{\mathrm{i}}{\theta_1(\tau,y^2)\,\eta(\tau)^{b_2(S)-1}}$$

Göttsche 1990

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- The partition function $h^S_{N,c_1,J}$ has simple transformations under blow up and wall-crossing. This can be used to compute it in principle for any rational surface.

Yoshioka 1994; Göttsche 1998; Manschot 2010-2016

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- The partition function $h^S_{N,c_1,J}$ has simple transformations under blow up and wall-crossing. This can be used to compute it in principle for any rational surface.

Yoshioka 1994; Göttsche 1998; Manschot 2010-2016

- Mock modular properties and holomorphic anomalies allow to computing the generating function of VW invariants for any del Pezzo surfaces at arbitrary rank directly.

Alexandrov 2020 (see previous talk)

DT invariants and VW invariants

- For $N = 1$, there are no non-holomorphic contributions, nor any dependence on J, and h_1 is truly modular,

$$h_1^S(\tau, y) = \frac{\mathrm{i}}{\theta_1(\tau, y^2)\,\eta(\tau)^{b_2(S)-1}}$$

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Alexandrov 2020 (see previous talk)

- I shall demonstrate that quivers provide an alternative way of computing these invariants. But first, some more background on wall-crossing and attractor indices is needed.

Outline

Wall-crossing and attractor indices

- The DT invariants $\bar{\Omega}(\vec{N}, \vec{\zeta}, y)$ jump on hyperplanes where stable representations become semi-stable. The discontinuity is given by the Konsevitch-Soibelman wall-crossing formula.

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Denef Moore 2007; Andriyash et al 2010

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- The KS formula can be derived using localisation in the black hole supersymmetric quantum mechanics. Rational invariants $\bar{\Omega}(\gamma, t)$ arise as effective indices for particles with Boltzmann statistics.

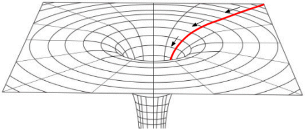
Manschot BP Sen 2010

Wall-crossing and attractor indices

- For fixed $\vec{N}$, there is a particular stability condition

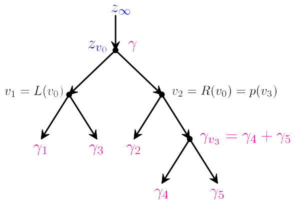
$$\zeta_i^\star(\vec{N}) = -\kappa_{ij} N^j$$

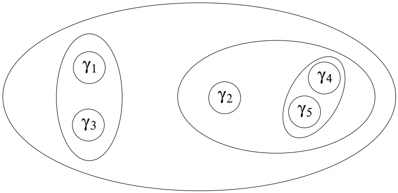
known as 'attractor point' or 'self-stability' where bound states are ruled out. This is analogous to the attractor point for spherically symmetric black holes in $\mathcal{N}=2$ supergravity.



Wall-crossing and attractor indices

- The full spectrum can be constructed as bound states of these attractor BPS states, labelled by attractor flow trees:





Denef '00; Denef Green Raugas '01; Denef Moore'07

Wall-crossing and attractor indices

- The 'flow tree formula' allows to express $\bar{\Omega}(\vec{N},\vec{\zeta},y)$ in terms of the attractor indices $\bar{\Omega}^\star(\vec{N}_i,y) := \bar{\Omega}(\vec{N}_i,\vec{\zeta}^\star(\vec{N}_i),y)$:

$$\bar{\Omega}(\vec{N},\vec{\zeta},y) = \sum_{\vec{N}=\sum_{i=1}^n \vec{N}_i} \frac{g_{\rm tr}(\{\vec{N}_i,\vec{\zeta}_i\},y)}{|{\rm Aut}\{\vec{N}_i\}|} \prod_{i=1}^n \bar{\Omega}_*(\vec{N}_i,y,t)$$

  where $g_{\rm tr}$ is a sum over all possible stable flow trees ending on the leaves $\gamma_1,\dots,\gamma_n$.

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  where $g_{\rm tr}$ is a sum over all possible stable flow trees ending on the leaves $\gamma_1,\dots,\gamma_n$.
- The flow tree formula is purely combinatoric, and does not require integrating the attractor flow !

Alexandrov BP 2018

Wall-crossing and attractor indices

- Remarkably, attractor indices for quivers coming from Fano surfaces have a special property:

$\Omega_\star(\vec{N}, y) = 0$ unless $\vec{N}$ is supported on a single node with height 1 (in which case $\Omega_\star = 1$) or $\vec{N} \propto \vec{N}_{D0}$ (for a pure D0-brane)

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- To see this, we exhibit a positive quadratic form $\mathcal{Q}(\vec{N})$ and rational coefficients $\lambda_i \in \mathbb{Q}$ such that the expected dimension of the moduli space $\mathcal{M}^Q_{\vec{N},\vec{\zeta}^\star(\vec{N})}$ in the attractor chamber can be written as

$$d^*_{\mathbb{C}} = 1 - \mathcal{Q}(\vec{N}) - \sum_i \lambda_i N_i \zeta_i^\star$$

  where $\lambda_i = 0$ or $\mathrm{sgn}(\lambda_i) = \mathrm{sgn}(\zeta_i^\star)$ for all i. The quadratic form is degenerate along $\vec{N}_{D0}$. $\mathcal{Q}(\vec{N})$ is found case-by-case.

Beaujard Manschot BP 2020

Wall-crossing and attractor indices

- Since $\langle \vec{N}_{D0}, \vec{N} \rangle = 0$ for any $\vec{N}$, the flow tree formula does not involve the unknown indices $\Omega_\star(p\vec{N}_{D0})$. Thus it can be used to compute $\bar{\Omega}(\vec{N}, \vec{\zeta}, y)$ for any $(\vec{N}, \vec{\zeta})$!

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- This sounds puzzling at first: multi-centered black hole are not supposed to appear at the large volume attractor point, but apparently the BPS spectrum at this point can still be interpreted as multi-particle bound states in the quiver quantum mechanics !
- Presumably this micro-structure is revealed as one travels from large volume to the genuine (finite volume) attractor point.

Attractor indices and pure Higgs indices

- While there are no genuine bound states at the attractor point $\vec{\zeta} = \vec{\zeta}^{\star}(\vec{N})$, from the Coulomb branch prospective there can still be contributions from 'scaling solutions', where several centers approach at arbitrary small distance.

Bena Wang Warner 2007; de Boer El-Showk Messamah Den Bleeken 2008

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- The Coulomb branch formula gives a (conjectural) general prescription for removing these scaling contributions. It expresses $\bar{\Omega}(\vec{N}, \vec{\zeta}, y)$ in terms of 'single-centered' or 'pure-Higgs' indices :

$$\bar{\Omega}(\vec{N}, \vec{\zeta}, y) = \sum_{\vec{N} = \sum_{i=1}^{n} \vec{N}_i} \frac{g_{\rm tr}(\{\vec{N}_i, \vec{\zeta}_i\}, y)}{|{\rm Aut}\{\vec{N}_i\}|} \prod_{i=1}^{n} \bar{\Omega}_S(\vec{N}_i, y, t)$$

Denef Moore 2007, Manschot BP Sen 2011, Lee Yang Yi 2012

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- In particular, $\Omega_S(\vec{N}, y) = \Omega_\star(\vec{N}, y)$ unless $\vec{N} \propto \vec{N}_{D0}$. This is surprising since scaling solutions do exist classically. However, they are removed by quantum effects, under the 'minimal modification hypothesis'.

Outline

Example 1: Local $\mathbb{P}^2$

- The projective plane admits a strong cyclic exceptional collection

$$\mathcal{C}_\vee = (\mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(2))$$

$$\begin{array}{l} \gamma_\vee^1 = [1,0,0] \\ \gamma_\vee^2 = [1,1,\frac{1}{2}] \\ \gamma_\vee^3 = [1,2,2] \end{array} \qquad S_\vee = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} ,$$

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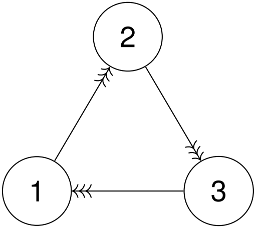
- The dual collection is (with $\Omega(1)$ the twisted cotangent bundle)

$$\mathcal{C} = (\mathcal{O}, \Omega(1)[1], \mathcal{O}(-1)[2])$$

$$\begin{array}{l} \gamma_1 = [1,0,0] \\ \gamma_2 = [-2,1,\frac12] \\ \gamma_3 = [1,-1,\frac12] \end{array} \qquad S = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 1: Local $\mathbb{P}^2$

- This leads to the familiar quiver for $\mathbb{C}^3/\mathbb{Z}_3$,



$$W = \sum_{(ijk)\in S_3} \mathrm{sgn}(ijk)\, \Phi^i_{12}\Phi^j_{23}\Phi^k_{31}$$

Douglas Fiol Romelsberger 2000

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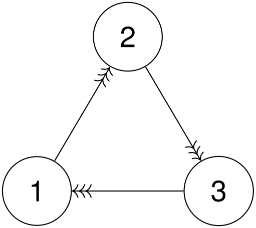

$$W = \sum_{(ijk)\in S_3} \mathrm{sgn}(ijk)\, \Phi^i_{12}\Phi^j_{23}\Phi^k_{31}$$

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- The dimension vectors are given in terms of $\mathrm{ch} = [N, c_1, \mathrm{ch}_2]$ by

$$\vec{N} = -\left(\tfrac{3}{2}c_1 + \mathrm{ch}_2 + N, \tfrac{1}{2}c_1 + \mathrm{ch}_2, -\tfrac{1}{2}c_1 + \mathrm{ch}_2\right)$$

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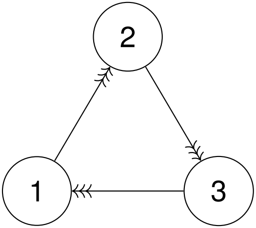

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- When $N_1 = 0$ or $N_3 = 0$, the 3-node quiver reduces to the Kronecker quiver K_3.

Example 1: Local $\mathbb{P}^2$

- The stability vector is

$$\vec{\zeta} = 3\rho\,(N_2 - N_3, N_3 - N_1, N_1 - N_2) + (-\tfrac{N_2+N_3}{2}, \tfrac{N_1+3N_3}{2}, \tfrac{N_1-3N_2}{2})$$
$$= -\rho\vec{\zeta}^{\star} + \mathcal{O}(1)$$

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$$= -\rho\vec{\zeta^\star} + \mathcal{O}(1)$$

- In the Beilinson chamber $\Phi_{31\alpha} = 0$, the expected dimensions of $\mathcal{M}^Q$ and $\mathcal{M}^S$ agree,

$$d_{\mathbb{C}} = 3(N_1N_2 + N_2N_3 - N_3N_1) - N_1^2 - N_2^2 - N_3^2 + 1 = c_1^2 - 2N\,\mathrm{ch}_2 - N^2 + 1$$

This requires $\zeta_1 \geq 0, \zeta_3 \leq 0$ hence $-N \leq c_1 \leq 0$.

Example 1: Local $\mathbb{P}^2$

- In the attractor chamber $\vec{\zeta} = \rho\vec{\zeta^\star}$, the expected dimension can be written as

$$d^*_{\mathbb{C}} = 1 - \mathcal{Q}(\vec{N}) + \begin{cases} \frac{2}{3}N_3\zeta^\star_3 - \frac{2}{3}N_1\zeta^\star_1 & \zeta^\star_1 \geq 0, \zeta^\star_3 \leq 0 \\ \frac{2}{3}N_1\zeta^\star_1 - \frac{2}{3}N_2\zeta^\star_2 & \zeta^\star_2 \geq 0, \zeta^\star_1 \leq 0 \\ \frac{2}{3}N_2\zeta^\star_2 - \frac{2}{3}N_3\zeta^\star_3 & \zeta^\star_3 \geq 0, \zeta^\star_2 \leq 0 \end{cases}$$

$$\mathcal{Q}(\vec{N}) = \frac{1}{2}(N_1 - N_2)^2 + \frac{1}{2}(N_2 - N_3)^2 + \frac{1}{2}(N_3 - N_1)^2$$

hence $d^*_{\mathbb{C}} < 0$ unless $\vec{N} \in \{(1,0,0),(0,1,0),(0,0,1),(p,p,p)\}$. Hence $\Omega_\star(\vec{N}) = 0$ except in those cases.

- Using the flow tree formula with $\Omega_\star = 0$, or the Coulomb branch formula with $\Omega_{\rm S} = 0$, we get expected results:

$[N; c_1; c_2]$	$\vec N$	$\Omega_c(\vec N)$
$[1;0;2]$	$(1,2,2)$	$y^4+2y^2+3+\dots$
$[1;0;3]$	$(2,3,3)$	$y^6+2y^4+5y^2+6+\dots$
$[2;0;3]$	$(1,3,3)$	$-y^9-2y^7-4y^5-6y^3-6y-\dots$
$[2;-1;2]$	$(1,2,1)$	$y^4+2y^2+3+\dots$
$[2;-1;3]$	$(2,3,2)$	$y^8+2y^6+6y^4+9y^2+12+\dots$
$[3;-1;3]$	$(1,3,2)$	$y^8+2y^6+5y^4+8y^2+10+\dots$
$[4;-2;4]$	$(1,3,1)$	$y^5+y^3+y+\dots$
$[4;-2;5]$	$(2,4,2)$	$-y^{13}-2y^{11}-6y^9-10y^7-17y^5-21y^3-24y-.$

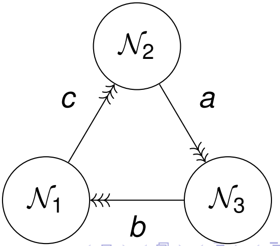
Example 2: Three-block collections I

- For $\mathbb{F}_0$ and all del Pezzo surfaces dP_k with $k \neq 1, 2$, Karpov and Nogin have constructed strong cyclic exceptional collections with three-blocks structure with $\alpha + \beta + \gamma = \chi(S)$

$$S = \left(\begin{array}{c|c|c} 1_\alpha & -c & b \\ \hline & 1_\beta & -a \\ \hline & & 1_\gamma \end{array}\right), \quad \kappa = \left(\begin{array}{c|c|c} 0_\alpha & c & -b \\ -c & 0_\beta & a \\ b & -a & 0_\gamma \end{array}\right),$$

where $\alpha x^2 + \beta y^2 + \gamma z^2 = xyz\sqrt{K_S^2\,\alpha\beta\gamma}$

$$\begin{aligned} a &= \alpha x\, K' \\ b &= \beta y\, K' \\ c &= \gamma z\, K' \\ K' &= \sqrt{K_S^2/(\alpha\beta\gamma)} \end{aligned}$$



Example 2: Three-block collections II

- In the Beilinson chamber where $\Phi_{31,\alpha} = 0$, the expected dimension of $\mathcal{M}^Q$ agrees with that of $\mathcal{M}^S$,

$$d_{\mathbb{C}} = c\,\mathcal{N}_1\mathcal{N}_2 + a\mathcal{N}_2\mathcal{N}_3 - b\mathcal{N}_1\mathcal{N}_3 - \sum_i N_i^2 + 1$$

$$\mathcal{N}_1 = \sum_{i=1}^{\alpha} N_i\,, \quad \mathcal{N}_2 = \sum_{i=\alpha+1}^{\alpha+\beta} N_i\,, \quad \mathcal{N}_3 = \sum_{i=\alpha+\beta+1}^{\alpha+\beta+\gamma} N_i$$

Example 2: Three-block collections III

- In the attractor chamber, one has instead

$$d^*_{\mathbb{C}} = 1 - \mathcal{Q}(\vec{N}) + \frac{2\mathcal{A}}{\mathcal{A}+\mathcal{B}+\mathcal{C}} \mathcal{N}_3 \varsigma_3^\star - \frac{2\mathcal{C}}{\mathcal{A}+\mathcal{B}+\mathcal{C}} \mathcal{N}_1 \varsigma_1^\star$$

  when $\varsigma_3^\star \leq 0, \varsigma_1^\star \geq 0$, or cyclic permutation thereof

- $\mathcal{Q}(\vec{N})$ is the positive quadratic form, degenerate along the direction $\vec{N}_{D0} = (x, \ldots; y, \ldots; z, \ldots)$

$$\sum_{i=1}^{r} N_i^2 - \frac{\mathcal{A}+\mathcal{B}-\mathcal{C}}{\mathcal{A}+\mathcal{B}+\mathcal{C}} c \mathcal{N}_1 \mathcal{N}_2 - \frac{\mathcal{B}+\mathcal{C}-\mathcal{A}}{\mathcal{A}+\mathcal{B}+\mathcal{C}} a \mathcal{N}_2 \mathcal{N}_3 - \frac{\mathcal{C}+\mathcal{A}-\mathcal{B}}{\mathcal{A}+\mathcal{B}+\mathcal{C}} b \mathcal{N}_3 \mathcal{N}_1$$

  Hence $\Omega_\star(\vec{N}) = 0$ except for simple representations or for D0-branes. Using flow tree formula we get agreement in other chambers with prediction from blow-up and wall-crossing formulae.

Example 3: Local toric surfaces

- Smooth toric surfaces are described by a toric fan spanned by vectors $v_1, \ldots, v_r \in \mathbb{Z}^2$ forming a convex polygon. Each vector corresponds to a toric divisor D_i, subject to linear equivalences

$$\sum_i (u, v_i)\, D_i = 0$$

The intersection $D_i \cdot D_j$ vanishes unless $i - j \in \{-1, 0, 1\} \pmod{r}$, and $D_i \cdot D_{i+1} = 1$, $D_i \cdot D_i = a_i$ where a_i are determined by

$$v_{i-1} + v_{i+1} + a_i\, v_i = 0 \; .$$

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$$v_{i-1} + v_{i+1} + a_i\, v_i = 0\ .$$

- Fano surfaces have $a_i \geq -1$ for all i, weak Fano surfaces have $a_i \geq -2$. There are 5 smooth toric Fano surfaces, and 11 weak Fano, related by blow-up/down.

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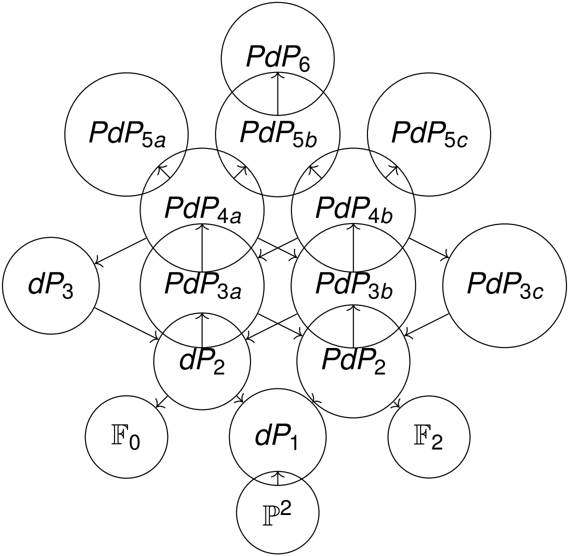

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- Toric Fano surfaces admit strongly cyclic exceptional collections.

$$\mathcal{O}(0), \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \dots, \mathcal{O}(D_1 + \dots + D_{r-1})$$

For weak Fano surfaces, this is not strongly exceptional but there is an alternative choice $D_i \to \tilde{D}_i$.

Hille Perling 2011

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- Alternatively, one may read off the quiver along with its superpotential from the brane tiling. The various branches are in one-to-one correspondance with the internal perfect matchings.

  Franco Hanany Kennaway Vegh Wecht 2005; Hanany Herzog Vegh 2006

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- In all these examples, the BPS indices computed using the attractor flow formula are in agreement with the result form the blow-up and wall-crossing formulae.

Outline

Summary and Outlook

- VW invariants of Fano surfaces S, or BPS indices counting D4-D2-D0 bound states on K_S, can be computed algorithmically at arbitrary rank through the flow tree formula.

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- VW invariants of Fano surfaces S, or BPS indices counting D4-D2-D0 bound states on K_S, can be computed algorithmically at arbitrary rank through the flow tree formula.
- Presumably this method should extend to any rational or ruled surface. How about K3 surfaces or surfaces of general type ? In general, VW invariants will also include contributions from the monopole branch, can this be described in the language of quivers ?

Summary and Outlook

- VW invariants of Fano surfaces S, or BPS indices counting D4-D2-D0 bound states on K_S, can be computed algorithmically at arbitrary rank through the flow tree formula.
- Presumably this method should extend to any rational or ruled surface. How about K3 surfaces or surfaces of general type ? In general, VW invariants will also include contributions from the monopole branch, can this be described in the language of quivers ?
- For $S = \mathbb{P}^2$, BPS indices can also be computed using scattering diagrams. Are those equivalent to the attractor flow trees ?

Gross Pandharipande Siebert 2010; Bridgeland 2017; Bousseau (2019)

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- It would be interesting to compute BPS indices in compact CY threefolds, where non-trivial single-centered black holes are expected to occur !

Thank you for your attention, and mind the wall !