1/8-BPS couplings in effective string theory

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Workshop on black holes:
BPS, BMS and integrability, Lisbon (online), 7 Sep 2020

Joint work with Guillaume Bossard and Boris Pioline

[arXiv:2001.05562, SciPost Phys. 8 (2020) 054]

String theory effective action

Type II string theory on T^{d-1}

$$\Big\downarrow \quad \alpha' \to 0$$

Max. supergravity in $D = 11 - d$, global $E_d(\mathbb{R})$ symmetry as complete low energy effective action (EFT)

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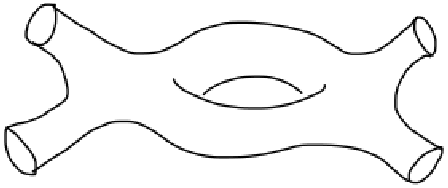

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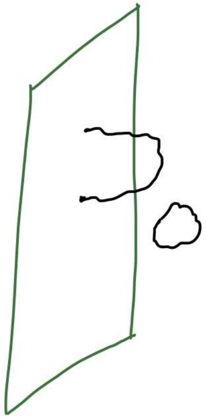

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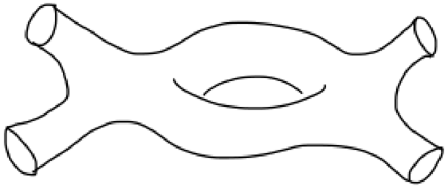

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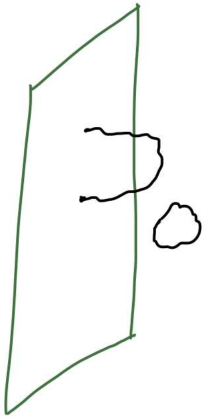

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$\alpha' = \ell_{\text{s}}^2$ $\quad$ $\Phi \in E_d/K(E_d)$ $\quad$ (Riemann)^4

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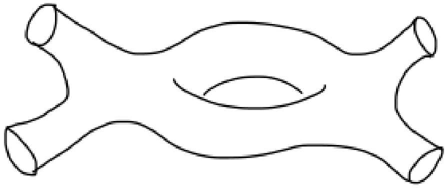

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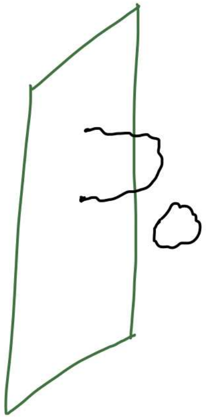

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Breaks SUGRA $E_d(\mathbb{R})$ to string U-duality $E_d(\mathbb{Z})$

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$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SUGRA}} + (\alpha')^3 f(\Phi)\sqrt{-g}R^4 + \dots$$

designed to capture string interactions up to given α' order

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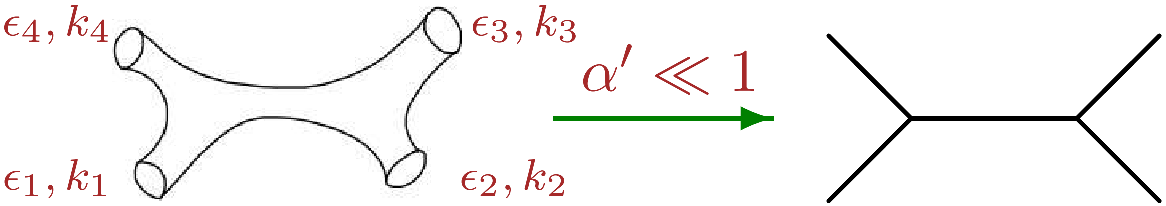

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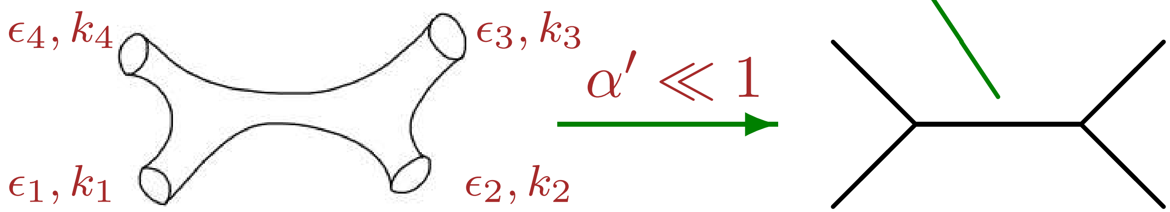

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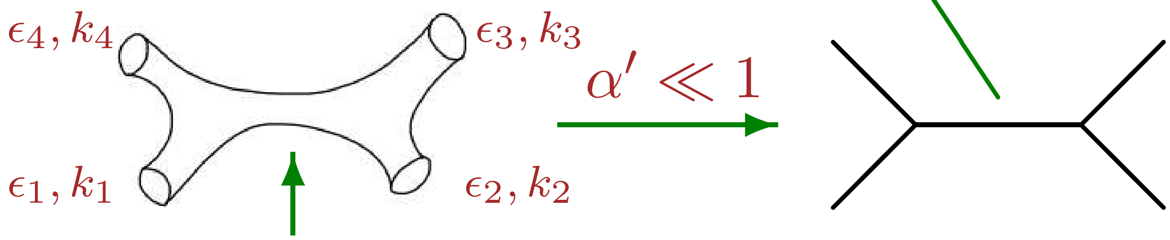

can also exchange
massive strings $m^2 \propto (\alpha')^{-1}$

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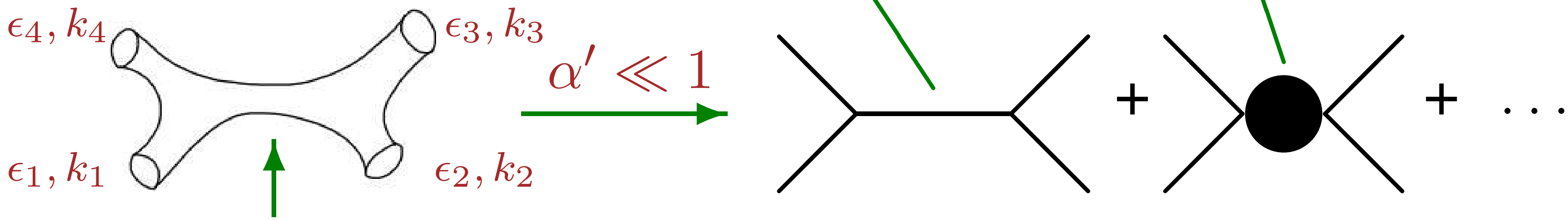

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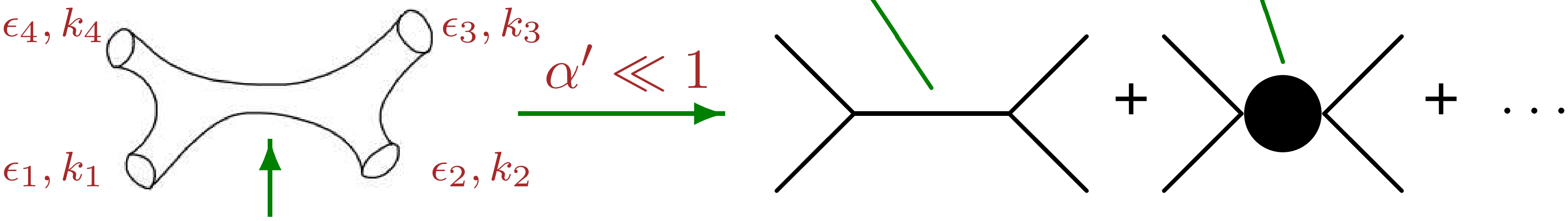

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Information about loop order (genus) and torus background in $f(\Phi)$.

$\Longrightarrow$ Determining these is hard and aim of the talk

String theory effective action (III)

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SUGRA}} + (\alpha')^3 f(\Phi)\sqrt{-g}R^4 + \dots$$

Determine $f(\Phi)$

1. directly: Compute (pert.) string amplitudes to given order in α'
2. indirectly: Use symmetry arguments and auxiliary systems. Particularly feasible when given coupling is BPS-protected

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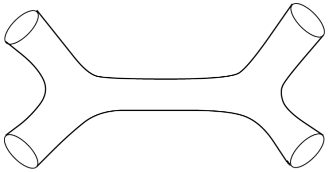

Tree level in $D = 10$

$$\mathcal{A}(s,t,u) = g_s^{-2}\frac{4}{stu}\frac{\Gamma(1-\alpha's)\Gamma(1-\alpha't)\Gamma(1-\alpha'u)}{\Gamma(1+\alpha's)\Gamma(1+\alpha't)\Gamma(1+\alpha'u)}\mathcal{R}^4$$

[Green Schwarz]

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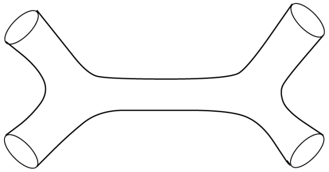

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Mandelstam variables

string coupling: tree level

absorbs polarisation tensors

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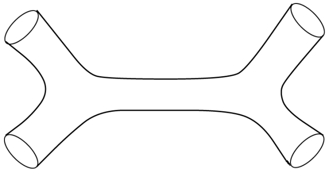

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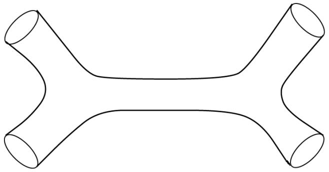

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Effective action (Einstein frame)

$$\sqrt{-g}^{-1}\mathcal{L} \sim R + (\alpha')^3 2\zeta(3) g_s^{-3/2} R^4 + (\alpha')^5 \zeta(5) g_s^{-5/2} \nabla^4 R^4 + \dots$$

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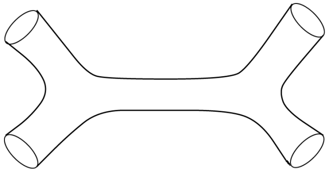

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only part of coefficient functions! Tree level...

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Higher derivative corrections

More precisely in $D = 11 - d$ dimensions

$E_d/K(E_d)$ scalar fields

$$e^{-1}\mathcal{L} = \ell^{2-D}\left[R - \frac{1}{2}G_{IJ}(\Phi)\partial\Phi^I\partial\Phi^J + \ldots\right]$$

Planck length

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- Moduli $\Phi = (g_{\rm s}, \ldots)$: couplings and shape of T^{d-1} etc.
- Analytic part $\mathcal{E}^D_{(p,q)}(\Phi)$ must expand for small coupling etc. to match perturbative results, e.g.

$$\mathcal{E}^{10B}_{(0,0)}(\Phi) = 2\zeta(3)g_{\rm s}^{-3/2} + \ldots$$

IIB: $\Phi = c_{(0)} + ig_{\rm s}^{-1}$

direct determination

Indirect determination

Coefficient fn.s $\mathcal{E}^{D}_{(p,q)}(\Phi)(s^2+t^2+u^2)^p(s^3+t^3+u^3)^q 4^{-2p-3q}$

- satisfy $\mathcal{E}^{D}_{(p,q)}(\gamma\Phi k)=\mathcal{E}^{D}_{(p,q)}(\Phi)$ for $\gamma\in E_d(\mathbb{Z})$, $k\in K(E_d)$

  $E_d(\mathbb{Z})$ U-duality invariance includes S-duality $g_s \to g_s^{-1}$
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- First few are BPS-protected

$$\mathcal{E}^{D}_{(0,0)}(\Phi)R^4 \qquad \mathcal{E}^{D}_{(1,0)}(\Phi)\nabla^4R^4 \qquad \mathcal{E}^{D}_{(0,1)}(\Phi)\nabla^6R^4$$

$\frac{1}{2}$-BPS $\frac{1}{4}$-BPS $\frac{1}{8}$-BPS

  - satisfy strong PDEs in Φ
  - only selected string theory states contribute

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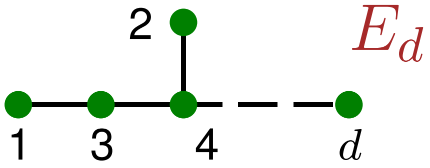
$\Longrightarrow$ Exceptional field theory loops

section constraint $\Leftrightarrow \frac{1}{2}$-BPS-constraint

Exceptional field theory

[**de Wit, Nicolai; Hull; Waldram et al.; Berman, Perry; Hohm, Samtleben; West; ...**]



Formalism to make hidden $E_d(\mathbb{R})$ (continuous!) ‘geometric’. Combine diffeomorphisms with gauge transformations.

Exceptional field theory

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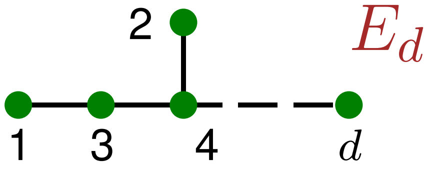

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Extended space-time ($D = 11 - d$)

$$\underline{\mathcal{M}^D} \times \underline{\mathcal{M}^{d(\alpha_d)}}$$

external space(-time)

internal space, $d(\alpha_d) = \dim \mathbf{R}_{\alpha_d}$
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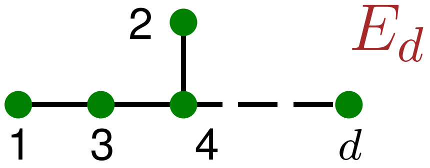

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Assume toroidal internal space $\Rightarrow$ quantised momenta

$\mathbf{R}_{\alpha_d}$ decomposes under ‘gravity line’ $GL(d, \mathbb{R}) \subset E_d(\mathbb{R})$

$$\Gamma_M = (n_m, n^{m_1 m_2}, n^{n_1 \ldots n_5}, \ldots) \in \mathbb{Z}^{d(\alpha_d)}$$

KK momenta

M2 wrappings

Interpretation as charges of (BPS)-states

Section constraint

ExFT has section constraint for closure of gauge algebra

$$\Gamma_M \times \Gamma_N \Big|_{\mathbf{R}_{\alpha_1}} = 0$$

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E_d	$\mathbf{R}_{\alpha_d}$	$\mathbf{R}_{\alpha_1}$
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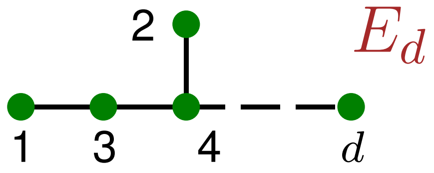
E_6	$\mathbf{27}$	$\overline{\mathbf{27}}$
E_7	$\mathbf{56}$	$\mathbf{133}$
E_8	$\mathbf{248}$	$\mathbf{3875} \oplus \mathbf{1}$



Section constraint

ExFT has section constraint for closure of gauge algebra

$$\Gamma_M \times \Gamma_N \Big|_{\mathbf{R}_{\alpha_1}} = 0 \qquad \left\{ \begin{array}{r} n^{m_1 m_2} n_{m_2} = 0 \\ 3 n^{[m_1 m_2} n^{m_3 m_4]} = n^{m_1 m_2 m_3 m_4 k} n_k \\ \ldots \end{array} \right.$$

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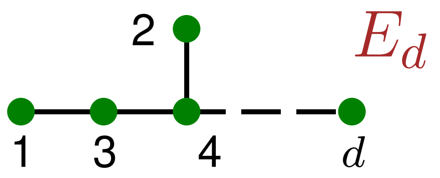

One solution: only KK momenta $n_k \neq 0$ (M-theory solution)

Section constraint

ExFT has section constraint for closure of gauge algebra

$$\Gamma_M \times \Gamma_N \Big|_{\mathbf{R}_{\alpha_1}} = 0 \qquad \left\{ \begin{array}{c} n^{m_1 m_2} n_{m_2} = 0 \\ 3 n^{[m_1 m_2} n^{m_3 m_4]} = n^{m_1 m_2 m_3 m_4 k} n_k \\ \ldots \end{array} \right.$$

E_d	$\mathbf{R}_{\alpha_d}$	$\mathbf{R}_{\alpha_1}$
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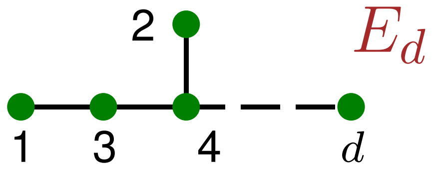

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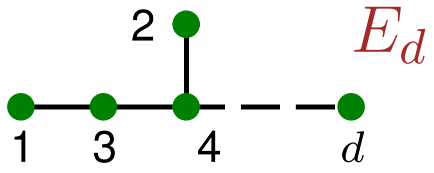

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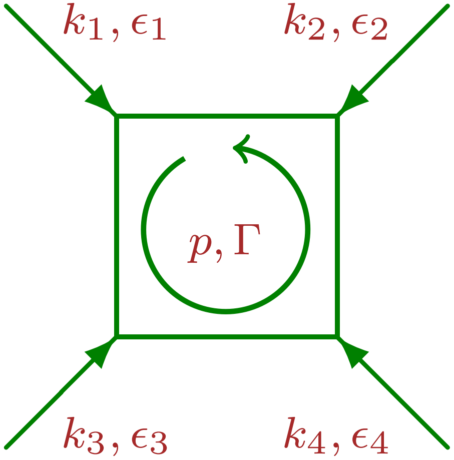
Use this set-up to compute amplitudes

ExFT amplitudes [Bossard AK]

Four-graviton one-loop amplitude reduces to scalar box [Bern et al.]

(avoid full Feynman rules)



contains polarisations

$$= \overset{\downarrow}{\mathcal{R}^4} A^{\text{1-loop}}(k_1, k_2, k_3, k_4)$$

with

$$A^{\text{1-loop}}(k_1, k_2, k_3, k_4) = \kappa^2 \int \frac{d^{11-d}p}{(2\pi)^{11-d}} \sum_{\substack{\Gamma \in \mathbb{Z}^{d(\alpha_d)} \\ \Gamma \times \Gamma = 0}} \frac{1}{((p-k_1)^2 + \ell^{-2}|Z(\Gamma)|^2)}$$

$$\times \frac{1}{(p^2+\ell^{-2}|Z(\Gamma)|^2)((p-k_1-k_2)^2+\ell^{-2}|Z(\Gamma)|^2)((p+k_4)^2+\ell^{-2}|Z(\Gamma)|^2)}$$

$$+ \text{ perms.}$$

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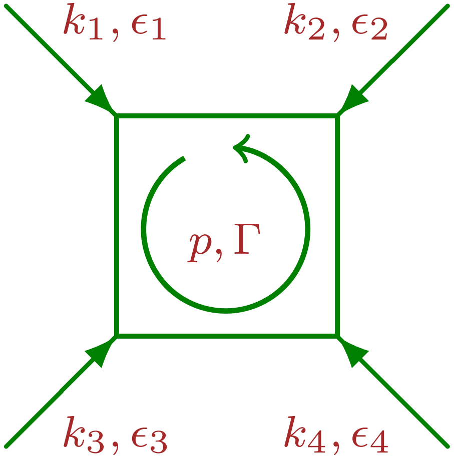

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loop momentum

loop BPS charge w/ $\frac{1}{2}$-BPS constraint

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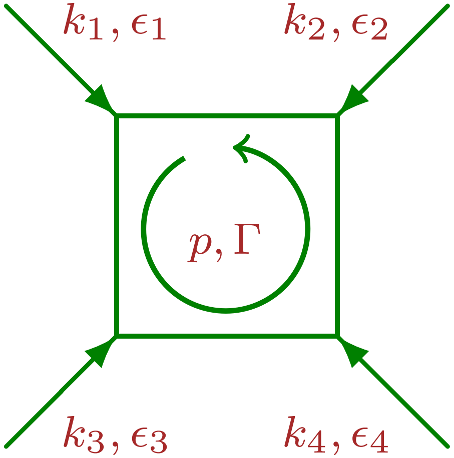

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w/ $\frac{1}{2}$-BPS constraint
'BPS-enhanced' SUGRA

$$A^{\text{1-loop}}(k_1, k_2, k_3, k_4) = \kappa^2 \int \frac{d^{11-d}p}{(2\pi)^{11-d}} \sum_{\substack{\Gamma \in \mathbb{Z}^{d(\alpha_d)} \\ \Gamma \times \Gamma = 0}} \frac{1}{((p-k_1)^2 + \ell^{-2}|Z(\Gamma)|^2)}$$

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$+$ perms.

$E_d/K(E_d)$ moduli Φ-dependent BPS mass

One-loop in ExFT

Treat loop integral over $d^{11-d}p$ with usual Schwinger and Feynman techniques:

$$A^{\text{1-loop}}(k_1, k_2, k_3, k_4) = 4\pi\ell^{9-d} \sum_{\substack{\Gamma\in\mathbb{Z}^{d(\alpha_d)} \\ \Gamma\times\Gamma=0}} \int_0^\infty \frac{dv}{v^{\frac{d-1}{2}}} \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3$$

$$\times \exp\left[\frac{\pi}{v}\left((1-x_1)(x_2-x_3)s + x_3(x_1-x_2)t - \ell^{-2}|Z|^2\right)\right] + \text{perms.}$$

Low energy from expanding in Mandelstam variables

$$s = -(k_1+k_2)^2, \quad t = -(k_1+k_4)^2, \quad u = -(k_1+k_3)^2.$$

$\Gamma = 0$ term corresponds to SUGRA in $D = 11 - d$; usual log threshold contribution ⇒ remove for analytic eff. action

Low energy correction terms

For lowest two orders

$$A^{\text{1-loop}}(s,t,u) = 4\pi\ell^6 \Bigg(\xi(d-3) E_{\alpha_d, \frac{d-3}{2}} + \frac{\pi^2 \ell^4 (s^2+t^2+u^2)}{720} \xi(d+1) E_{\alpha_d, \frac{d+1}{2}} + \dots \Bigg)$$

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R^4 correction

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$\nabla^4 R^4$ correction

Notation

- $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ [completed Riemann zeta]
- $E_{\alpha_d,s} = \frac{1}{2\zeta(2s)} \sum_{\substack{\Gamma\neq 0\\ \Gamma\times\Gamma=0}} |Z(\Gamma)|^{-2s}$ [Eisenstein series]

  Restricted lattice sum rewritable as single U-duality orbit! E.g. of KK momentum states.

Interpretation

- The 1-loop ExFT result for the $\frac{1}{2}$-BPS coupling R^4

$$\mathcal{E}^{D}_{(0,0)} = 4\pi\xi(d-3)E_{\alpha_d,\frac{d-3}{2}} \qquad ✓$$

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- The 1-loop ExFT result

  $$\mathcal{E}^{D}_{(1,0)} = \frac{4\pi^3}{45}\xi(d+1)E_{\alpha_d,\frac{d+1}{2}} \qquad (✗)$$

  for the $\frac{1}{4}$-BPS coupling $\nabla^4 R^4$ diverges for some d and requires renormalisation. Difficulty not fully surprising:
  - ExFT 1-loop calculation only contained $\frac{1}{2}$-BPS states
  - Should have 2-loop $\frac{1}{2}$-BPS contribution

Two loops in EFT (I)

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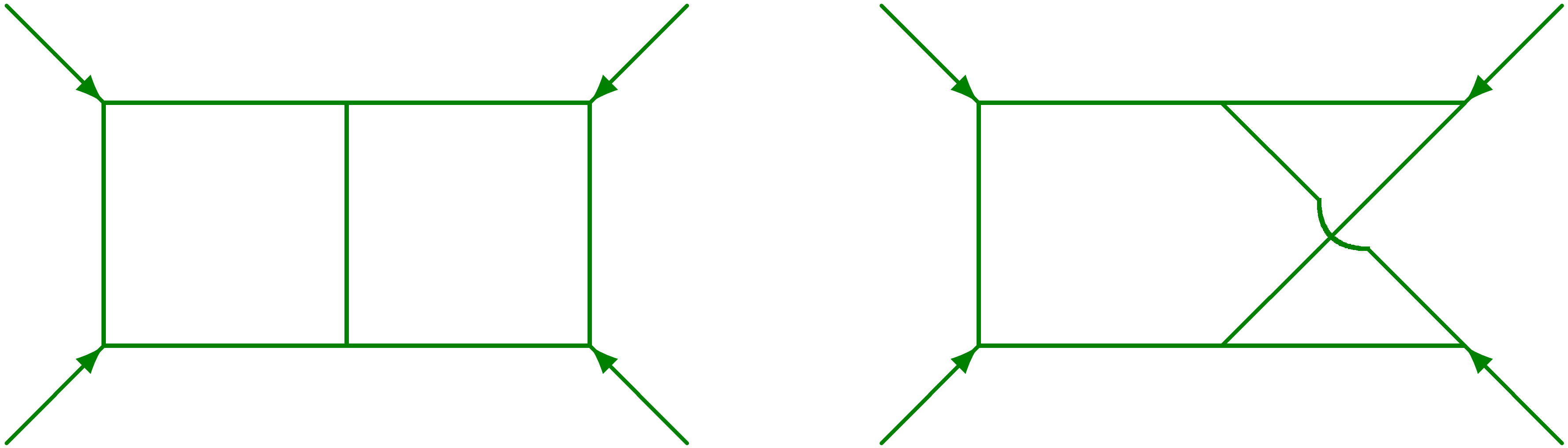
[Bern et al.] : combination of planar and non-planar scalar diagram at $L = 2$



After a few pages of calculation

$$A^{\text{2-loop}}(s,t,u) \sim \ell^6 \sum_{\substack{\Gamma_1,\Gamma_2 \\ \Gamma_i\times\Gamma_j=0}} \int_0^\infty \frac{d^3\Omega}{(\det\Omega)^{\frac{7-d}{2}}} e^{-\Omega^{ij}\langle Z(\Gamma_i)|Z(\Gamma_j)\rangle}$$

$$\times \left[\frac{\ell^4(s^2+t^2+u^2)}{6} + \frac{\ell^6(s^3+t^3+u^3)}{72}\Phi_{(0,1)}(\Omega) + \dots \right]$$

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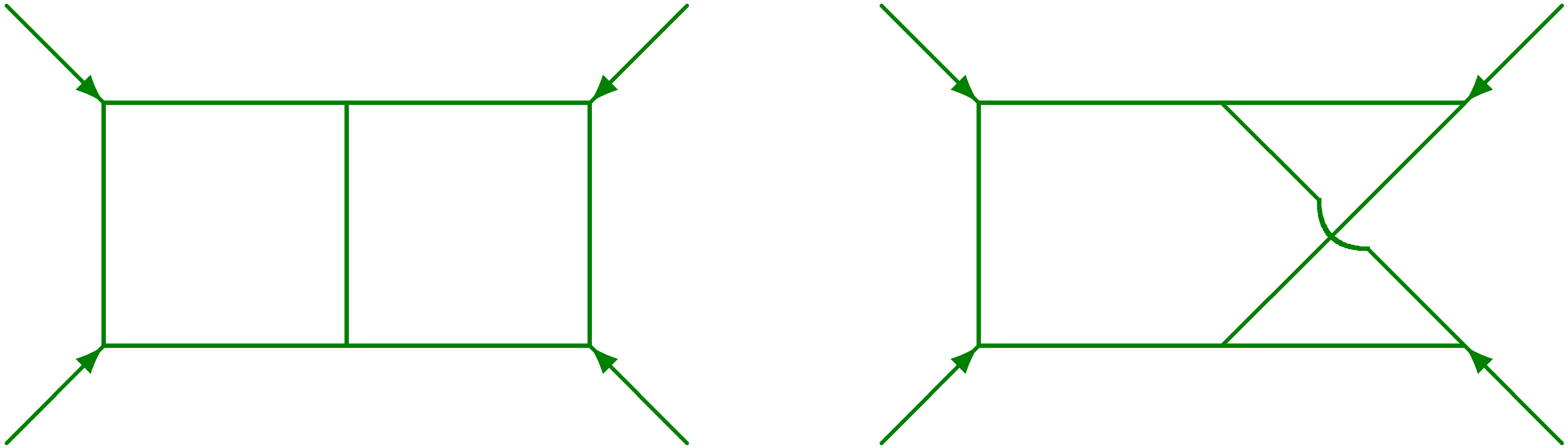

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Two loops in ExFT (II)

Focus first on $\nabla^4 R^4$ contribution. Need to understand

$$\sum_{\substack{\Gamma_1,\Gamma_2 \\ \Gamma_i\times\Gamma_j=0}} \int_0^\infty \frac{d^3\Omega}{(\det\Omega)^{\frac{7-d}{2}}} e^{-\Omega^{ij}\langle Z(\Gamma_i)|Z(\Gamma_j)\rangle}$$

where $\Omega^{ij} = \Omega = \begin{pmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{pmatrix}$

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Sum is restricted to pairs of charges Γ_1, Γ_2 satisfying

$$\Gamma_i \times \Gamma_j|_{\mathbf{R}_{\alpha_1}} = 0$$

Solutions can be parametrised by suitable parabolic decompositions, organised into orbits [Bossard, AK].

Two loops in ExFT (III)

Putting everything together (non-deg. orbit)

$$A^{\text{2-loop},\nabla^4 R^4}(s,t,u) = 8\pi\ell^{10}\xi(d-4)\xi(d-5)E_{\alpha_{d-1},\frac{d-4}{2}}$$

- This gives the correct function and coefficient for $3 \leq d \leq 8$ with the right coefficient. Case $d=5$ ($D=6$) trickier due to IR divergences.
- Other orbits subdominant at low energies except $d=5$.
- Certain doubling of $\frac{1}{2}$-BPS contributions from one loop and two loops. Need to discuss renormalisation and $\frac{1}{4}$-BPS states!

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$\frac{1}{4}$-BPS states violate the section constraint and do not have direct ExFT description.

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- ExFT: Saw above that divergences arise from 1-loop in ExFT. Adopt prescription that the corresponding 1-loop piece is removed. [Bossard, AK 2015]
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Both approaches lead to the same result:

Fully correct $\frac{1}{4}$-BPS coupling $\mathcal{E}^{D}_{(1,0)}$ for $\nabla^4 R^4$ ✓

$\frac{1}{8}$-BPS coupling $\nabla^6 R^4$ function $\mathcal{E}^D_{(0,1)}$

[String answer in $D{=}10$ [Green Vanhove], conj. in $D{=}8,9$ [Green,Russo Vanhove] [Basu] and in $D{=}6,7$ [Pioline]]

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The ExFT 2-loop calculation gives at order $\nabla^6 R^4$

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[D'Hoker, Green; Pioline]

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[D'Hoker, Green; Pioline]

- satisfies the correct inhomogeneous PDE ✓

$$\left(\Delta_{E_d/K(E_d)} - \frac{6(D-6)(14-D)}{D-2} \right) \mathcal{E}^{D}_{(0,1)} \simeq -\left(\mathcal{E}^{D}_{(0,0)} \right)^2$$

[Green, Russo, Vanhove] [Pioline], tensorial generalisation [Bossard, Verschinin]

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- further (homogeneous) contributions from ExFT at 1-loop and 3-loop
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Putting together provides concrete candidate $\mathcal{E}^D_{(0,1)}$ for $D \geq 4$

$\frac{1}{8}$-BPS coupling $\nabla^6 R^4$ function $\mathcal{E}^D_{(0,1)}$ (II)

To check candidate

$$\mathcal{E}^{D,\text{ExFT}}_{(0,1)} = \frac{2\pi^{5-d}}{9} \sum_{\substack{\Gamma_i\in\mathbb{Z}^{2d(\alpha_d)}\\ \Gamma_i\times\Gamma_j=0}}{}' \int_{\mathbb{R}^3_+} \frac{d^3\Omega}{(\det\Omega)^{\frac{7-d}{2}}} \left(L_1+L_2+L_3-5\frac{L_1L_2L_3}{\det\Omega}\right) e^{-\Omega^{ij}g(\Gamma_i,\Gamma_j)}$$

have to look at perturbative expansions in moduli space $\Phi\in E_d/K(E_d)$, e.g. $g_\text{s}\ll 1$

- Decompose the Γ-sum under $SO(d-1,d-1)$, e.g. for E_7

$$\underset{\Gamma_i}{\mathbf{56}} = \underset{q_i}{\mathbf{12}^{(1)}} \oplus \underset{\chi_i}{\mathbf{32}^{(0)}} \oplus \underset{N_i}{\mathbf{12}^{(-1)}}$$

Sec constr. $\Gamma_i\times\Gamma_j=0$: $(q_i,q_j)=0\,,\quad q^a_{(i}\gamma_a\chi_{j)}=0\,,\ldots$

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$$\begin{array}{ccccccc} \mathbf{56} & = & \mathbf{12}^{(1)} & \oplus & \mathbf{32}^{(0)} & \oplus & \mathbf{12}^{(-1)} \\ \Gamma_i & & q_i & & \chi_i & & N_i \\ & & \text{momentum/} & & \text{RR} & & \text{dual} \\ & & \text{winding} & & & & \end{array}$$

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- ‘Populate’ Γ-sum layer by layer and carry out sums and integrals (requires analytic continuation)

$\frac{1}{8}$-BPS coupling $\nabla^6 R^4$ function $\mathcal{E}^D_{(0,1)}$ (III)

Many hours of work later: [Bossard AK, Pioline]

Produces exactly the expected perturbative terms! ✓

also need to include homogeneous pieces

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also need to include homogeneous pieces

$$\mathcal{E}^D_{(0,1)} = g_s^{\frac{2D-28}{D-2}} \left[\frac{2}{3}\zeta(3)^2 g_s^{-2} + \mathcal{E}^{D,\text{1-loop}}_{(0,1)} + g_s^2 \mathcal{E}^{D,\text{2-loop}}_{(0,1)} + g_s^4 \mathcal{E}^{D,\text{3-loop}}_{(0,1)} + O(e^{-1/g_s}) \right]$$

$\mathcal{E}^{D,h\text{-loop}}_{(p,q)}$ is perturbative string computation on genus-h surface, only T-duality invariant and automorphic form on $SO(d,d)/SO(d)\times SO(d)$ ($\Rightarrow$ Narain theta lift)

E.g.

$$\mathcal{E}^{D,\text{1-loop}}_{(0,1)} = \frac{4\pi\zeta(3)}{3}\xi(d-3)E^{D_{d-1}}_{\frac{d-3}{2}\Lambda_1} + \frac{8\pi^4}{567}\xi(d+3)E^{D_{d-1}}_{\frac{d+3}{2}\Lambda_1}$$

Requires renormalisation, also including $\frac{1}{4}$-BPS states

$\frac{1}{8}$-BPS coupling $\nabla^6 R^4$ function $\mathcal{E}^{D}_{(0,1)}$ (IV)

Other results in [**Bossard, AK Pioline**] on $\mathcal{E}^{D}_{(0,1)}$

- verified correct divergence structure and $\log g_{\mathrm{s}}$ terms
- checked decompactification limit $D \to D+1$, i.e. $E_{d-1} \subset E_d$
- showed equivalence with ‘string multiplet’ conjecture of [**Pioline**] in $D{=}6,7$
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- calculated non-perturbative effects in some cases (non-trivial Fourier coefficients). E.g. $\frac{1}{8}$-BPS stack of 1D5+ 1D1+ 1D1+ND1 on T^6 gives index generated by

$$-\frac{\theta_4(2\tau)}{\eta(4\tau)^6} = \sum_{n\geq -1} \tilde{c}(n) q^n$$ (inherited from Kawazumi–Zhang)

✓ 4D black hole counting [Maldacena, Moore Strominger] [Shih Strominger, Yin] [Pioline]

Summary

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- Instances of BPS states counting
- ExFT and its quantum loops powerful tool to compute U-duality invariant terms in string effective action
- Useful tools for dealing with section constraint
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Thank you for your attention!