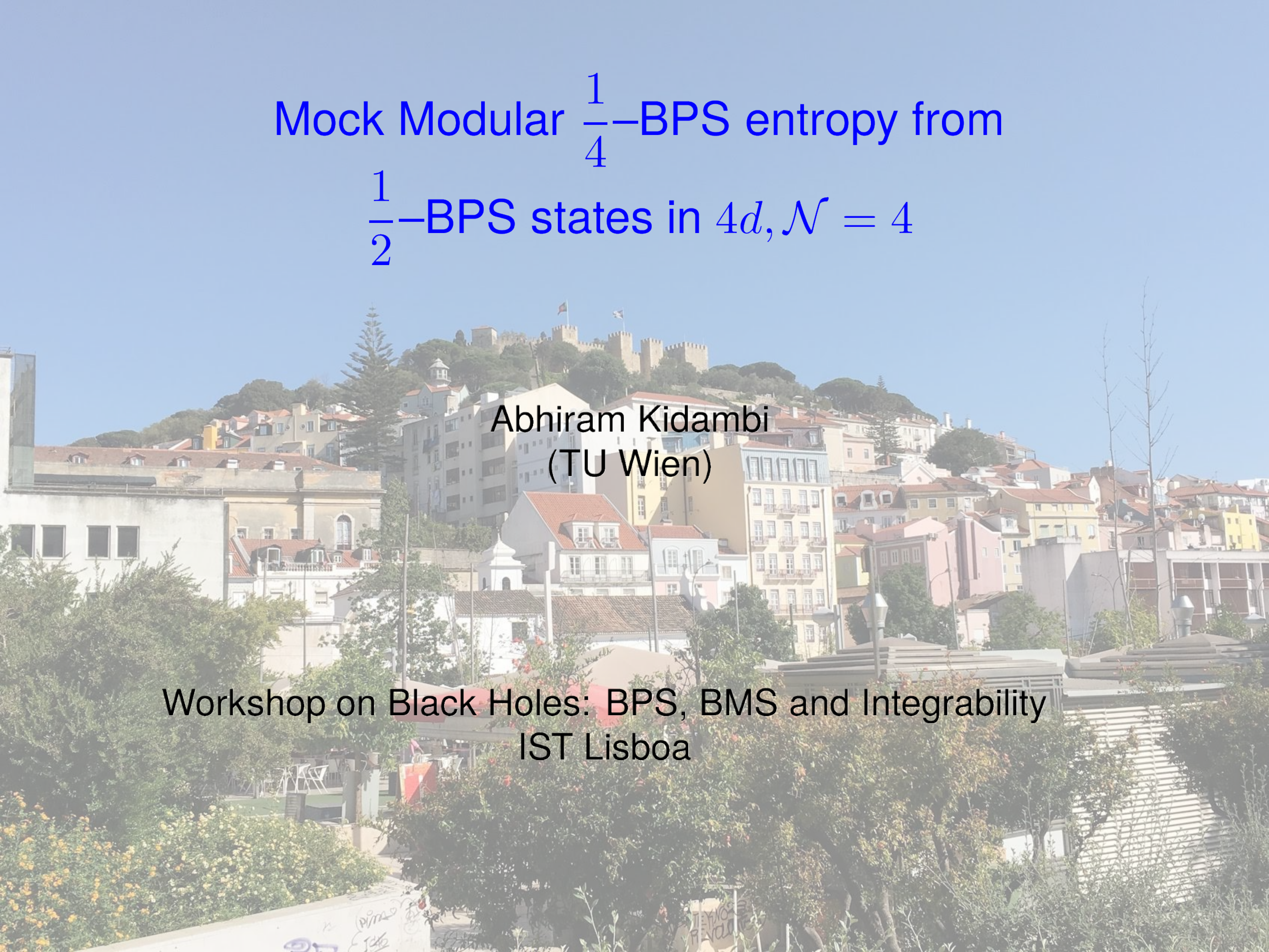
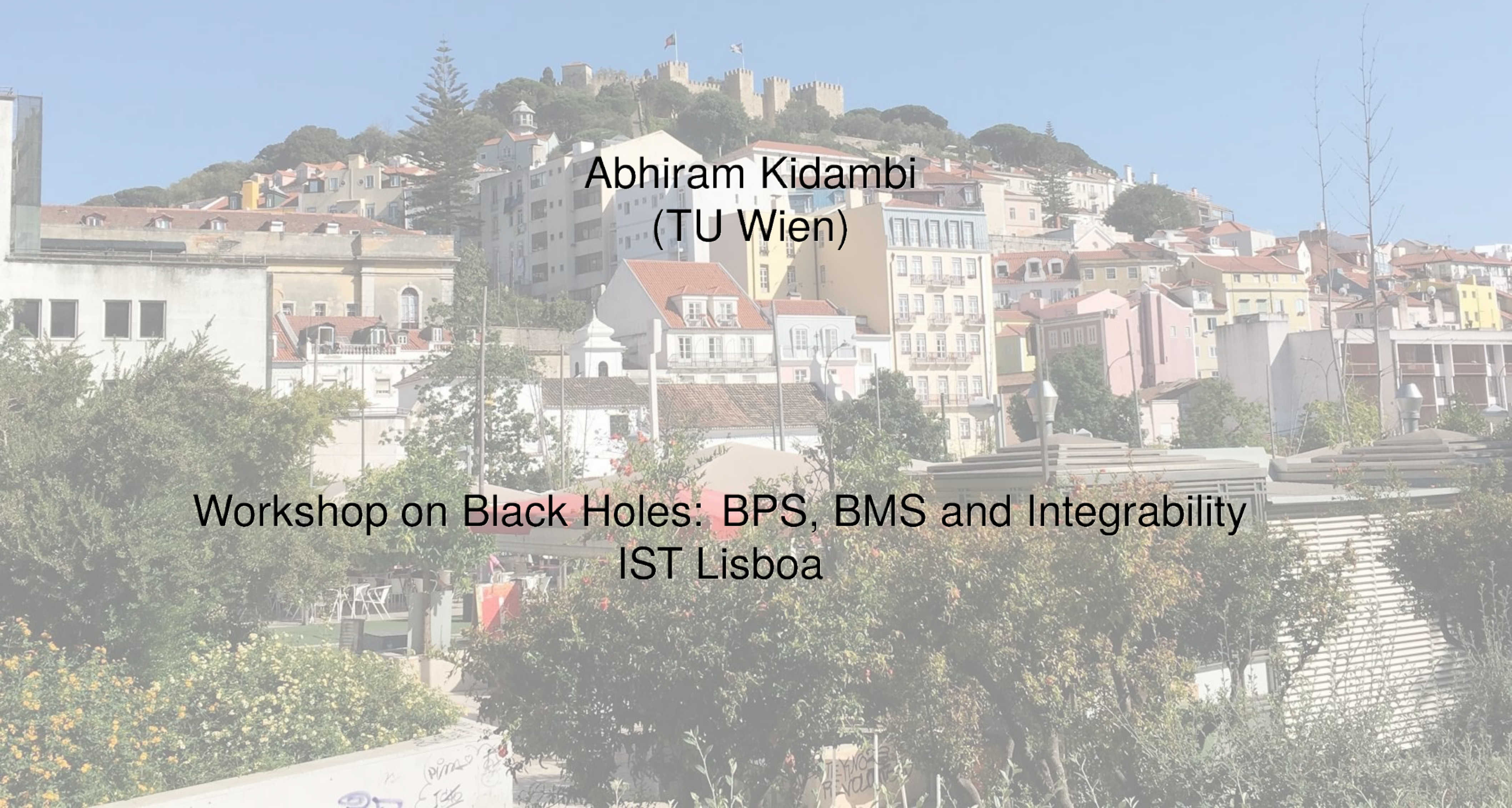
Mock Modular $\frac{1}{4}$–BPS entropy from $\frac{1}{2}$–BPS states in $4d, \mathcal{N} = 4$



Abhiram Kidambi
(TU Wien)

Workshop on Black Holes: BPS, BMS and Integrability
IST Lisboa

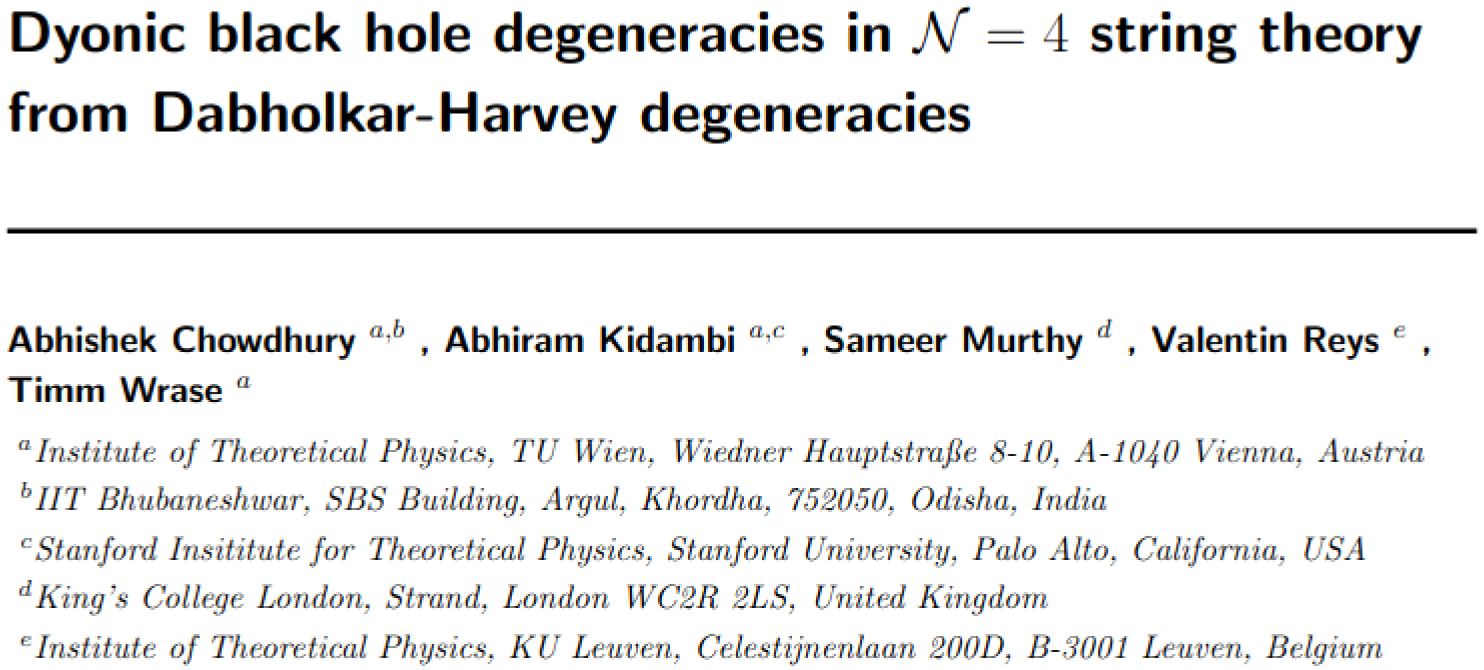


Re-re-recounting all dyons that don't die, for hopefully one last time

Abhiram Kidambi
(TU Wien)

Workshop on Black Holes: BPS, BMS and Integrability
IST Lisboa

Based on 1912.06562



Dyonic black hole degeneracies in $\mathcal{N} = 4$ string theory from Dabholkar-Harvey degeneracies

Abhishek Chowdhury [a,b] , **Abhiram Kidambi** [a,c] , **Sameer Murthy** [d] , **Valentin Reys** [e] ,
Timm Wrase [a]

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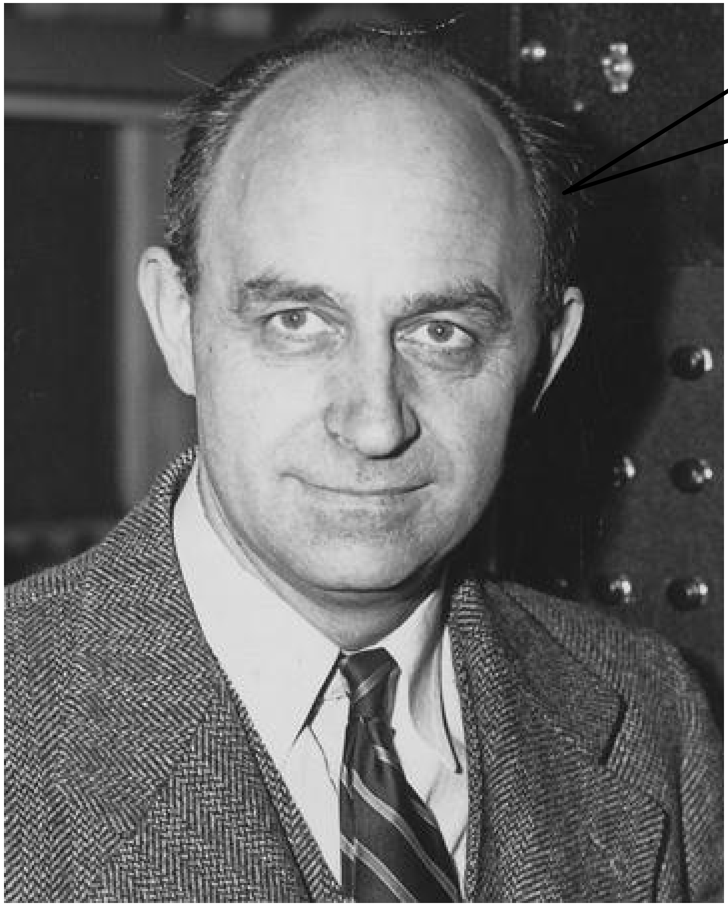
[b] *IIT Bhubaneshwar, SBS Building, Argul, Khordha, 752050, Odisha, India*
[c] *Stanford Insititute for Theoretical Physics, Stanford University, Palo Alto, California, USA*
[d] *King's College London, Strand, London WC2R 2LS, United Kingdom*
[e] *Institute of Theoretical Physics, KU Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium*



Never underestimate the joy that people derive from hearing something they already know.

Idea of talk

String compactification: Type II on $K3 \times T^2 \equiv$ Het on T^6

Dyonic BH solutions:	$\frac{1}{2}$–BPS	$\frac{1}{4}$–BPS
Area:	Zero area	Large BH
Partition function:	$\frac{1}{\eta(\tau)^{24}}$ *[Dabholkar, Harvey]*	$\frac{1}{\Phi_{10}(\tau,\sigma,z)}$ *[Dijkgraaf, Verlinde, Verlinde]*

$\frac{1}{\Phi_{10}(\tau,\sigma,z)}$ counts *all* $\frac{1}{4}$–BPS states $\xrightarrow{\text{Remove Wall Crossing effects}}$ Mock modular Jacobi forms that count single center states

[Dabholkar, Murthy, Zagier]

Idea of talk

String compactification: Type II on $K3 \times T^2 \equiv$ Het on T^6

Dyonic BH solutions:	$\frac{1}{2}$–BPS	$\frac{1}{4}$–BPS
Area:	Zero area	Large BH
Partition function:	$\frac{1}{\eta(\tau)^{24}}$ *[Dabholkar, Harvey]*	$\frac{1}{\Phi_{10}(\tau,\sigma,z)}$ *[Dijkgraaf, Verlinde, Verlinde]*

Fourier expansion coefficients of $\frac{1}{\eta(\tau)^{24}}$ $\xrightarrow{\text{Exact formula + tecnhiques}^*}$ Mock modular Jacobi forms that count single center states

[CMKRW]

$\ast =$ Black hole metamorphosis, Rademacher expansion for MJF's

[Sen; Sen et.al; , Denef, Moore, Andriyash, Jafferis]

[Murthy, Bringmann; Reys, Ferrari]

Automorphic forms and BH's

Key Mantra 1: BPS black hole "partition functions" are automorphic forms that can be computed from the SCFT.

Ex: $\frac{1}{2}$–BPS states $$\frac{1}{\eta(\tau)^{24}} = q^{-1} \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^{24}}, \quad q = e^{2\pi i\tau}$$

Key Mantra 2: Black hole degeneracies are extracted charge wise from the Fourier expansion coefficients of the automorphic forms.

$$\frac{1}{\eta(\tau)^{24}} = \sum_{n\geq -1} c(n) q^n$$

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Degeneracy of $\frac{1}{2}$–BPS BH's with electric or magnetic T–dual invariant n

Key Mantra 3: Need only know the degeneracies of "*polar terms*" to compute all BPS degeneracies (Rademacher expansion)

Mock modular dyonic BH entropy

Dyonic black holes with electric charge Q and magnetic charge P.

$\frac{1}{4}$–BPS states counted by inverse of Igusa cusp form $\frac{1}{\Phi_{10}(\tau,\sigma,z)}$

$$\Delta = 4mn - \ell^2 = P^2Q^2 - (P\cdot Q)^2$$

$$\frac{1}{\Phi_{10}(\tau,\sigma,z)} = (qyp)^{-1} \prod_{(n,m,\ell)>0} (1-q^n p^m y^\ell)^{-2C_{K3}(\Delta)}$$

$$q = e^{2\pi i\tau}, \; y = e^{2\pi i z}, \; p = e^{2\pi i\sigma}$$

$$(n,m,\ell) > 0 \Rightarrow \begin{cases} \ell \in \mathbb{Z}_-, & \text{if } m = n = 0 \\ \ell \in \mathbb{Z}, & \text{if } m \geq 0, n > 0 \end{cases}$$

$$\mathbf{EG}_{K3}(\tau,z) = 2\phi_{0,1}(\tau,z) = 8\sum_{i=2}^{4}\left(\frac{\vartheta_i(\tau,z)^2}{\vartheta_i(\tau,0)^2}\right) \xrightarrow{\text{Fourier Exp.}} \mathbf{EG}_{K3}(\tau,z) = \sum_{n,\ell} c(n,\ell)\, q^n y^\ell, \;\; c(n,\ell) \equiv C_{K3}(\Delta)$$

$$\Delta = 4n - \ell^2$$

[DVV]

Remark: Discrimininat $\Delta = 4mn - \ell^2$ is U–duality invariant quantity built out of T–dual invariants.

Mock modular dyonic BH entropy

Fourier expansion: $$\frac{1}{\Phi_{10}(\tau,\sigma,z)} = \sum_{n,m,\ell} c(n,m,\ell)\, q^n p^m y^\ell$$

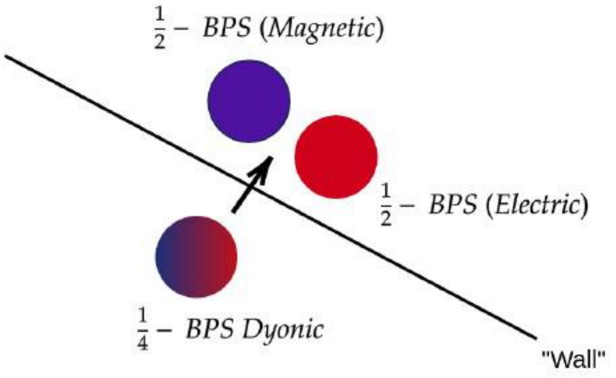
$$\Updownarrow$$

Degeneracy of $\frac{1}{4}$–BPS state $(Q^2/2, P^2, P\cdot Q) = (n,m,\ell)$ → $$d(n,m,\ell) = (-1)^{\ell+1}\int_{\mathcal{C}} d\tau\, dz\, d\sigma\, \frac{e^{-i\pi(\tau n+2\ell z+\sigma m)}}{\Phi_{10}(\tau,\sigma,z)}$$

$$\frac{1}{\Phi_{10}(\tau,\sigma,z)}$$ Double pole at $z=0$ on Siegel UHP and $Sp(2,\mathbb{Z})$ images

Integral picks up residue at pole if $\mathcal{C}$ encompasses it [Cheng, Verlinde]

Discontinuous jump in the index of $\frac{1}{4}$–BPS states



[Denef]

Mock modular dyonic BH entropy

Fourier–Jacobi expansion: $$\frac{1}{\Phi_{10}(\tau,\sigma,z)} = \sum_{m\geq -1} \psi_m(\tau,z)\, p^m$$

Meromorphic! ← $\psi_m(\tau,z)$ is a JF of weight -10 and index m

Remove the residue contribution to the integral → Single center $\frac{1}{4}$–BPS degeneracies

[Dabholkar, Murthy, Zagier]

Averaging function: $$\mathrm{Av}^{(m)}\left[\frac{y}{(y-1)^2}\right] = \sum_{\lambda} q^{m\lambda^2} y^{2m\lambda} \frac{q^\lambda y}{(y-1)^2}$$

Appell–Lerch sum: $$\mathcal{A}_{2,m}(\tau,z) = \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s} y^{2ms+1}}{(1-q^s y)^2}$$

[DMZ]

Polar piece: $$\psi_m^P(\tau,z) = \frac{p_{24}(m+1)}{\eta(\tau)^{24}} \mathcal{A}_{2,m}(\tau,z)$$

Coeff of q^m in Fourier exp of $\frac{1}{\eta(\tau)^{24}}$

Single center $\frac{1}{4}$–BPS PF: $$\psi_m^F(\tau,z) = \psi_m(\tau,z) - \psi_m^P(\tau,z)$$

Mock modular dyonic BH entropy

$\psi_m^F(\tau, z)$ is a holomorphic, mock–modular Jacobi form.

Fourier Expansion: $\psi_m^F(\tau, z) = \sum_{n,\ell} c_m^F(n, \ell)\, q^n y^\ell$

Mock modular dyonic BH entropy

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Fourier Expansion: $$\psi_m^F(\tau,z) = \sum_{n,\ell} c_m^F(n,\ell)\, q^n y^\ell$$

Deg. of single center $\frac{1}{4}$–BPS BH

Mock modular dyonic BH entropy

$\psi^F_m(\tau, z)$ is a holomorphic, mock–modular Jacobi form.

Fourier Expansion: $\psi^F_m(\tau, z) = \sum_{n,\ell} c^F_m(n, \ell)\, q^n y^\ell$

Deg. of single center $\frac{1}{4}$–BPS BH

MMJF’s are weakly holomorphic: Finite number of terms with $\Delta = 4mn - \ell^2 < 0$

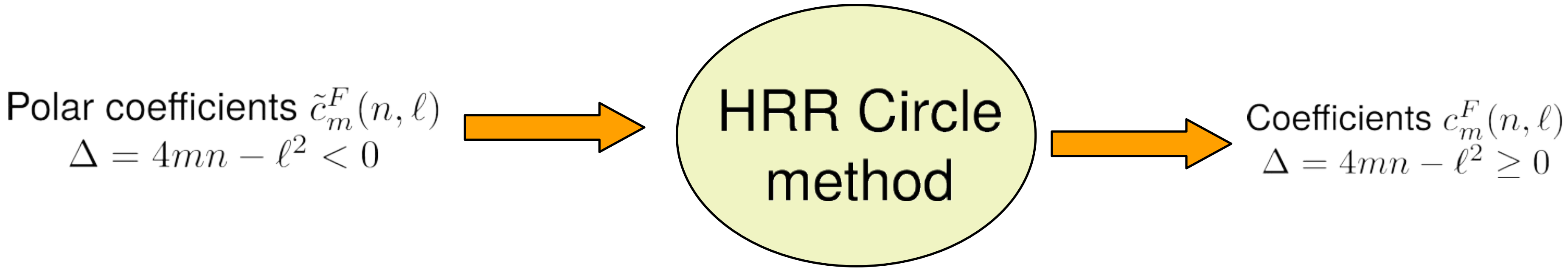
(Polar terms or negative discriminant states)

$\Delta < 0$ states “seed" large black hole entropy

Hardy–Ramanujan growth of FEC’s
(Cardy formula)

Polar coefficients are required input for
Hardy–Ramanujan–Rademacher circle method

Hardy-Ramananujan-Rademacher circle method



Polar coefficients $\tilde{c}^F_m(n,\ell)$
$\Delta = 4mn - \ell^2 < 0$

HRR Circle method

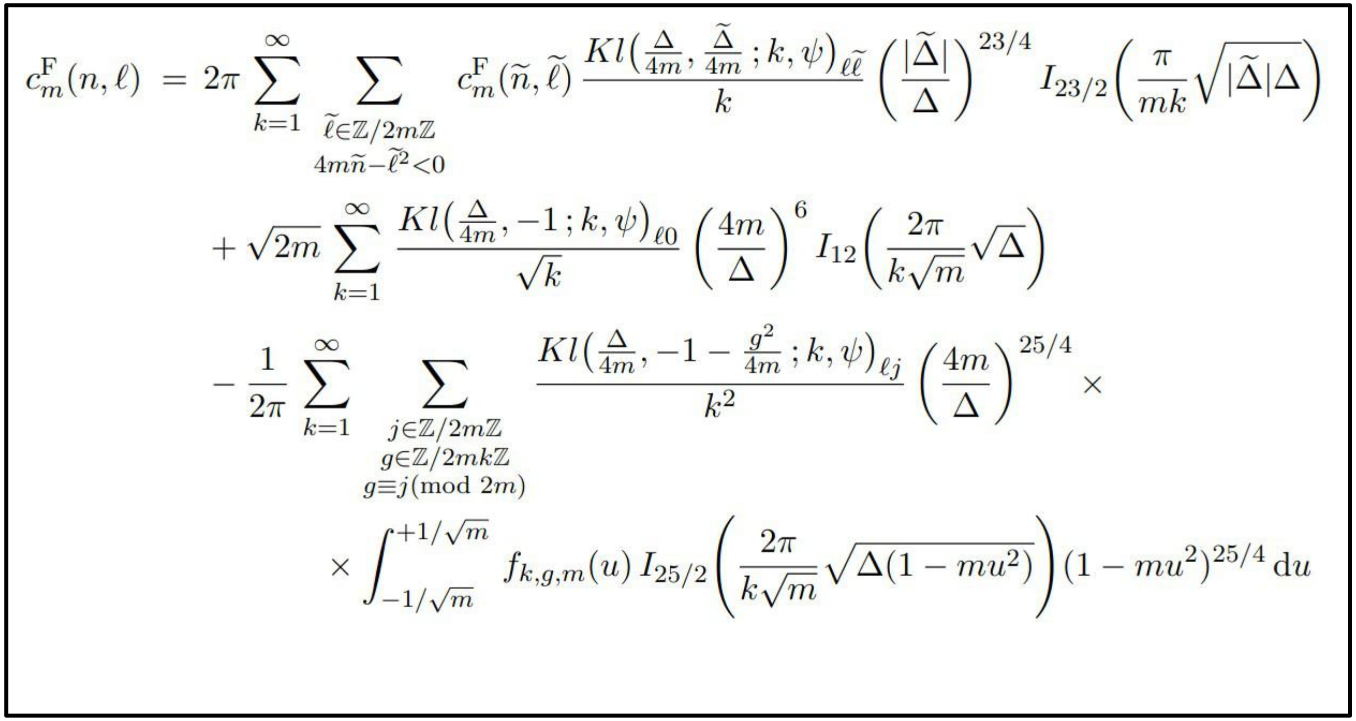
Coefficients $c^F_m(n,\ell)$
$\Delta = 4mn - \ell^2 \geq 0$

[Dijkgraaf, Maldacena, Moore, Verlinde; Moore, Manschot; Murthy, Bringmann; Ferrari, Reys]

Modularity and polar coefficients are enough to fully determine all FEC's

Mock–modularity

Hardy-Ramananujan-Rademacher circle method



$$\begin{aligned}
c^{\mathrm{F}}_m(n,\ell) \;=\; & 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ 4m\widetilde{n}-\widetilde{\ell}^2<0}} c^{\mathrm{F}}_m(\widetilde{n},\widetilde{\ell})\, \frac{Kl\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m}\,;k,\psi\big)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\!\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right) \\
& + \sqrt{2m}\sum_{k=1}^{\infty} \frac{Kl\big(\frac{\Delta}{4m},-1\,;k,\psi\big)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\!\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) \\
& - \frac{1}{2\pi}\sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g\in\mathbb{Z}/2mk\mathbb{Z} \\ g\equiv j (\mathrm{mod}\ 2m)}} \frac{Kl\big(\frac{\Delta}{4m},-1-\frac{g^2}{4m}\,;k,\psi\big)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\
& \qquad \times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\!\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\,\mathrm{d}u
\end{aligned}$$

[Murthy, Bringmann; Ferrari, Reys]

Hardy-Ramananujan-Rademacher circle method

$$\begin{aligned} c^{\mathrm{F}}_m(n,\ell) \;=\; & 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ 4m\widetilde{n}-\widetilde{\ell}^2<0}} c^{\mathrm{F}}_m(\widetilde{n},\widetilde{\ell})\, \frac{Kl\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m}\,;k,\psi\big)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\!\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right) \\ & + \sqrt{2m}\sum_{k=1}^{\infty} \frac{Kl\big(\frac{\Delta}{4m},-1\,;k,\psi\big)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\!\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) \\ & - \frac{1}{2\pi}\sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g\in\mathbb{Z}/2mk\mathbb{Z} \\ g\equiv j(\mathrm{mod}\ 2m)}} \frac{Kl\big(\frac{\Delta}{4m},-1-\frac{g^2}{4m}\,;k,\psi\big)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\ & \qquad \times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\!\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\,\mathrm{d}u \end{aligned}$$

Kloosterman sum

$$Kl(\mu,\nu\,;k,\psi)_{\ell\widetilde{\ell}} \;\coloneqq\; \sum_{\substack{0\le h<k \\ (h,k)=1}} e^{2\pi\mathrm{i}\left(-\frac{h}{k}\mu+\frac{h'}{k}\nu\right)}\,\psi(\gamma)_{\ell\widetilde{\ell}}$$

ψ is an appropriate multiplier system

$$\gamma = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix} \in SL(2,\mathbb{Z}) \text{ and } hh' \equiv -1 \ (\mathrm{mod}\ k)$$

Hardy-Ramananujan-Rademacher circle method

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\begin{aligned}
c^{\mathrm{F}}_m(n,\ell) \;=\; & 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\tilde\ell\in\mathbb{Z}/2m\mathbb{Z}\\ 4m\tilde n-\tilde\ell^2<0}} c^{\mathrm{F}}_m(\tilde n,\tilde\ell)\, \boxed{\frac{Kl\big(\frac{\Delta}{4m},\frac{\tilde\Delta}{4m};k,\psi\big)_{\ell\tilde\ell}}{k}} \left(\frac{|\tilde\Delta|}{\Delta}\right)^{23/4} \boxed{I_{23/2}}\left(\frac{\pi}{mk}\sqrt{|\tilde\Delta|\Delta}\right) \\
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\end{aligned}
$$

$I-$Bessel function

$$
I_\rho(x) \;=\; \frac{1}{2\pi\mathrm{i}}\left(\frac{x}{2}\right)^{\rho} \int_{\epsilon-\mathrm{i}\infty}^{\epsilon+\mathrm{i}\infty} t^{-\rho-1}\, e^{t+\frac{x^2}{4t}}\,\mathrm{d}t
$$

Kloosterman sum

$$
Kl(\mu,\nu\,;k,\psi)_{\ell\tilde\ell} \;:=\; \sum_{\substack{0\le h<k\\ (h,k)=1}} e^{2\pi\mathrm{i}\left(-\frac{h}{k}\mu+\frac{h'}{k}\nu\right)}\,\psi(\gamma)_{\ell\tilde\ell}
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Hardy-Ramananujan-Rademacher circle method

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I–Bessel function

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$$f_{k,g,m}(u) \;:=\; \begin{cases} \dfrac{\pi^2}{\sinh^2\!\big(\frac{\pi u}{k}-\frac{\pi i g}{2mk}\big)} & \text{if } g\not\equiv 0\,(\mathrm{mod}\ 2mk)\,, \\[2ex] \dfrac{\pi^2}{\sinh^2\!\big(\frac{\pi u}{k}\big)} - \dfrac{k^2}{u^2} & \text{if } g\equiv 0\,(\mathrm{mod}\ 2mk)\,. \end{cases}$$

Kloosterman sum

$$Kl(\mu,\nu\,;k,\psi)_{\ell\widetilde{\ell}} \;:=\; \sum_{\substack{0\le h<k\\ (h,k)=1}} e^{2\pi\mathrm{i}\left(-\frac{h}{k}\mu+\frac{h'}{k}\nu\right)}\,\psi(\gamma)_{\ell\widetilde{\ell}}$$

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$$\gamma = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix} \in SL(2,\mathbb{Z}) \text{ and } hh' \equiv -1 \ (\mathrm{mod}\ k)$$

Hardy-Ramananujan-Rademacher circle method

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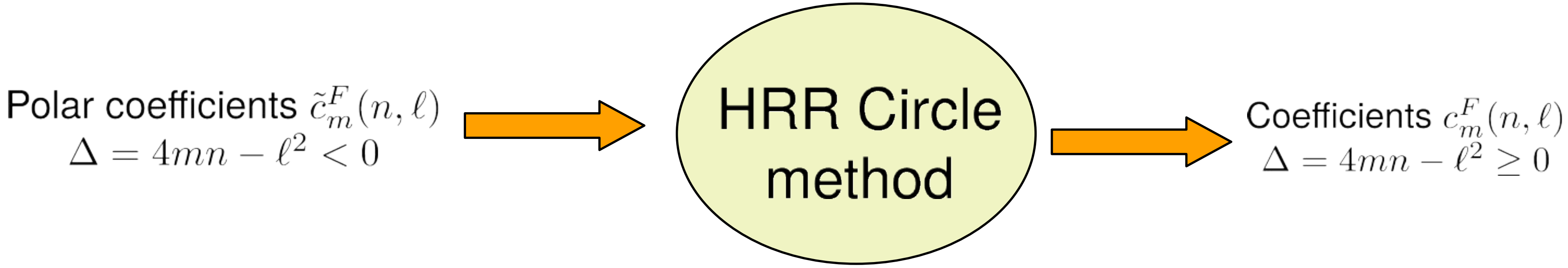
\end{aligned}
$$



Polar coefficients $\tilde{c}^F_m(n,\ell)$
$\Delta = 4mn - \ell^2 < 0$

HRR Circle method

Coefficients $c^F_m(n,\ell)$
$\Delta = 4mn - \ell^2 \geq 0$

Issue from holography

Localization of $\frac{1}{4}$–BPS states in $\mathcal{N} = 4$ SUGRA $\xleftrightarrow{\text{AdS/CFT}}$ Microscopic degeneracies

Does not match for a certain $\Delta < 0$ states

[Murthy, Reys]

BUT

$\Delta < 0$ states “seed" large black hole entropy

→ Hardy–Ramanujan growth of FEC’s (Cardy formula)

→ Polar coefficients are required input for Hardy–Ramanujan–Rademacher circle method

Attractors and walls

Degeneracy of $\frac{1}{4}$–BPS states is contour dependent

Choose contour on SUHP that corresponds to choosing attractor values

[Ferrara, Kallosh, Strominger; Cheng-Verlinde; Sen; CMKRW]

Fourier–Jacobi expansion: SUHP reduces to UHP

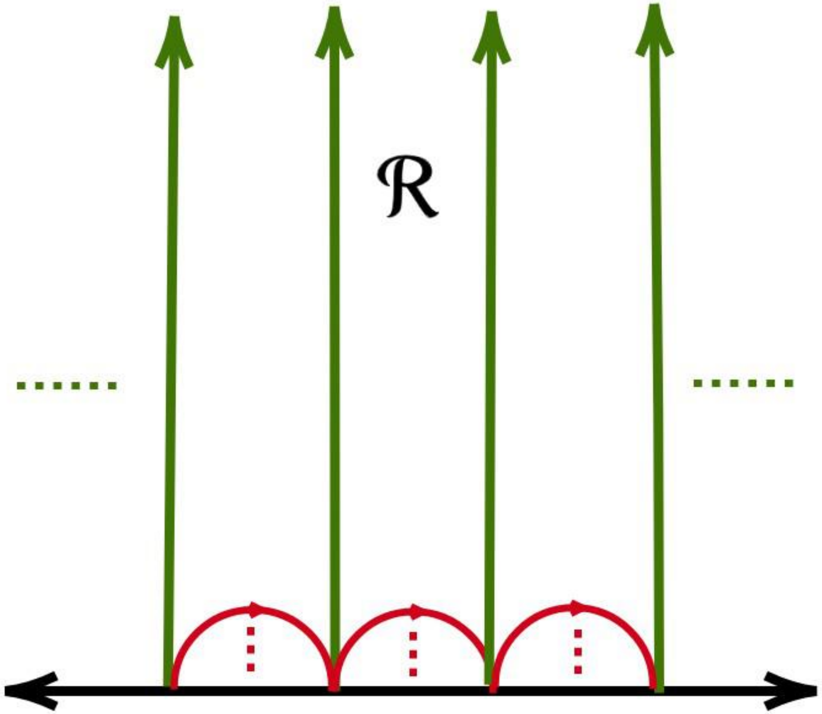
Appell–Lerch sum: $\mathcal{A}_{2,m}(\tau, z) = \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s}y^{2ms+1}}{(1-q^s y)^2}$

AL sum has different Fourier expansions in different strips of the UHP

Chamber in “axion–dilaton" moduli space corresponding to attractor contour

[Sen; CMKRW]

Attractors and walls



[Sen; CMKRW]

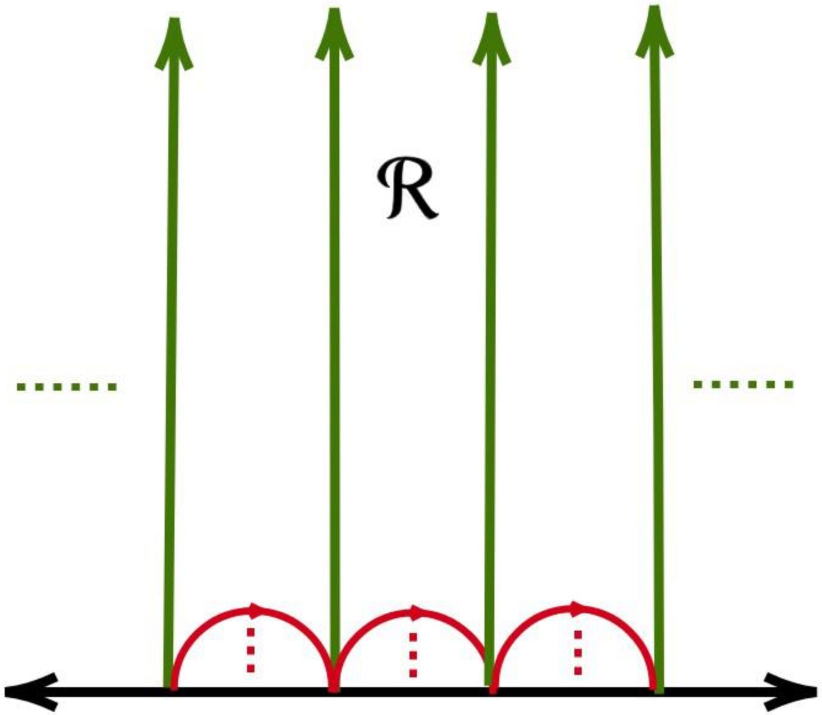
Chamber $\mathcal{R}$: All $\Delta > 0$ states are single center $\frac{1}{4}$–BPS states

Subtraction of $\psi_m^P(\tau, z)$ removes bound states from vertical 'T-walls'

Negative discriminant 'bound states' contribute from semicircular 'S-walls'

$\Delta < 0$ states in $\psi_m^F(\tau, z)$ are residual bound states of $\frac{1}{2}$–BPS "monopoles/instantons"

Attractors and walls

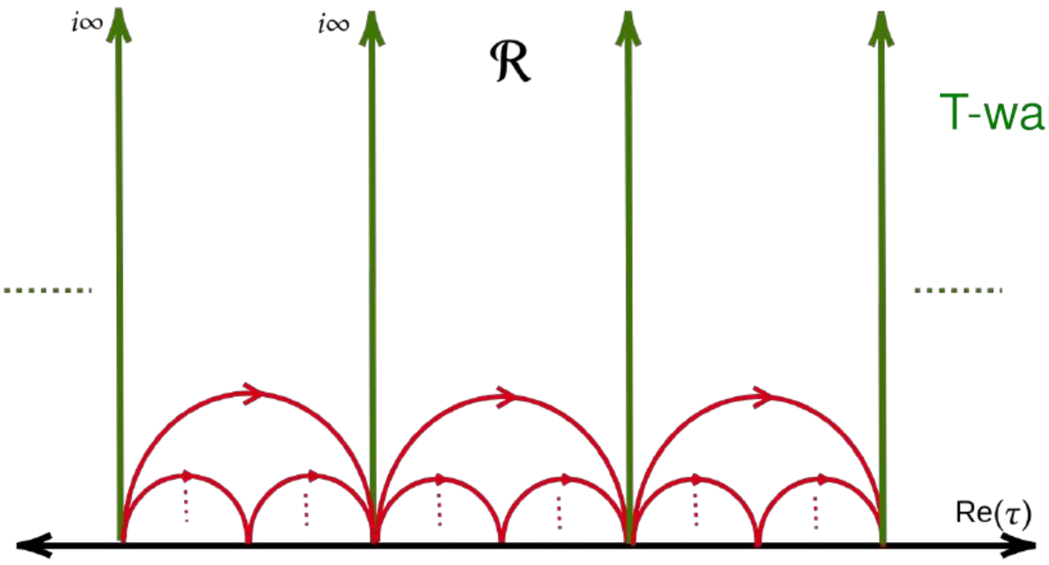


Problem of obtaining correct single center $\frac{1}{4}$–BPS degeneracies

↓

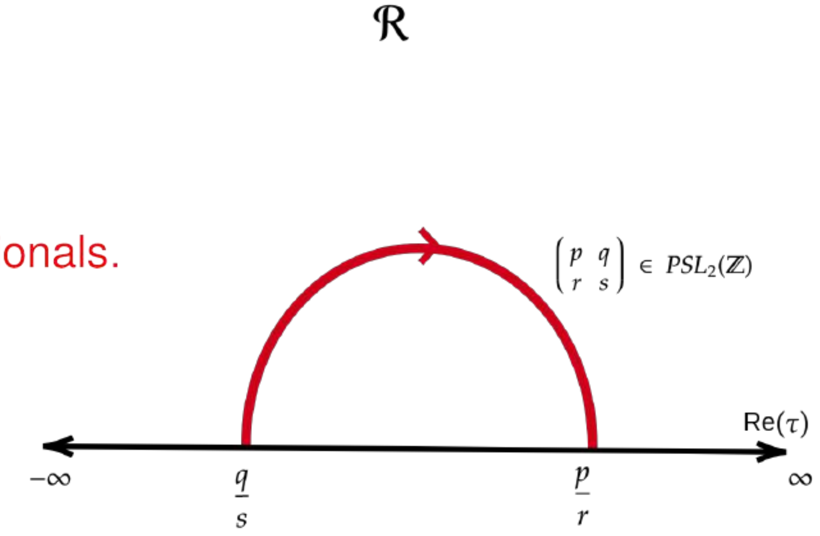
Problem of counting precisely all $\Delta < 0$ states in $\mathcal{R}$

Counting $\Delta < 0$ bound states: Walls

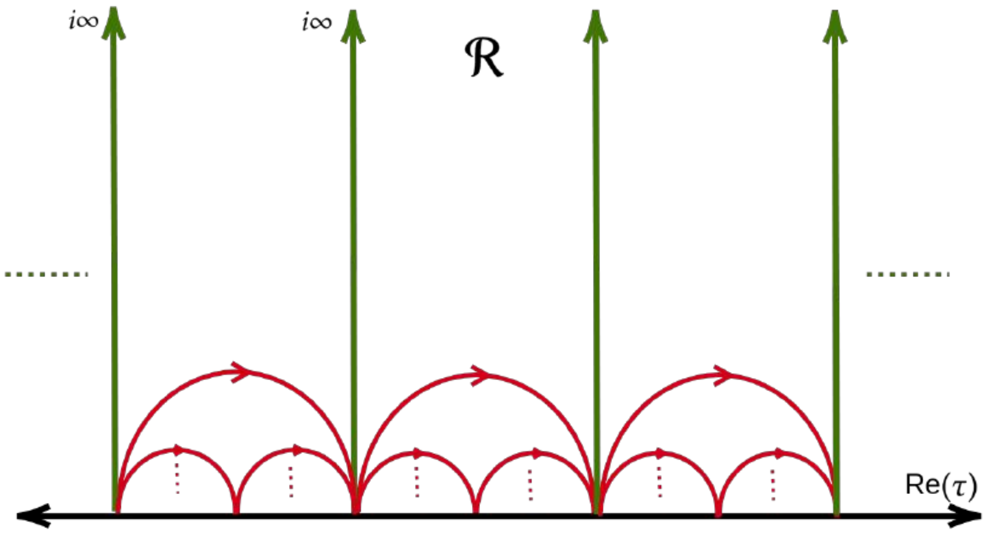


T-walls start from an integer and go to $i\infty$

S-walls are semicircles that start and end on rationals.



Counting $\Delta < 0$ bound states: Walls



$p/r > q/s$, if $\gamma \in \Gamma_S^+$

$p/r < q/s$, if $\gamma \in \Gamma_S^-$

$$\Gamma_S^- \cup \Gamma_S^+ \cup \Gamma_T = PSL(2,\mathbb{Z})$$

Each wall has an associated $SL(2,\mathbb{Z})$ matrix, γ.

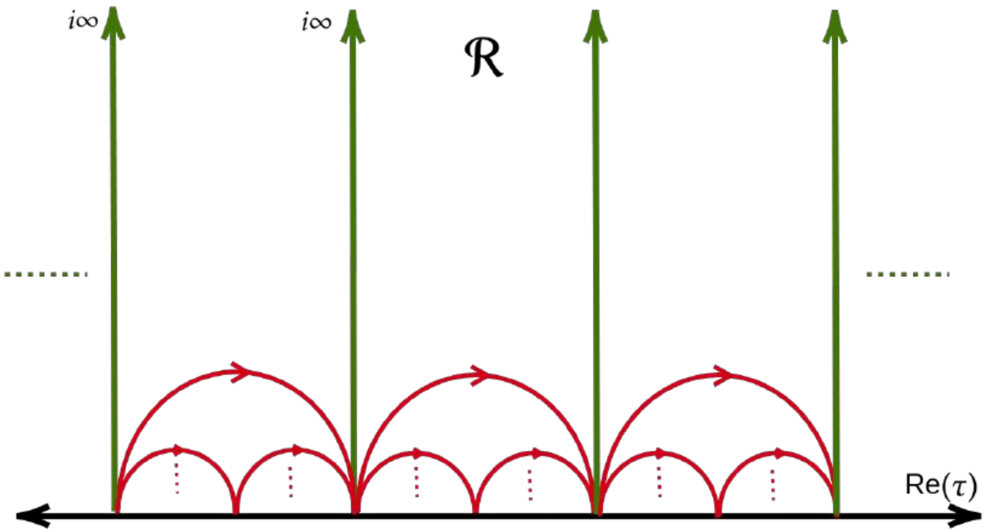
Set of all walls in UHP:

$$\Gamma_S^+ := \left\{ \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z}) \;\middle|\; r>0,\, s>0 \right\},$$

$$\Gamma_S^- := \left\{ \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z}) \;\middle|\; r>0,\, s<0 \right\},$$

$$\Gamma_T := \left\{ \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z}) \;\middle|\; rs=0 \right\}.$$

Counting $\Delta < 0$ bound states: Walls



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$p/r > q/s$, if $\gamma \in \Gamma_S^+$

$p/r < q/s$, if $\gamma \in \Gamma_S^-$

$$\Gamma_S^- \cup \Gamma_S^+ \cup \Gamma_T = PSL(2,\mathbb{Z})$$

Counting $\Delta < 0$ bound states

$\psi^P_m(\tau, z)$ removes contribution from T-walls. $\Delta < 0$ bound states can only come from S-walls.

Consider $\Delta < 0$ state $(n, m, \ell) = (Q^2/2, P^2/2, P \cdot Q)$

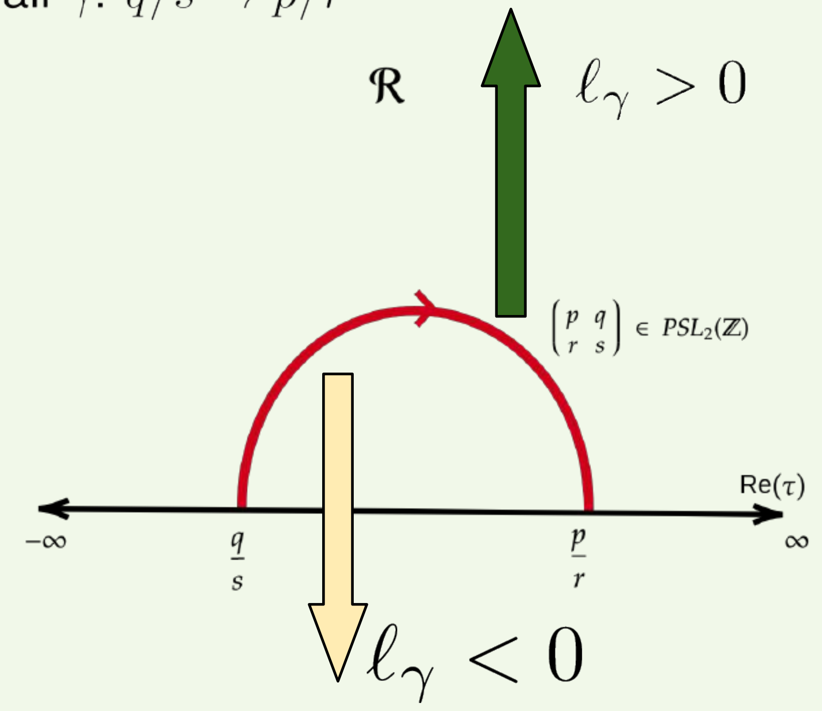
At a wall $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

$$\Longrightarrow \quad \begin{pmatrix} Q \\ P \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} p(sQ - qP) \\ r(sQ - qP) \end{pmatrix}}_{\text{Electric center}} + \underbrace{\begin{pmatrix} q(-rQ + pP) \\ s(-rQ + pP) \end{pmatrix}}_{\text{Magnetic center}}$$

$$(n_\gamma, \ell_\gamma, m_\gamma) = (Q_\gamma^2/2,\, Q_\gamma \cdot P_\gamma,\, P_\gamma^2/2) \quad \Longrightarrow \quad \begin{aligned} n_\gamma &= s^2 n + q^2 m - sq\ell \,, \\ \ell_\gamma &= -2srn - 2pqm + \ell(ps + qr) \,, \\ m_\gamma &= r^2 n + p^2 m - pr\ell \,. \end{aligned}$$

Counting $\Delta < 0$ bound states

Orientation of the wall γ: $q/s \to p/r$



Counting $\Delta < 0$ bound states

Define: $\theta(\gamma,\mathcal{R}) = \left|\frac{\mathcal{O}(\gamma,\mathcal{R}) + \mathrm{sgn}(\ell_\gamma)}{2}\right|, \qquad \mathcal{O}(\gamma,\mathcal{R}) = \begin{cases} +1, & \gamma \in \Gamma_S^+ \\ -1, & \gamma \in \Gamma_S^- \end{cases}$

$\tilde{c}_m(n,\ell,\Delta<0)$ contribution: $\frac{1}{2}\sum_{\gamma\in\Gamma_S} (-1)^{\ell_\gamma+1}\,\theta(\gamma,\mathcal{R})\,|\ell_\gamma|\,d(m_\gamma)\,d(n_\gamma)$

$$\frac{1}{\eta(\tau)^{24}} = \sum_{n\geq -1} d(n) q^n$$

BUT $\frac{1}{2}\sum_{\gamma\in\Gamma_S} (-1)^{\ell_\gamma+1}\,\theta(\gamma,\mathcal{R})\,|\ell_\gamma|\,d(m_\gamma)\,d(n_\gamma)$

is a sum over an infinite set

Try to constrain it

Case 1: $\Delta < 0, (n_\gamma \geq 0, m_\gamma \geq 0)$

$$
\begin{aligned}
n_\gamma &= s^2 n + q^2 m - sq\ell\,, \\
\ell_\gamma &= -2srn - 2pqm + \ell(ps + qr)\,, \\
m_\gamma &= r^2 n + p^2 m - pr\ell\,.
\end{aligned}
$$

Solve for bounds on γ ↓

[CMKRW]

Contribution from only $\gamma \in \Gamma_S$ such that

$$\gamma = \begin{pmatrix} p & q \\ r & r \end{pmatrix}, -m < p, q, r, s < m$$

Bound state metamorphosis

Subtlety in case n_γ and/or $m_\gamma = -1$

Consider: n_γ or $m_\gamma = -1$ (electric or magnetic BSM)

For a wall γ *with* $n_\gamma = -1,\ m_\gamma \geq 0$, *we define its metamorphic dual as*

$$\tilde{\gamma} \ = \ \gamma \cdot \begin{pmatrix} 1 & 0 \\ -\ell_\gamma & 1 \end{pmatrix} .$$

Given (n, m, ℓ), γ and $\tilde{\gamma}$ have the same contribution to polar coefficients $\tilde{c}_m(n, \ell)$

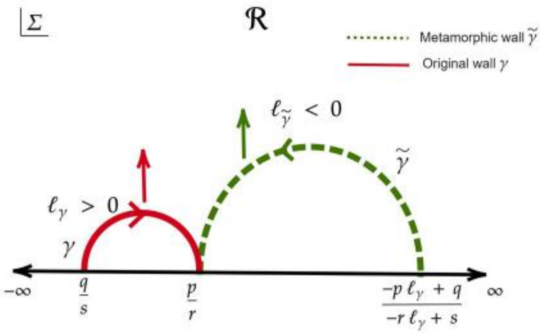
For a wall γ *with* $m_\gamma = -1, n_\gamma \geq 0$, *we define its metamorphic dual as*

$$\tilde{\gamma} \ := \ \gamma \cdot \begin{pmatrix} 1 & -\ell_\gamma \\ 0 & 1 \end{pmatrix} .$$

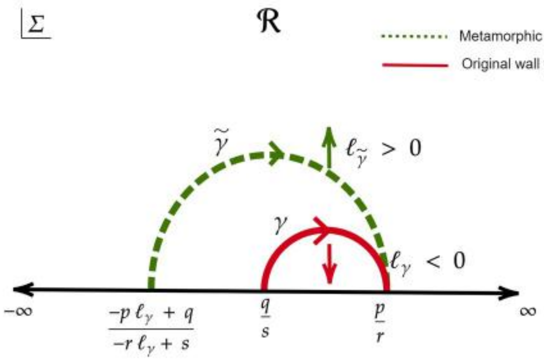
A wall γ at which BSM occurs contributes to the degeneracy of $\Delta < 0$ states in $\mathcal{R}$ iff $\tilde{\gamma}$ also contributes. Both their contributions are identified and counted only once.

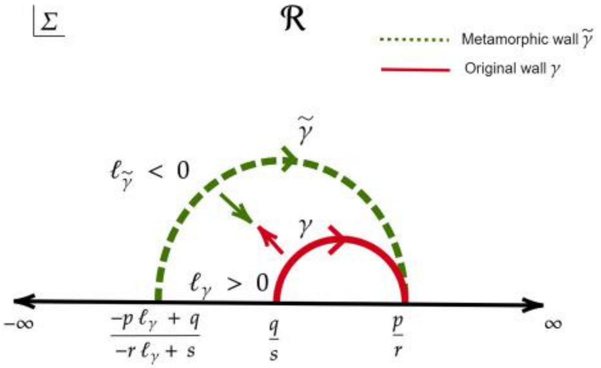
Case 2: $\Delta < 0, (n_\gamma \geq 0, m_\gamma = -1$ **or** $n_\gamma = -1, m_\gamma \geq 0)$

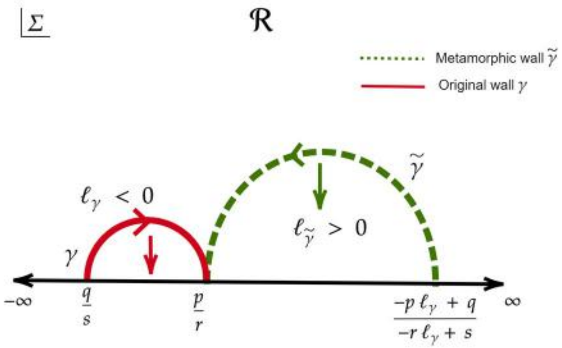
[CMKRW]

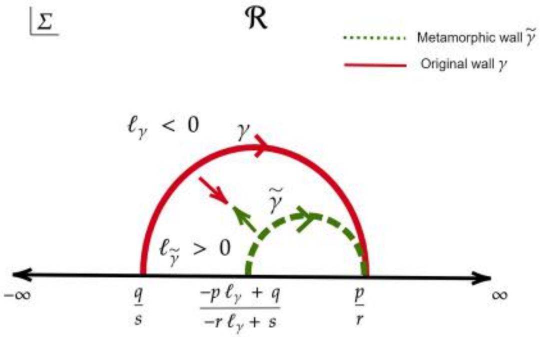


Only contributing case

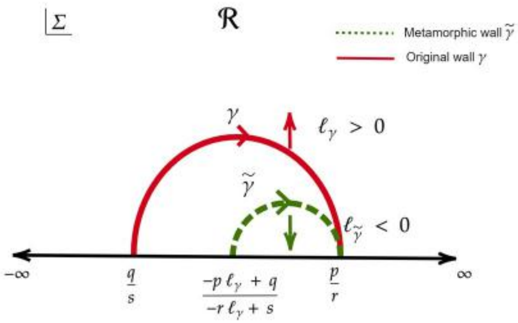








Non-contributing



Case 2: $\Delta < 0, (n_\gamma \geq 0, m_\gamma = -1$ or $n_\gamma = -1, m_\gamma \geq 0)$

$$
\begin{aligned}
n_\gamma &= s^2 n + q^2 m - sq\ell\,, \\
\ell_\gamma &= -2srn - 2pqm + \ell(ps + qr)\,, \\
m_\gamma &= r^2 n + p^2 m - pr\ell\,.
\end{aligned}
$$

Solve for bounds on γ

Contribution from only $\gamma \in \Gamma_S$ such that

$$\gamma = \begin{pmatrix} p & q \\ r & r \end{pmatrix}, -(m+1) < p, q, r, s < m+1$$

[CMKRW]

Case 3: $\Delta < 0, (n_\gamma = m_\gamma = -1)$

Dyonic metamorphosis

Let γ *be a wall at which* $m_\gamma = n_\gamma = -1$. *The metamorphic duals are*

$$\tilde{\gamma}_i = \tilde{\gamma}_{i-1} \cdot M_{(i \bmod 2)} \quad \mathit{for} \quad i > 0\,, \quad \mathit{and} \quad \tilde{\gamma}_0 = \gamma\,,$$

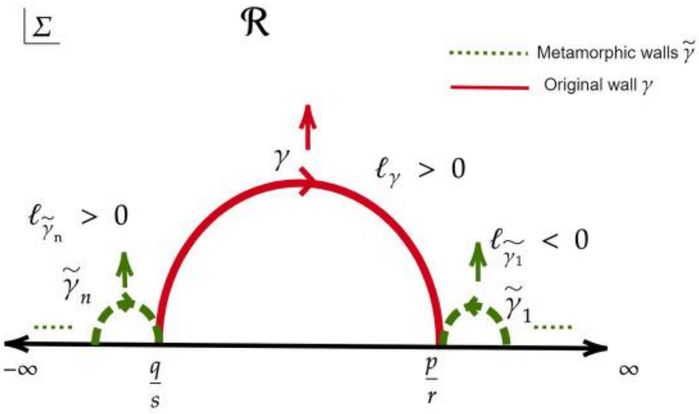
where M_1, M_0 *are defined as* $M_1 := \begin{pmatrix} 1 & -\ell_\gamma \\ 0 & 1 \end{pmatrix}, \qquad M_0 := \begin{pmatrix} 1 & 0 \\ \ell_\gamma & 1 \end{pmatrix}.$

Given (n, m, ℓ), γ, all $\tilde{\gamma}_i$ have the same contribution to polar coefficients $\tilde{c}_m(n, \ell)$

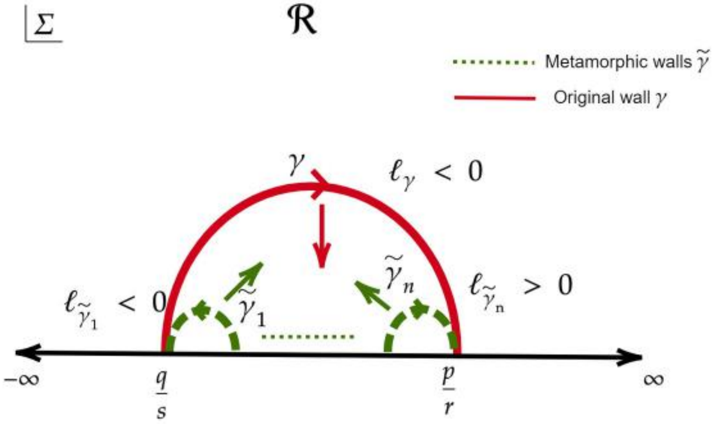
A wall γ at which dyonic BSM occurs contributes to the degeneracy of $\Delta < 0$ states in $\mathcal{R}$ iff all $\tilde{\gamma}_i$ also contribute. All their contributions are identified and counted only once.

Case 3: $\Delta < 0, (n_\gamma = m_\gamma = -1)$

Dyonic metamorphosis



Contributes



Does not contribute

Contribution from only $\gamma \in \Gamma_S$ such that

$$\gamma = \begin{pmatrix} p & q \\ r & r \end{pmatrix}, -(m+1) \leq p, q, r, s \leq m+1$$

Comment: Very interesting number theory here. Refer CMKRW.

Orbits of metamorphosis

Metamorphosis splits walls contributing to $\tilde{c}_m(n,\ell)$
into ‘orbits’ of length 1, 2, ∞

Let $\Gamma_{\text{BSM}}(n,m,\ell) = PSL(2,\mathbb{Z})/BSM$

Define: $$\Theta(\mu) = \prod_{\gamma\in\mu} \theta(\gamma,\mathcal{R}), \qquad \mu \in \Gamma_{\mathrm{BSM}}(n,\ell,m)$$

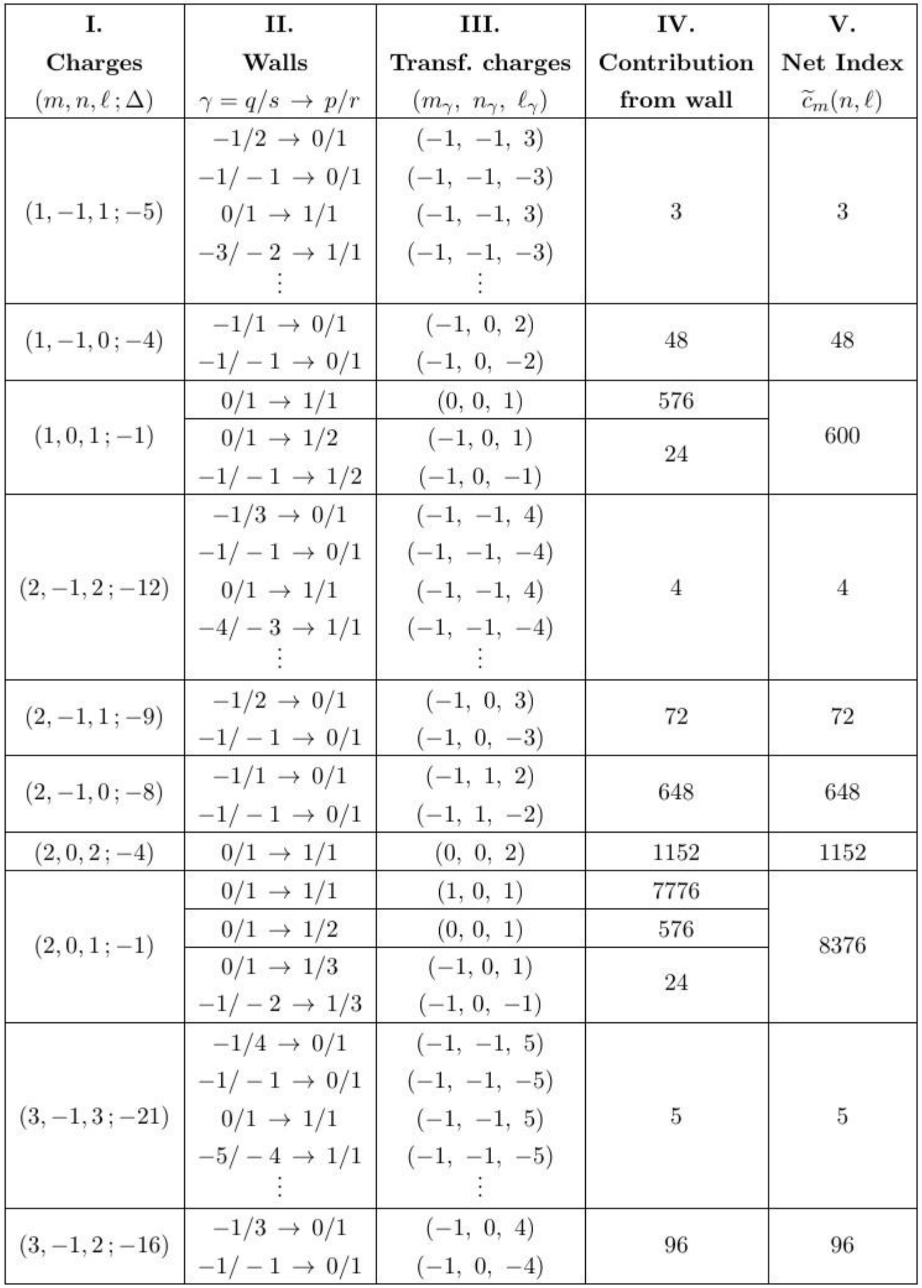
$$\Theta(\gamma) = \Theta(\mu), \quad \gamma \in \mu$$

$$\boxed{\tilde{c}_m(n,\ell) = \frac{1}{2} \sum_{\gamma\in\Gamma_{\mathrm{BSM}}(n,\ell,m)} (-1)^{\ell_\gamma+1}\,\Theta(\gamma)\,|\ell_\gamma|\,d(m_\gamma)\,d(n_\gamma)}$$

Contribution from only $\gamma \in \Gamma_{\text{BSM}}$ such that

$$\gamma = \begin{pmatrix} p & q \\ r & r \end{pmatrix}, -(m+1) \leq p,q,r,s \leq m+1$$

Agreement with SUGRA Localization



I. Charges $(m,n,\ell\,;\Delta)$	II. Walls $\gamma = q/s \to p/r$	III. Transf. charges $(m_\gamma,\ n_\gamma,\ \ell_\gamma)$	IV. Contribution from wall	V. Net Index $\widetilde{c}_m(n,\ell)$
$(1,-1,1\,;-5)$	$-1/2 \to 0/1$ $-1/-1 \to 0/1$ $0/1 \to 1/1$ $-3/-2 \to 1/1$ $\vdots$	$(-1,\ -1,\ 3)$ $(-1,\ -1,\ -3)$ $(-1,\ -1,\ 3)$ $(-1,\ -1,\ -3)$ $\vdots$	3	3
$(1,-1,0\,;-4)$	$-1/1 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 0,\ 2)$ $(-1,\ 0,\ -2)$	48	48
$(1,0,1\,;-1)$	$0/1 \to 1/1$	$(0,\ 0,\ 1)$	576	600
	$0/1 \to 1/2$ $-1/-1 \to 1/2$	$(-1,\ 0,\ 1)$ $(-1,\ 0,\ -1)$	24	
$(2,-1,2\,;-12)$	$-1/3 \to 0/1$ $-1/-1 \to 0/1$ $0/1 \to 1/1$ $-4/-3 \to 1/1$ $\vdots$	$(-1,\ -1,\ 4)$ $(-1,\ -1,\ -4)$ $(-1,\ -1,\ 4)$ $(-1,\ -1,\ -4)$ $\vdots$	4	4
$(2,-1,1\,;-9)$	$-1/2 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 0,\ 3)$ $(-1,\ 0,\ -3)$	72	72
$(2,-1,0\,;-8)$	$-1/1 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 1,\ 2)$ $(-1,\ 1,\ -2)$	648	648
$(2,0,2\,;-4)$	$0/1 \to 1/1$	$(0,\ 0,\ 2)$	1152	1152
$(2,0,1\,;-1)$	$0/1 \to 1/1$	$(1,\ 0,\ 1)$	7776	8376
	$0/1 \to 1/2$	$(0,\ 0,\ 1)$	576	
	$0/1 \to 1/3$ $-1/-2 \to 1/3$	$(-1,\ 0,\ 1)$ $(-1,\ 0,\ -1)$	24	
$(3,-1,3\,;-21)$	$-1/4 \to 0/1$ $-1/-1 \to 0/1$ $0/1 \to 1/1$ $-5/-4 \to 1/1$ $\vdots$	$(-1,\ -1,\ 5)$ $(-1,\ -1,\ -5)$ $(-1,\ -1,\ 5)$ $(-1,\ -1,\ -5)$ $\vdots$	5	5
$(3,-1,2\,;-16)$	$-1/3 \to 0/1$ $-1/-1 \to 0/1$	$(-1,\ 0,\ 4)$ $(-1,\ 0,\ -4)$	96	96

Many more checks in *[CMKRW]*

Conclusions

1. $\Delta < 0$ states in $\psi_m^F(\tau, z)$ are $\frac{1}{2}$–BPS bound states

2. Should be counted exactly since they seed $\frac{1}{4}$–BPS BH entropy

3. Presented an exact formula which counts $\Delta < 0$ states in an appropriate chamber

4. Subtleties from metamorphosis constrain the set of contributing walls

5. Implication of $m+1$ being the bound is that BH entropy is controlled by a finite number of worldsheet instantons

6. Exact formula is computationally advantageous

7. OSV type argument?

Outlook

1. Gravitational interpretation of $\Delta < 0$ states? *[Dabholkar, Gaiotto, Nampuri]*

2. CHL extension *[Cardoso, Nampuri, Rossello]*

Obrigado!