Mock modularity and refinement: from BPS black holes to Vafa-Witten theory

Sergei Alexandrov

Laboratoire Charles Coulomb, CNRS, Montpellier

S.A., B.Pioline arXiv:1808.08479
S.A., J.Manschot, B.Pioline arXiv:1910.03098

S.A. arXiv:2005.03680
arXiv:2006.10074

S.A., S.Nampuri arXiv:2009.01172

Workshop on Black Holes: BPS, BMS and Integrability
September 10, 2020

Motivation

The main object of interest:

BPS indices $\Omega(\gamma)$ — (signed) number of BPS states in theories with extended SUSY

- degeneracies of BPS black holes
- spectrum of states in supersymmetric gauge theories
- weights of instanton corrections to the effective action
- Donaldson-Thomas invariants of Calabi-Yau manifolds
- Vafa-Witten invariants of complex surfaces – topologically twisted SYM

It is useful to study *generating functions*

$$h_{\dots}(\tau) = \sum_{q_0>0} \Omega(\gamma)\, e^{2\pi \mathrm{i} q_0 \tau}$$

Sometimes they possess non-trivial *modular* properties:

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

they can be *modular forms,*
mock modular forms,
higher depth mock modular forms…

$$h\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w h(\tau)$$

Motivation

The goal: understand modular properties of the generating functions of BPS indices $\Omega(\gamma)$

- Can be used to extract the asymptotic behavior of BPS indices to compare with the macroscopic entropy of BPS black holes

- Can be used to find them exactly!

- They encode a non-trivial geometry of the quantum corrected moduli spaces affected by BPS instantons

The plan of the talk

1. D4-D2-D0 black holes in Type IIA/CY and their BPS indices

2. Modularity of the generating functions of (*refined*) BPS indices and their modular completions

3. Applications:

   a) BPS dyons in N=4 string theory

   b) Vafa-Witten theory on projective surfaces

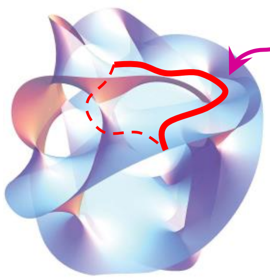
   c) Holomorphic anomaly for BPS partition function

   d) Quantization of the moduli space

4. Conclusions

D4-D2-D0 black holes in Type IIA/CY

These are ½ BPS black holes in 4d N=2 SUGRA with electro-magnetic charge



$$\gamma = (0, p^a, q_a, q_0) \qquad\qquad a = 1, \dots, b_2(CY)$$

label 4- and 2-dim cycles wrapped by D4 and D2-branes

BPS index $\Omega(\gamma)$ — black hole degeneracy = generalized Donaldson-Thomas invariant of CY

Natural generating function $\quad h^{\mathrm{DT}}_{p^a,q_a}(\tau) = \sum_{q_0>0} \Omega(\gamma)\, e^{2\pi \mathrm{i} q_0 \tau}$

but:
- the generating function depends on too many charges
- DT invariants depend on CY moduli: $\Omega(\gamma; z^a)$ — *wall-crossing*

(BPS *bound* states can form or decay)

no nice modular properties expected

MSW invariants

Solution: consider *MSW invariants*
count states in SCFT constructed
in Maldacena,Strominger,Witten '97

$$\boxed{\Omega^{\rm MSW}_\gamma = \Omega(\gamma, z^a_\infty(\gamma))}$$

large volume
attractor point

$$z^a_\infty(\gamma) = \lim_{\lambda\to\infty}(-q^a + \mathrm{i}\lambda p^a)$$

Properties:

- independent of CY moduli
- invariant under *spectral flow symmetry*

$$\Omega^{\rm MSW}_\gamma = \Omega_p(\hat q_0)$$

$\hat q_0 \equiv q_0 - \frac12\,\kappa^{ab} q_a q_b$ – invariant charge bounded from above

spectral flow

$$q_a \mapsto q_a - \kappa_{ab}\epsilon^b$$

$$q_0 \mapsto q_0 - \epsilon^a q_a + \tfrac12\,\kappa_{ab}\epsilon^a\epsilon^b$$

$\kappa_{ab} = \kappa_{abc}p^c$ – quadratic form, given by intersection numbers of 4-cycles, of *indefinite* signature $(1, b_2-1)$

generating function
of MSW invariants

$$h_p(\tau) = \sum_{\hat q_0 \le \hat q_0^{\rm max}} \bar\Omega_p(\hat q_0)\, e^{-2\pi\mathrm{i}\hat q_0\tau}$$

where

$$\bar\Omega(\gamma) := \sum_{d|\gamma}\frac{1}{d^2}\,\Omega(\gamma/d)$$

rational invariants

The origin of modular symmetry

modular group

S-duality

M-theory / CY ×T^2 ⟷ Type IIA / CY×S^1 ⟷ Type IIB / CY×S^1

T-duality

M5-brane / divisor ⟷ D4-D2-D0 bl.h. ⟷ D3-D1-D(-1) inst.

affect the metric on the QK moduli space $\mathcal{M}$

$\mathcal{M}$ carries an isometric action of $SL(2,\mathbb{Z})$ preserved by $\boxed{\text{non-pert. corrections}}$

Twistor description of D-instantons
S.A.,Pioline,Saueressig,Vandoren '08
S.A. '09

A function $\mathcal{G}$ on $\mathcal{M}$ (called "*instanton generating potential*") constructed from DT-invariants is *modular* of weight $\left(-\frac{3}{2},\frac{1}{2}\right)$

Restriction on (the generating function of) BPS indices $\Omega(\gamma)$

The functions $h_p(\tau)$ have a modular anomaly, but one can construct an explicit expression for a non-holomorphic modular completion $\widehat{h}_p(\tau,\bar\tau)$

Refinement

There is a *refined* version of BPS indices $\Omega(\gamma, y) \sim \mathrm{Tr}_{\mathcal{H}_\gamma}(-y)^{2J_3}$

y – refinement parameter conjugate to the angular momentum

generating function of refined MSW invariants

$$h_p^{\mathrm{ref}}(\tau, y) = \sum_{\hat{q}_0 \le \hat{q}_0^{\max}} \frac{\bar{\Omega}_p(\hat{q}_0, y)}{y - y^{-1}}\, e^{-2\pi \mathrm{i} \hat{q}_0 \tau}$$

single order pole at $y \to 1$

The claim: the unrefined construction of the modular completion $\widehat{h}_p(\tau, \bar{\tau})$ has a natural generalization to the refined case provided the (log of the) refinement parameter transforms as Jacobi elliptic variable:

If $y = e^{2\pi \mathrm{i} w}$ then $w \mapsto \dfrac{w}{c\tau + d}$

or $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for $w = \alpha - \tau\beta$

The unrefined construction can be obtained by taking the limit $y \to 1$

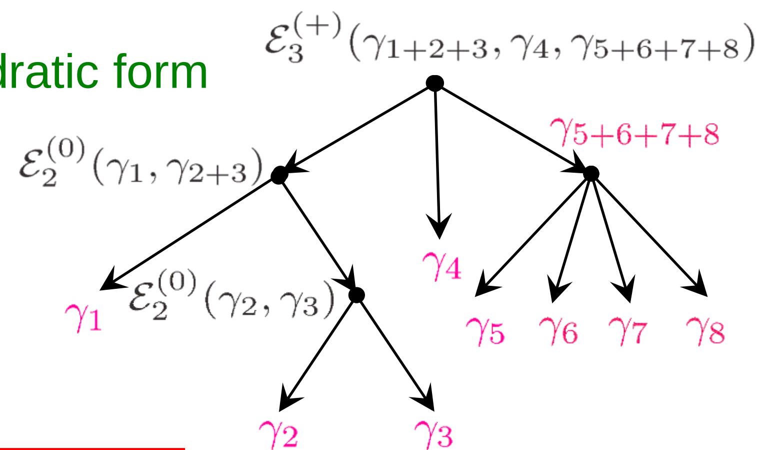
S.A.,Manschot,Pioline ‘19

Modular completion

$$\widehat{h}_p^{\rm ref} = h_p^{\rm ref} + \sum_{n=2}^{\infty} \sum_{\sum_{i=1}^n \gamma_i=\gamma} R_n^{\rm ref}(\{\gamma_i\};\tau_2)\,(-y)^{\sum_{i<j}\gamma_{ij}}\, e^{\pi \mathrm{i}\tau Q_n(\{\gamma_i\})} \prod_{i=1}^{n} h_{p_i}^{\rm ref}$$

Jacobi modular form of weight $-\frac{1}{2}\,b_2$

- $\gamma_{ij}=\langle\gamma_i,\gamma_j\rangle$ — Dirac skew-symmetric product
- $Q_n=\kappa^{ab}q_a q_b-\sum_{i=1}^{n}\kappa_i^{ab}q_{i,a}q_{i,b}$ — indefinite quadratic form
- $R_n^{\rm ref}$ — sum over (*Schröder)* trees of products of signs and generalized error functions of order *n*-1 assigned to vertices of the trees, with parameters defined by charges



Generalized error functions

$$\Phi_n^E(\{\boldsymbol{v}_i\},\boldsymbol{x}) = \int_{\mathrm{Span}\{\boldsymbol{v}_i\}} \mathrm{d}\boldsymbol{x}'\, e^{\pi(P(\boldsymbol{v})\cdot\boldsymbol{x}-\boldsymbol{x}')^2} \prod_{i=1}^{n} \mathrm{sgn}(\boldsymbol{v}_i\cdot\boldsymbol{x}')$$

$P(\boldsymbol{v})$ — projector on $\mathrm{Span}\{\boldsymbol{v}_i\}$

For *n*=1 reduces to the usual error function

$$\Phi_1^E(\boldsymbol{v},\boldsymbol{x}) = \mathrm{Erf}\left(\sqrt{\pi}\,\frac{\boldsymbol{x}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}\right)$$

Modular completion – unrefined limit

The unrefined limit $y \to 1$

It is *well-defined* because the sum over charges produces a zero of order *n*-1 which cancels the poles of the refined generating functions

The unrefined construction differs only by the factors assigned to vertices of the trees which now involve sums of *derivatives* of the generalized error functions

Example: $n = 2$

$$R_2^{\rm ref} = -\frac{1}{2}\,\mathrm{sgn}(\gamma_{12})\,\mathrm{Erfc}\left(\sqrt{\frac{2\pi\tau_2}{(pp_1p_2)}}\,|\gamma_{12}|\right) \qquad R_2 = -\frac{|\gamma_{12}|}{8\pi}\,\beta_{\frac{3}{2}}\left(\frac{2\tau_2\gamma_{12}^2}{(pp_1p_2)}\right)$$

where $(pp_1p_2) = \kappa_{abd}p^a p_1^b p_2^c$

$$\beta_{\frac{3}{2}}(x^2) = \frac{2}{|x|}\,e^{-\pi x^2} - 2\pi\mathrm{Erfc}(\sqrt{\pi}|x|)$$

The origin of the completion

These results follow from

indefinite theta series of signature

$(nb_2 - n + 1, n - 1)$

$$\mathcal{G} \sim \frac{1}{\sqrt{\tau_2}} \sum_{n=1}^{\infty} \left[\prod_{i=1}^{n} \sum_{p_i} h_{p_i}(\tau) \right] \theta_{\boldsymbol{p}}^{(n)}(\tau, t^a, b^a, \dots) \longrightarrow$$

Modular properties of the theta series determine the properties of $h_p(\tau)$

for $n \geq 2$ there is a modular *anomaly*

rearrange the expansion

$$= \frac{1}{\sqrt{\tau_2}} \sum_{n=1}^{\infty} \left[\prod_{i=1}^{n} \sum_{p_i} \widehat{h}_{p_i}(\tau, \bar{\tau}) \right] \widehat{\theta}_{\boldsymbol{p}}^{(n)}(\tau, t^a, b^a, \dots)$$

modular completion

modular invariant

How to construct modular completions of indefinite theta series?

Indefinite theta series

$$\vartheta_{\boldsymbol{p}}(\tau,\boldsymbol{z}) = \sum_{\boldsymbol{q}\in\boldsymbol{\Lambda}+\frac12\boldsymbol{p}} (-1)^{\boldsymbol{q}\cdot\boldsymbol{p}}\, e^{\pi i\tau\boldsymbol{q}^2+2\pi i\boldsymbol{q}\cdot\boldsymbol{z}}$$

d-dim. lattice

If the quadratic form $\boldsymbol{q}^2$ is *positive definite*, the theta series is known to be a *Jacobi modular form*

$$\vartheta_{\boldsymbol{p}}\left(\frac{a\tau+b}{c\tau+d},\frac{\boldsymbol{z}}{c\tau+d}\right)\sim\vartheta_{\boldsymbol{p}}(\tau,\boldsymbol{z})$$

What if the quadratic form is *indefinite*?

⟶ the theta series diverges!

But one can make it convergent by restricting to the wedge where $\boldsymbol{q}^2>0$

Example: Lorentzian signature $(d-1,1)$

$$\vartheta_{\boldsymbol{p}}(\tau,\boldsymbol{z}) = \sum_{\boldsymbol{q}\in\boldsymbol{\Lambda}+\frac12\boldsymbol{p}} (-1)^{\boldsymbol{q}\cdot\boldsymbol{p}}\Big[\mathrm{sgn}((\boldsymbol{q}+\boldsymbol{b})\cdot\boldsymbol{v})-\mathrm{sgn}((\boldsymbol{q}+\boldsymbol{b})\cdot\boldsymbol{v}')\Big]\, e^{\pi i\tau\boldsymbol{q}^2+2\pi i\boldsymbol{q}\cdot\boldsymbol{z}}$$

$$\boldsymbol{z}=\boldsymbol{c}-\tau\boldsymbol{b}$$

Converges if $\boldsymbol{v}^2,\ \boldsymbol{v}'^2,\ \boldsymbol{v}\cdot\boldsymbol{v}'<0$. But the sign functions spoil modularity!

Can it be cured? — **Yes!**

The modular completion of $\vartheta_{\boldsymbol{p}}(\tau,\boldsymbol{z})$ is obtained by replacement

$$\mathrm{sgn}(\boldsymbol{x}\cdot\boldsymbol{v})\to\mathrm{Erf}\left(\sqrt{\pi}\,\frac{\boldsymbol{x}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}\right)$$

$$\boldsymbol{x}=\sqrt{2\tau_2}(\boldsymbol{q}+\boldsymbol{b})$$

This is an example of *mock modular form*

Important property: $\mathrm{Erf}\left(\sqrt{\pi}\,\frac{\boldsymbol{x}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}\right)\underset{|v|\to 0}{\to}\mathrm{sgn}(\boldsymbol{x}\cdot\boldsymbol{v})$ ⟶ *null vectors don't require completion*

Mock modularity

Problem: what are the modular properties and the modular completion for generic signature?

Solution came from the twistorial formulation of D-instantons

S.A.,Banerjee,Manschot,Pioline; Nazaroglu '16

The kernel for signature (d-n,n) is given by combinations of products of n signs. The completion is obtained by

$$\prod_{i=1}^{n} \mathrm{sgn}(\boldsymbol{x}\cdot\boldsymbol{v}_i) \to \Phi_n^E(\{\boldsymbol{v}_i\},\boldsymbol{x})$$

generalized error functions

Indefinite theta series of signature $(d-n,n)$ are examples of mock modular forms of depth n

$\partial_{\bar\tau}\widehat{h}(\tau)$ is expressed via mock modular forms of lower depth

The generating function of (refined) MSW invariants is a higher depth mock Jacobi form

$$\mathcal{D}=\mathcal{D}_1+\cdots+\mathcal{D}_n$$

irreducible divisors

Depth = n-1

S.A.,Nampuri '20

Helicity supertraces

Can these results be applied in cases with N>2 SUSY?

BPS indices counting $\frac{1}{r}$-BPS states in theory with *N* extended supersymmetry are all *different*

Helicity supertrace

$$B_K(\mathcal{R}) = \mathrm{Tr}_{\mathcal{R}}\left[(-1)^{2J_3} J_3^K\right]$$

soaks up 2*K* fermionic zero modes

the first non-vanishing supertrace

$$K = \frac{2N}{r}(r-1)$$

$$\left(\tfrac{1}{2}y\partial_y\right)^K B|_{y=1}$$

$$\Omega^{(N|r)}(\gamma) = \frac{B_K(\mathcal{H}^N_{\gamma,j})}{B_K(\mathcal{R}_{K,j})}$$

center of mass contribution

Helicity generating function

$$B(\mathcal{R}, y) = \mathrm{Tr}_{\mathcal{R}}(-y)^{2J_3}$$

on $\mathcal{R}_{K,j}$ has zero of order *K* at $y = 1$

Refined BPS index

$$\Omega(\gamma, y) = \frac{B(\mathcal{H}_{\gamma,j}, y)}{B(\mathcal{R}_{2,j}, y)} \sim \frac{y\, B(\mathcal{H}_{\gamma,j}, y)}{(y-1)^2}$$

All BPS indices can be expressed through the refined index

$$\boxed{\Omega^{(N|r)}(\gamma) = \frac{\partial_y^{K-2}\Omega(\gamma, y)|_{y=1}}{(K-2)!}}$$

Holomorphic anomaly

Conjecture: $h_p^{\rm ref}$ **is a mock Jacobi form of weight** $w_{\rm ref} = -\frac{1}{2}\,{\rm rank}(\kappa_{AB})$ **with the completion satisfying a similar holomorphic anomaly equation in string compactifications with any amount of SUSY**

$$(y-y^{-1})\partial_{\bar\tau}\widehat{h}_p^{\rm ref} = \sum_{n=2}^{\infty} \sum_{\sum_{i=1}^{n}\gamma_i=\gamma} \Theta_n(\{\gamma_i\};y)\prod_{i=1}^{n}\Omega(\gamma_i,y)$$

$\kappa_{AB} = \kappa_{ABC}p^C$

where $A = 1,\dots,b_2+2b_1$ and κ_{ABC} generalizes intersection numbers

κ_{AB} can be degenerate!

The unrefined limit to get $\Omega^{(N|r)}$:

apply $(y\partial_y)^{K-2}$ and set $y=1$

$\longrightarrow$ $h_p^{(N|r)}$ is a *mock modular* form of weight $w = K-3-\frac{1}{2}\,{\rm rank}\kappa_{AB}$

$\longrightarrow$ # of derivatives: $K-2 = 2N-2-2N/r \leq n(N-2)$ for $n, N\geq 2$

for $N>2$ equality holds only for $\underline{r=N,\ n=2}$

For all $r<N$, the generating functions of $\frac{1}{r}$-BPS states are usual *modular* forms

Only bound states of *two half-BPS* states contribute to the anomaly

N=4 dyons

D4 on K3

Appear in Type IIA/K3×T² or in Het/T⁶

E.-m. charge in **(2,28)** of $SL(2,\mathbb{Z})\times O(22,6,\mathbb{Z})$

$$\gamma = \begin{pmatrix} Q^I \\ P^I \end{pmatrix} = \begin{pmatrix} q_0, & -p^\flat, & q_\alpha \\ q_\flat, & 0, & p^\alpha \end{pmatrix}$$

D6-brane charge

26d vector

- **½ BPS states** $(Q^I \parallel P^I) \longrightarrow q_\flat = p^\alpha = 0$

  $K=4$, $\mathrm{rank}(\kappa_{AB}) = 26$ $\longrightarrow$ Gen. function – modular form of weight -12 $\quad h_p^{(4|2)}(\tau) = \eta^{-24}(\tau)$

- **¼ BPS states**

  $K=6$, $\mathrm{rank}(\kappa_{AB}) = 27$ $\longrightarrow$ Gen. function – mock modular form of weight -21/2

  $p^\flat = \gcd(p^\alpha) = 1 \longrightarrow I(\gamma) = \gcd(Q\wedge P) = 1$ – *torsion* (U-duality invariant)

  The holomorphic anomaly for the gen. function is *identical* to

$$\tau_2^{3/2}\partial_{\bar\tau}\widehat{\psi}_m^F = \frac{\sqrt{m}}{8\pi \mathrm{i}}\,\frac{\Omega^{(4|2)}(m)}{\eta(\tau)^{24}}\sum_{\ell=0}^{2m-1}\overline{\theta_{m,\ell}(\tau,0)}\,\theta_{m,\ell}(\tau,z)$$

Dabholkar,Murthy,Zagier '12

$$\theta_{m,\ell}(\tau,z) = \sum_{r\in 2m\mathbb{Z}+\ell} \mathrm{q}^{\frac{r^2}{4m}}\, y^r$$

generating function of immortal dyons

Generalization (for $p^\flat = 1$):
includes contributions for various values of torsion

$$\tau_2^{3/2}\partial_{\bar\tau}\widehat{\psi}_p = \sum_{d|\gcd(p^\alpha)} d\,\mathcal{A}_{m/d^2}(\tau, dz)$$

S.A.,Manschot,Pioline ‘19

Relation to Vafa-Witten

Consider a CY given by an elliptic fibration over a projective surface *S* and take a *local limit* where the elliptic fiber becomes large

- local CY – the canonical bundle over *S*

  non-compact!

(refined) DT invariant of local CY = (refined) VW invariant of *S*

- The only surviving divisor is the base of the fibration [*S*] → $p^a = N p_0^a$ ← charge corresponding to [*S*]

All D4-brane charges are collinear!

degree of reducibility of the divisor — N — rank of the VW gauge group *U*(*N*)

Prediction for all ranks and surfaces with

$b_2^+(S) = 1, \; b_1(S) = 0$

$$\boxed{\widehat{h}^{\mathrm{VW,ref}}_{N,J=-K_S} = \widehat{h}^{\mathrm{ref}}_{N p_0}}$$

canonical polarization (attractor chamber)

Check: for $S = \mathbb{P}^2$ the modular completions have been explicitly computed for *N*=2 and 3 (*only!*). They perfectly coincide with our predictions!

Rank *N* Vafa-Witten invariants

The formula for the completion allows to find the VW invariants themselves!

Example: $N=2$

$$\widehat{h}_2 = h_2 + \frac{h_1^2}{2} \sum_{k\in\Lambda_S} \left(\mathrm{Erf}\left(2\sqrt{\frac{\pi\tau_2}{K_S^2}}\,(K_S\cdot k+\beta K_S^2)\right) - \mathrm{sgn}(K_S\cdot k)\right) \mathrm{q}^{-k^2}\, y^{2K_S\cdot k}$$

where for all surfaces

$$h_1 = \frac{\mathrm{i}}{\theta_1(\tau,2z)\,\eta(\tau)^{b_2(S)-1}}$$

modularity requires the kernel

$$\mathrm{Erf}(\sqrt{\tau_2}v\cdot(k+\beta K_S)) - \mathrm{Erf}(\sqrt{\tau_2}v'\cdot(k+\beta K_S))$$

Rank *N* Vafa-Witten invariants

The formula for the completion allows to find the VW invariants themselves!

Example: $N=2$

$$\widehat{h}_2 = h_2 + \frac{h_1^2}{2}\sum_{k\in\Lambda_S}\left(\mathrm{Erf}\left(2\sqrt{\frac{\pi\tau_2}{K_S^2}}\,(K_S\cdot k+\beta K_S^2)\right) - \mathrm{sgn}(K_S\cdot k)\right) \mathrm{q}^{-k^2}\, y^{2K_S\cdot k}$$

where for all surfaces

$$h_1 = \frac{\mathrm{i}}{\theta_1(\tau,2z)\,\eta(\tau)^{b_2(S)-1}}$$

modularity requires the kernel

$$\mathrm{Erf}(\sqrt{\tau_2}v\cdot(k+\beta K_S)) - \mathrm{sgn}(v_0\cdot(k+\beta K_S))$$

holomorphic! $v_0^2=0$

$$\boxed{h_{2,-K_S} = H_2 + \frac{h_1^2}{2}\sum_{k\in\Lambda_S}\left(\mathrm{sgn}(K_S\cdot k) - \mathrm{sgn}(v_0\cdot(k+\beta K_S))\right)\mathrm{q}^{-k^2}\, y^{2K_S\cdot k}}$$

holomorphic & modular

H_2 is found by requiring well-defined unrefined limit

$\longrightarrow$ $h_1^{-2}h_2$ must have a zero at $y=\pm 1$

for *Hirzebruch* and *del Pezzo*

$$H_2 \sim \frac{\mathrm{i}\eta(\tau)}{\theta_1(\tau,4z)\theta_1(\tau,2z)^2}$$

Explicit expressions for generating functions of refined VW invariants and their completions for all *N*

Rank *N* Vafa-Witten invariants

S.A. 2005.03680
2006.10074

The formula for the completion allows to find the VW invariants themselves!

Example: $N=2$

$$\widehat{h}_2 = h_2 + \frac{h_1^2}{2}\sum_{k\in\Lambda_S}\left(\mathrm{Erf}\left(2\sqrt{\frac{\pi\tau_2}{K_S^2}}\,(K_S\cdot k+\beta K_S^2)\right)-\mathrm{sgn}(K_S\cdot k)\right)\mathrm{q}^{-k^2}\,y^{2K_S\cdot k}$$

where for all surfaces

$$h_1 = \frac{\mathrm{i}}{\theta_1(\tau,2z)\,\eta(\tau)^{b_2(S)-1}}$$

modularity requires the kernel

$$\mathrm{Erf}(\sqrt{\tau_2}v\cdot(k+\beta K_S))-\mathrm{sgn}(v_0\cdot(k+\beta K_S))$$

holomorphic! $v_0^2=0$

$$\boxed{h_{2,\,J} = H_2 + \frac{h_1^2}{2}\sum_{k\in\Lambda_S}\left(\mathrm{sgn}((-J)\cdot k)-\mathrm{sgn}(v_0\cdot(k+\beta K_S))\right)\mathrm{q}^{-k^2}\,y^{2K_S\cdot k}}$$

holomorphic & modular

H_2 is found by requiring well-defined unrefined limit

$\longrightarrow$ $h_1^{-2}h_2$ must have a zero at $y=\pm 1$

for *Hirzebruch* and *del Pezzo*

$$H_2 \sim \frac{\mathrm{i}\eta(\tau)}{\theta_1(\tau,4z)\theta_1(\tau,2z)^2}$$

Explicit expressions
for generating functions of refined
VW invariants and their completions
for all *N* and all *J*

Comments

- This construction requires only two ingredients:
  1) a *unimodular* charge (second homology) lattice Λ_S
  2) a *null* vector $v_0 \in \Lambda_S$

  What if there are several null vectors in the lattice?

- The null vectors satisfying $v_0 \cdot v_0' = 1$ & $v_0 \cdot K_S = v_0' \cdot K_S$
  give the *same* generating functions ← example of *fiber-base duality*

  Katz,Mayr,Vafa '97

  Requires very non-trivial identities between theta functions!

  simplest example $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

- Other null vectors give different generating functions → **new invariants?**
  simplest example $S = \mathbb{F}_1$

- What if the lattice does not have a null vector? simplest example $S = \mathbb{P}^2$
  Then one can extend the lattice by multiplying the formula for the completion by a theta series

Explicit results at all *N* for $S = \mathbb{P}^2$

An improved version of the *blow-up formula*

Holomorphic anomaly of VW partition function

The formula for the completion allows to prove an old conjecture for *U(N)* Vafa-Witten theory (checked before for $S=\frac{1}{2}K3$ and $\mathbb{P}^2$ up to $N\leq 3$)

$$\boxed{\overline{\mathcal{D}}\widehat{\mathcal{Z}}_N=\frac{\sqrt{2\tau_2}}{32\pi\mathrm{i}}\sum_{N_1+N_2=N}N_1N_2\,\widehat{\mathcal{Z}}_{N_1}\widehat{\mathcal{Z}}_{N_2}}$$

$$\widehat{Z}_N=\frac{\sqrt{p_0^3}}{\sqrt{N}}\,\widehat{h}_N(\tau)\vartheta_N^{\mathrm{SN}}(\tau,v)$$

$$\overline{\mathcal{D}}=\tau_2^2\left(\partial_{\bar\tau}-\frac{\mathrm{i}}{4\pi}\left(\partial_{c_+}+2\pi\mathrm{i}b_+\right)^2\right)$$

Minahan,Nemeschansky, Vafa,Warner '98

such that $\overline{\mathcal{D}}\vartheta_N^{\mathrm{SN}}=0$

Siegel-Narain theta series

reduction to the quadratic term due to collinearity of charges $p^a=Np_0^a$

A similar anomaly equation for the refined partition function can be written only using a non-commutative star product!

$$\boxed{f\star g=f\exp\left[\frac{1}{2\pi\mathrm{i}}\left(\overleftarrow{\mathscr{D}_a}\,\overrightarrow{\partial}_{\tilde c_a}-\overleftarrow{\partial}_{\tilde c_a}\overrightarrow{\mathscr{D}_a}\right)\right]g}$$

$$\begin{aligned}\mathscr{D}_a&=w\partial_{v^a}+\bar w\partial_{\bar v^a}\\&=\alpha\partial_{c^a}+\beta\partial_{b^a}\end{aligned}$$

$\tilde c_a, c^a$ – RR-fields coupled to D4 and D2-brane charges

$$\boxed{\overline{\mathcal{D}}\widehat{\mathcal{Z}}_N^{\mathrm{ref}}=\frac{\mathrm{i}(2\tau_2)^{3/2}}{32\pi^2N}\sum_{N_1+N_2=N}\left(\partial_{\bar v^a}\widehat{\mathcal{Z}}_{N_2}^{\mathrm{ref}}\star\partial_{\tilde c_a}\widehat{\mathcal{Z}}_{N_1}^{\mathrm{ref}}-\partial_{\tilde c_a}\widehat{\mathcal{Z}}_{N_2}^{\mathrm{ref}}\star\partial_{\bar v^a}\widehat{\mathcal{Z}}_{N_1}^{\mathrm{ref}}\right)}$$

Instanton generating potential & TBA equations

The geometry of the moduli space is encoded in $H_\gamma(z)$, *Darboux coordinates* on the twistor space of $\mathcal{M}$ satisfying a *TBA-like equation*

large volume limit of the integral equation of Gaiotto-Moore-Neitzke for N=2 SYM / S^1

$$\boxed{H_\gamma(z) = H_\gamma^{\rm cl}(z)\, \exp\left[\sum_{\gamma'\in\Gamma_+}\int_{\ell_{\gamma'}} \mathrm{d}z'\, K_{\gamma\gamma'}(z,z')\, H_{\gamma'}(z')\right]}$$

where $\quad H_\gamma^{\rm cl} \sim \bar\Omega(\gamma), \quad K_{\gamma_1\gamma_2} = 2\pi\left(\kappa_{abc}t^a p_1^b p_2^c + \frac{\mathrm{i}\langle\gamma_1,\gamma_2\rangle}{z_1-z_2}\right)$

Instanton generating potential:

$$\mathcal{G} = \sum_{\gamma\in\Gamma_+}\int_{\ell_\gamma} \mathrm{d}z\, H_\gamma - \frac{1}{2}\sum_{\gamma_1,\gamma_2\in\Gamma_+}\int_{\ell_{\gamma_1}}\mathrm{d}z_1\int_{\ell_{\gamma_2}}\mathrm{d}z_2\, K_{\gamma_1\gamma_2}\, H_{\gamma_1} H_{\gamma_2} = \frac{1}{\sqrt{\tau_2}}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\sum_{p_i}\widehat{h}_{p_i}(\tau)\right]\widehat{\theta}_{\boldsymbol{p}}(\tau,\boldsymbol{v})$$

refinement

$$\sum_{\gamma\in\Gamma_+}\int_{\ell_\gamma}\mathrm{d}z\, H_\gamma^{\rm ref} = \frac{1}{\sqrt{\tau_2}}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\sum_{p_i}\widehat{h}^{\rm ref}_{p_i}(\tau,y)\right]\widehat{\vartheta}^{\rm ref}_{\boldsymbol{p}}(\tau,\boldsymbol{v},y) = \mathcal{G}^{\rm ref}(y)$$

Refined potential

Jacobi form of weight $(-\frac{1}{2},\frac{1}{2})$.

where

$$\boxed{H_\gamma^{\rm ref}(z) = H_\gamma^{\rm ref,cl}(z) \star \left[1+\sum_{\gamma'}\int_{\ell_{\gamma'}}\frac{\mathrm{d}z'}{z-z'}\, H_{\gamma'}^{\rm ref}(z')\right]}$$

non-commutative TBA-like equation

The refinement effectively quantizes the moduli space consistently with S-duality

Conclusions

Main result: *Explicit* form of the *modular completion* of the generating function of (refined) black hole degeneracies (DT invariants) at large volume attractor point for *arbitrary* divisor of CY

$\longrightarrow$ $h_p(\tau)$ – higher depth mock modular form

Numerous applications: N=4 dyons, VW invariants for arbitrary rank, fiber-base duality, blow-up formula, holomorphic anomaly, quantization of the moduli space consistent with S-duality….

Open problems:

- Extension of this technique to evaluation of DT invariants for *compact* CYs
- Understanding the non-commutative geometry of the refined moduli space and relations to previous constructions

  $\longrightarrow$ relation to *twistorial topological string*? [Ceccotti-Neitzke-Vafa '14]
- Geometric or physical meaning of the (refined) instanton generating potential
- Geometric of physical meaning of the invariants generated by "wrong" null vectors
- Compactifications with N=8 supersymmetry