Arithmetic of decay walls through continued fractions

A new exact dyon counting solution in $\mathcal{N} = 4$ CHL models

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with Gabriel Cardoso and Suresh Nampuri, arXiv:2007.10302

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Introduction

Understanding the microscopic origin of Black Hole entropy remains a central question in Quantum Gravity.

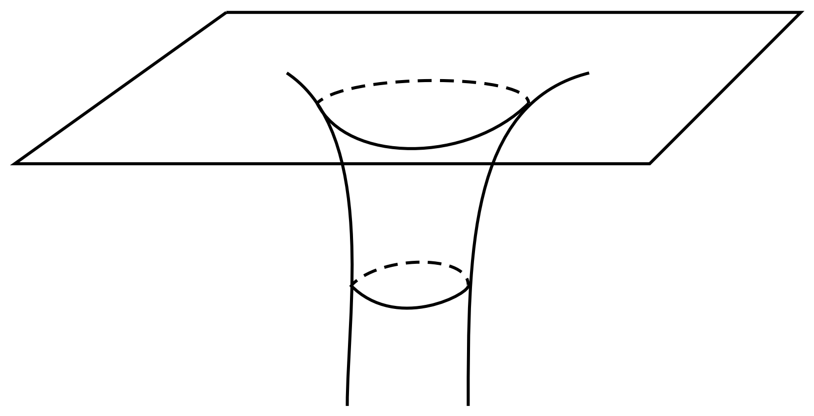
$$S_{stat}(Q) = \ln d(Q) \leftrightarrow S_{BH}(Q)$$

Address it in the context of $\mathcal{N} = 4$ supersymmetric String Theory. Concretely: study the degeneracies of a special type of $1/4-$BPS dyons, namely decadent dyons.

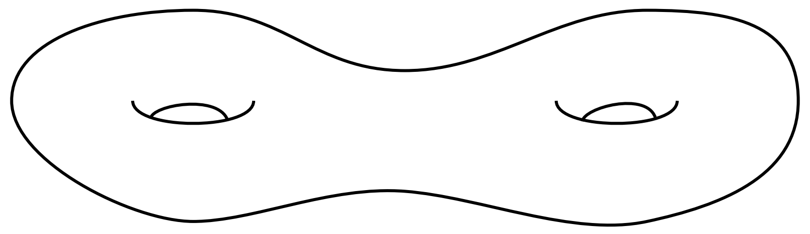
Our work has been inspired by the recent results of [Chowdhury, Kidambi, Murthy, Reys, Wrase '19]. Here we propose a new systematic way to tackle these issues.

Introduction

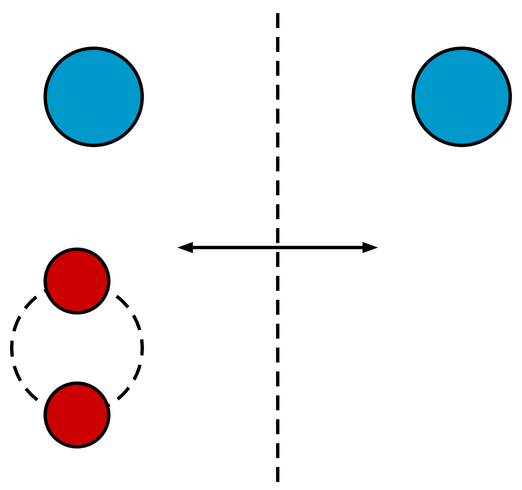
- Dyonic degeneracies



- Siegel modular forms



- Wall-crossing



- Continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

CHL models

Consider $\mathcal{N} = 4$ CHL models obtained by compactifying heterotic string theory on $T^5 \times S^1/\mathbb{Z}_N$ with $N = 1,2,3,5,7$. These models have $r = \dfrac{48}{N+1} + 4$ Abelian gauge fields. The $S-$duality group is

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \ s.t. \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

and the $T-$duality group is a subgroup $T(\mathbb{Z})$ of $O(r-6, 6; \mathbb{Z})$.

The $U-$duality group is

$$\Gamma_1(N) \times T(\mathbb{Z})$$

[Chaudhuri, Hockney, Lykken ’95]

[…]

Dyons in CHL models

A generic $1/4$-BPS state carries electric, $\vec{Q}$, and magnetic, $\vec{P}$, charge w.r.t. the r Abelian gauge fields. Dyonic degeneracies are functions of the $T-$duality invariants

$$m = P \cdot P/2 \in \mathbb{Z}, \qquad n = Q \cdot Q/2 \in \mathbb{Z}/N, \qquad \ell = P \cdot Q \in \mathbb{Z}$$

$$d(\vec{P}, \vec{Q}) = d(m, n, \ell)$$

We differentiate between two types of dyons:

- Single centre $1/4$-BPS dyonic black holes with finite or zero horizon area in two-derivative gravity

Immortal

- Two-centred bound states of $1/2$-BPS constituents

Can decay

[Cheng, Verlinde '07]

Dyons in CHL models

Dyons in $\mathbb{Z}_N$ CHL models with $N = 1,2,3,5,7$ have two discrete $U-$duality invariants

Area $\sim \sqrt{\Delta}$ $\quad$ $\Delta = Q^2P^2 - (Q \cdot P)^2 = 4mn - \ell^2$ $\quad$ Discriminant

(In this talk $I = 1$) $\quad$ $I = \gcd(Q_iP_j - Q_jP_i), \quad 1 \leq i,j \leq r$ $\quad$ Torsion

[Banerjee, Sen ’07]

Single centre $1/4$-BPS black holes with finite horizon area have $\Delta > 0$. Will focus on

$$\Delta \leq 0$$

$\Delta < 0$ are always two-centred states

$\Delta = 0$ can be two-centred and single centred states

Siegel modular forms

The generating function for dyonic degeneracies in these CHL models is a modular form of a subgroup of the genus-2 modular group $Sp(2,\mathbb{Z})$

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1} \qquad \Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \quad Im(\Omega) > 0$$

$$\Phi_k(\Omega) \to \det(C\Omega + D)^k \, \Phi_k(\Omega) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{G} \subset Sp(2,\mathbb{Z})$$

[Dijgkraaf, Verlinde, Verlinde '96]
[Jatkar, Sen '05]

$$k = \frac{24}{N+1} - 2$$

For $N = 1,2,3,5,7,$
$k = 10,6,4,2,1$

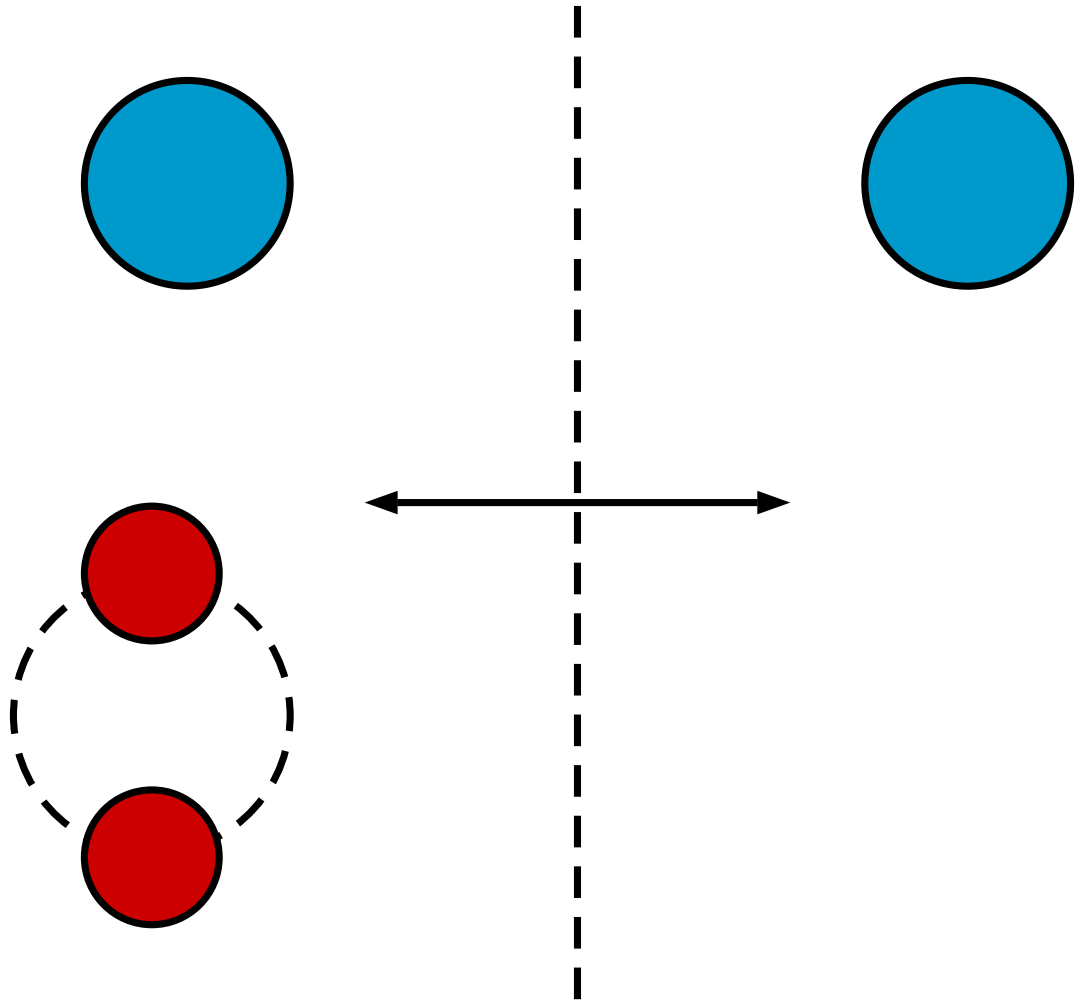
$$\frac{1}{\Phi_k(\rho,\sigma,v)} = \sum_{\substack{m,nN \geq -1 \\ m,nN,\ell \in \mathbb{Z}}} (-1)^{\ell+1} d(m,n,\ell) e^{2\pi i(m\rho + n\sigma + \ell v)}$$

Invariant under $\Gamma_0(N)$

Wall-crossing

Wall of marginal stability

Single centre



Two-centred bound state

Poles in the Siegel modular form

$$d(m,n,\ell) = (-1)^{\ell+1}\int_C d\rho d\sigma dv\, p^{-m}q^{-n}y^{-\ell}\Phi_k^{-1}$$

$$p = e^{2\pi i\rho}, q = e^{2\pi i\sigma}, y = e^{2\pi i v}$$

Changing the contour C

$|x| < 1$

Ex: $\dfrac{1}{1-x} = \sum_{n\geq 0} x^n$ or $-\sum_{n\geq 1} x^{-n}$

$|x| > 1$

[Sen, '07]

[Dabholkar, Gaiotto Nampuri '07]

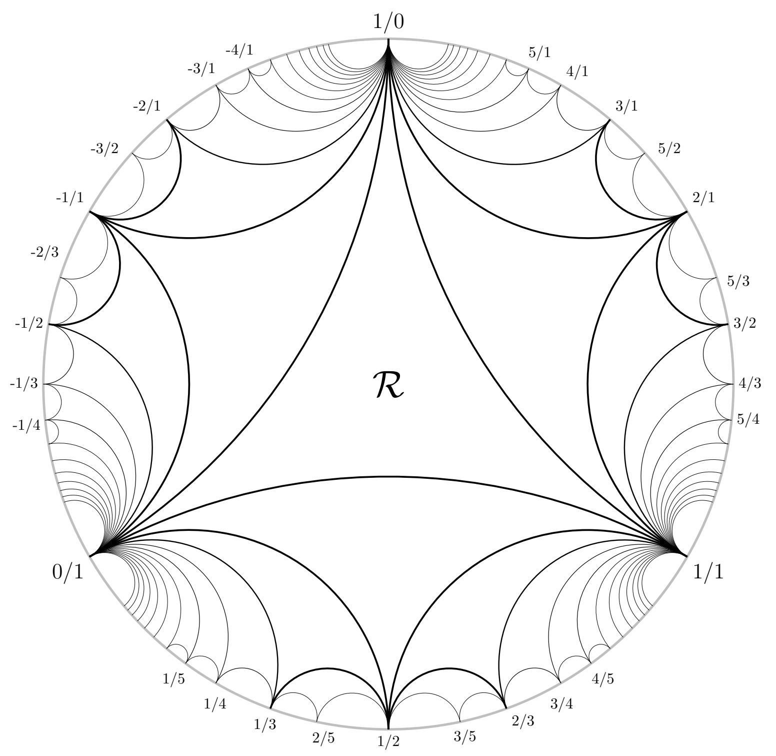
Poles and walls

$\dfrac{1}{\Phi_k}$ has an infinite family of second order poles in the (ρ, σ, v) space

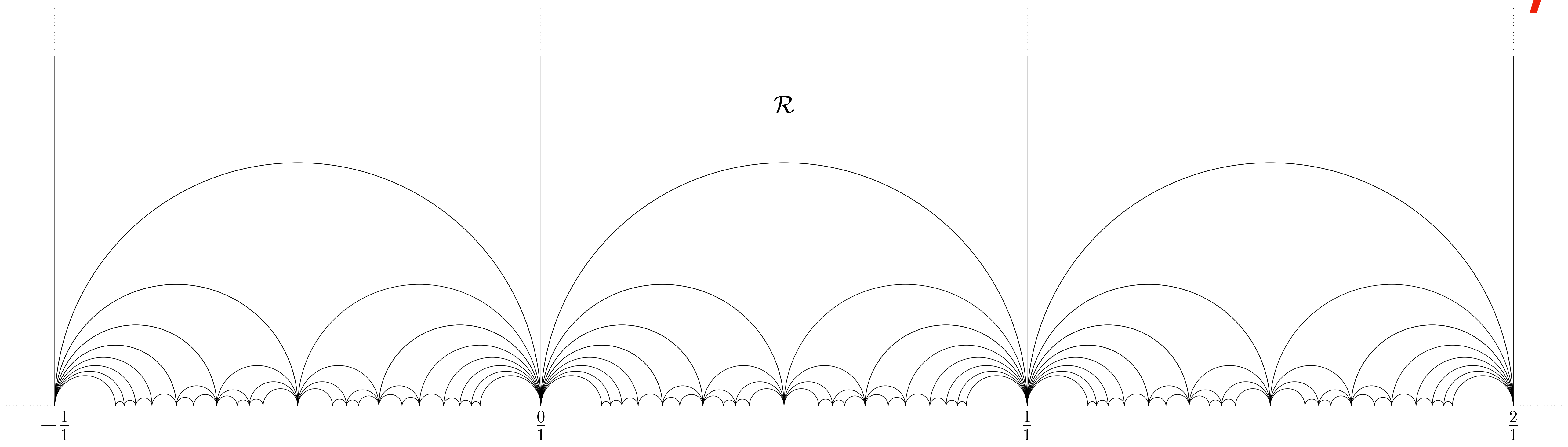
$$pq\sigma_2 + rs\rho_2 + (ps+qr)v_2 = 0, \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma_0(N)$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z}) \mid r \equiv 0 \mod N \right\}$$

Represent walls in the $(v_2/\sigma_2, \rho_2/\sigma_2)$ plane by lines joining $\dfrac{p}{r}$ and $\dfrac{q}{s}$



$N = 1$



[Sen, '07]

Dyonic decay

The decay modes at each wall of marginal stability are determined by the corresponding matrix in $\Gamma_0(N)$

$$\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma_0(N) : \begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} p\,(s\,Q - q\,P) \\ r\,(s\,Q - q\,P) \end{pmatrix} + \begin{pmatrix} q\,(-r\,Q + p\,P) \\ s\,(-r\,Q + p\,P) \end{pmatrix}.$$

The decay corresponding to the identity matrix is the 'elementary' split

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} Q \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ P \end{pmatrix}, \; v_2 = 0$$

[Sen, '07]

‘Elementary’ split

The change in the degeneracy from the ‘elementary’ split

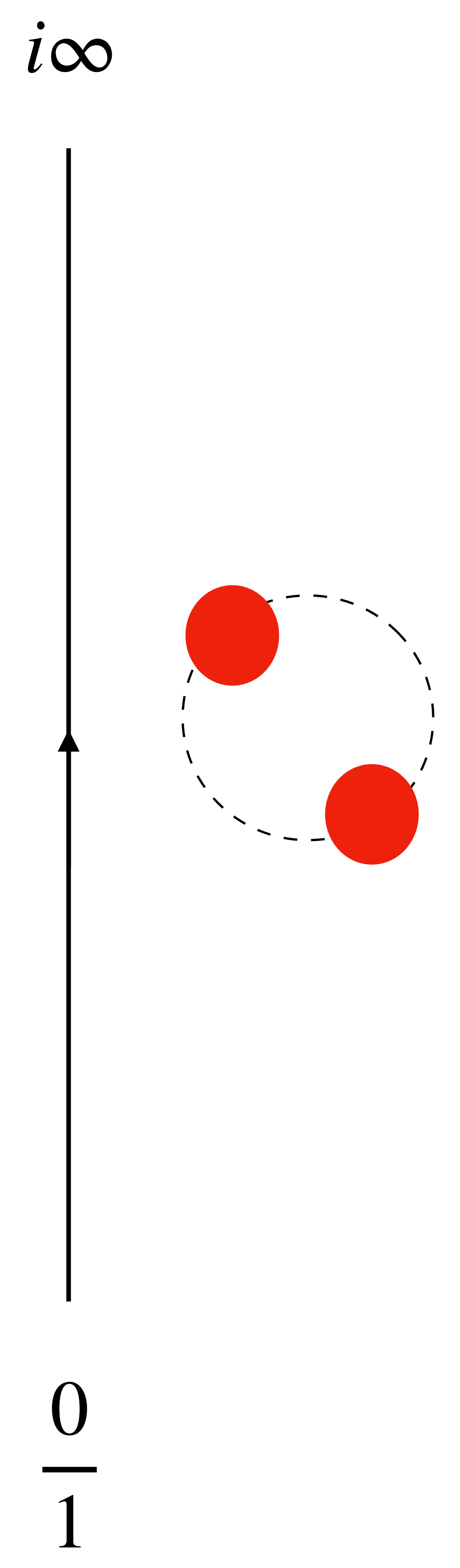
$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} Q \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ P \end{pmatrix}, \; v_2 = 0$$

$$\frac{1}{\Phi_k(\rho, \sigma, v)} \xrightarrow{v \to 0} \frac{1}{v^2} \frac{1}{f^{(k)}(\rho)} \frac{1}{f^{(k)}(\frac{\sigma}{N})}$$

where $f^{(k)}(\rho) = \eta(\rho)^{k+2} \eta(N\rho)^{k+2}$,

$$\frac{1}{f^{(k)}(\rho)} = \sum_{m=-1}^{\infty} d_1(m)\, e^{2\pi i m \rho} \quad , \quad \frac{1}{f^{(k)}(\sigma/N)} = \sum_{n=-1/N}^{\infty} d_2(n)\, e^{2\pi i n \sigma} \, .$$

is $$\Delta\, d(m, n, \ell) \;=\; (-1)^{\ell+1}\, |\,\ell\,|\, d_1(m)\, d_2(n)$$



[Sen, ’07]

Generic split

This is extended to the other walls by mapping a generic dyon decay to the elementary T-wall

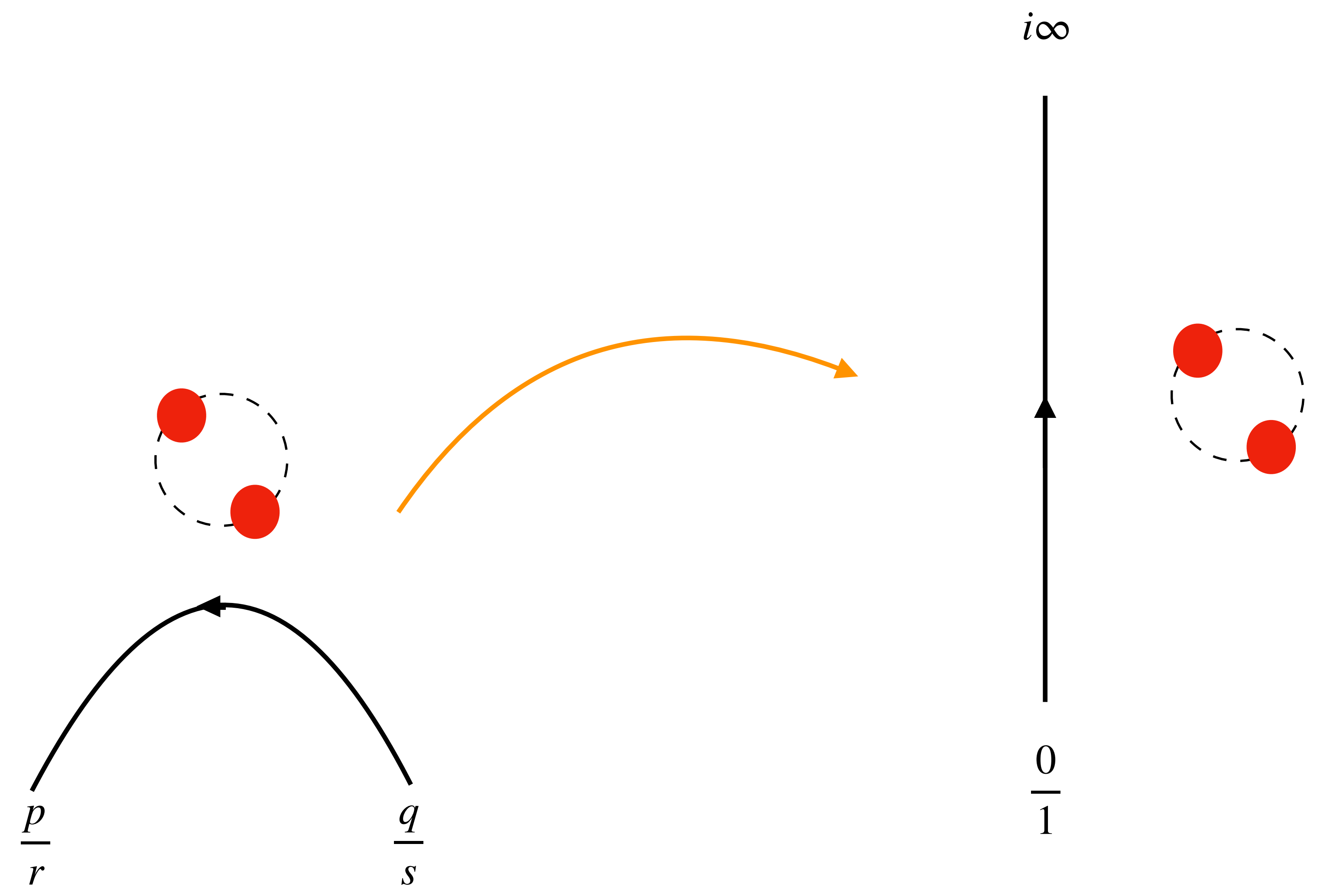
$$\gamma^{-1}\begin{pmatrix}Q\\P\end{pmatrix}=\begin{pmatrix}Q_\gamma\\P_\gamma\end{pmatrix}\to\begin{pmatrix}sQ-qP\\0\end{pmatrix}+\begin{pmatrix}0\\-rQ+pP\end{pmatrix}=\begin{pmatrix}Q_\gamma\\0\end{pmatrix}+\begin{pmatrix}0\\P_\gamma\end{pmatrix}.$$

The charge bilinears (m,n,ℓ) transform as

$$Q_\gamma^2/2=n_\gamma=s^2n+q^2m-qs\,\ell,$$

$$P_\gamma^2/2=m_\gamma=r^2n+p^2m-pr\,\ell$$

$$Q_\gamma\cdot P_\gamma=\ell_\gamma=-2rs\,n-2pq\,m+(ps+qr)\,\ell\ .$$



[Sen, '07] [Cheng, Verlinde '07] [Sen, '11]

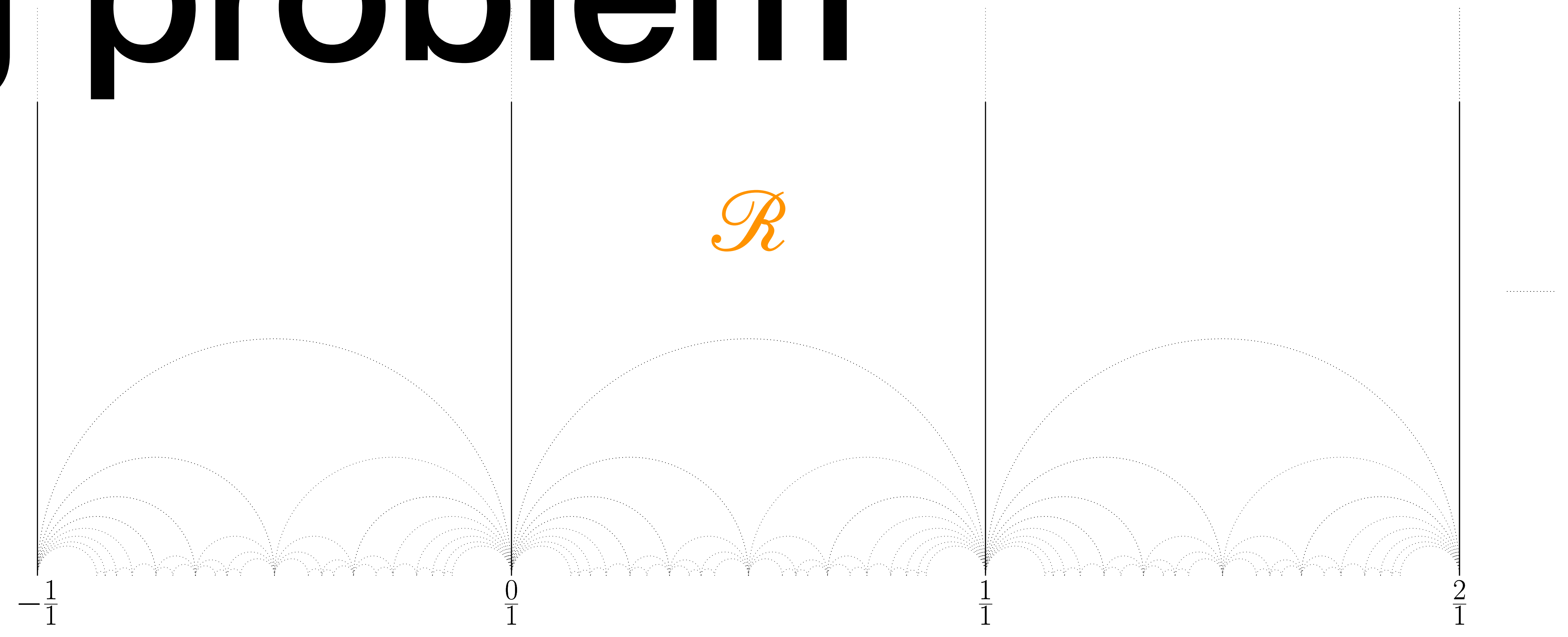
Wall-crossing formula

The wall-crossing jump contribution of a dyon $\begin{pmatrix} Q \\ P \end{pmatrix}$ across a generic line of marginal stability, labelled by a $\Gamma_0(N)$ matrix γ, to the dyonic degeneracy formula is equal to the jump contribution of the dyon $\begin{pmatrix} Q_\gamma \\ P_\gamma \end{pmatrix}$ across the elementary T-wall

$$\boxed{\Delta_\gamma\, d(m,n,\ell) = (-1)^{\ell_\gamma+1}\,|\,\ell_\gamma\,|\,d_1(m_\gamma)\,d_2(n_\gamma)\ .}$$

[Sen, '11]

Dyon counting problem



Consider the dyonic charge bilinears (m, n, ℓ) satisfying $\Delta = 4mn - \ell^2 \leq 0$ and $0 \leq \ell \leq m$. Want to compute $d(m, n, \ell)$ in the $\mathscr{R}$-chamber.

Want to find a decay path in the upper-half. Given (m, n, ℓ) construct a sequence of walls $W(m, n, \ell)$ crossed when going from the $\mathscr{R}$-chamber to a point $*$. Then,

$$d(m,n,\ell) = d_* + \sum_{i=1}^{k} \Delta_i = d_* + (-1)^{\ell+1} \sum_{\substack{i=1 \\ \gamma_i \in W(m,n,\ell)}}^{k} |\ell_{\gamma_i}| \, d_1(m_{\gamma_i}) \, d_2(n_{\gamma_i}) \ ,$$

Hopefully known (i.e. 0) or computable

$k = 10$

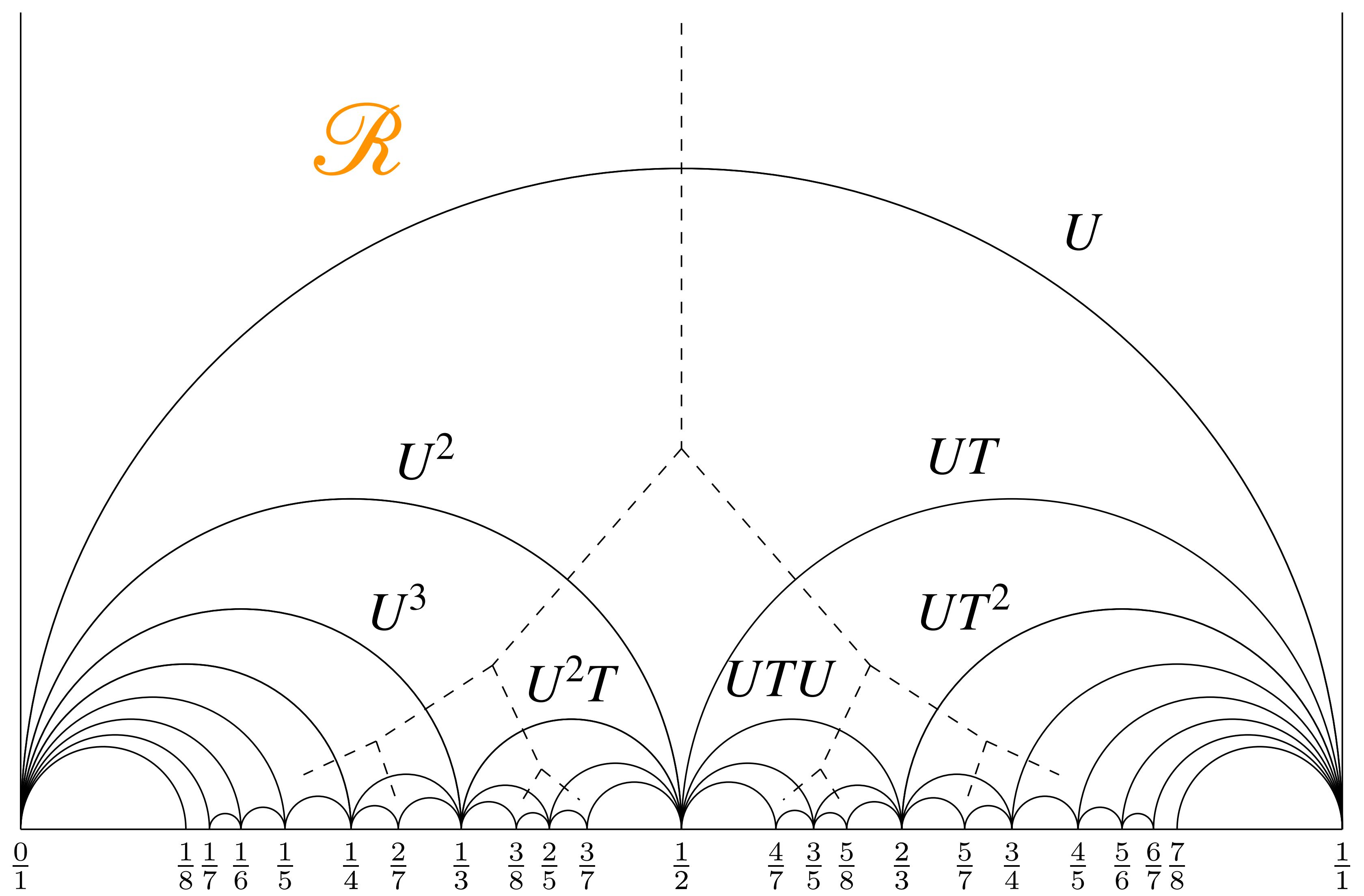
Solution for $N = 1$

$\Delta < 0$

Downward trajectory given by consecutive left-right choice associated to the matrices $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

This defines our arithmetic of decay walls: multiply T and U matrices to generate the walls of marginal stability. Decompose matrix in $W(m, n, \ell)$ as

$$\gamma = U^{s_1} T^{s_2} U^{s_3} \cdots T^{s_r}, \quad s_r \geq 0$$



$$W(m, n, \ell) = \left\{ U, U^2, \ldots, U^{s_1}, U^{s_1} T, \ldots, U^{s_1} T^{s_2}, U^{s_1} T^{s_2} U, \ldots, U^{s_1} T^{s_2} U^{s_3}, \ldots, \gamma_* \right\}$$

γ_* determines all s_i: Only need to determine γ_*

Solution for $N = 1$

$\Delta < 0$

To find γ_*, for $\Delta = 4mn - \ell^2 < 0$ and $0 \leq \ell \leq m$, we know that there is a γ_* such that

$$m_{\gamma_*} < -1 \text{ or } n_{\gamma_*} < -1$$

[Sen '11]

which implies $d_* = 0$.

Consider $m_{\gamma_*} < 0 \implies m_{\gamma_*} = r^2 n + p^2 m - pr\ell < 0$

$$\frac{\ell}{2m} - \frac{\sqrt{-\Delta}}{2m} < \frac{p}{r} < \frac{\ell}{2m} + \frac{\sqrt{-\Delta}}{2m}$$

For the conditions $r, s > 0$ and $\ell_{\gamma_*} > 0$ it is sufficient

$$0 \leq \frac{\ell}{2m} - \frac{q}{s} \leq \frac{1}{rs}$$

Solution for $N = 1$

$\Delta < 0$

Two conditions

$$\frac{\ell}{2m} - \frac{\sqrt{-\Delta}}{2m} < \frac{p}{r} < \frac{\ell}{2m} + \frac{\sqrt{-\Delta}}{2m}$$

$$0 \leq \frac{\ell}{2m} - \frac{q}{s} \leq \frac{1}{rs}$$

Solved by

$$\begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} \ell/g \\ 2m/g \end{pmatrix}, \quad g = \gcd(\ell, 2m)$$

Find $\begin{pmatrix} q \\ s \end{pmatrix}$ satisfying $ps - qr = 1$

$$\gamma_* = \begin{pmatrix} \ell/g & q \\ 2m/g & s \end{pmatrix} \text{ with } \begin{aligned} m_{\gamma_*} &= m\,\Delta/g^2 \\ \ell_{\gamma_*} &= -s\Delta/g \\ n_{\gamma_*} &= q^2 m + s^2 n - qs\ell \end{aligned}$$

Continued fractions $N = 1$ $\Delta < 0$

Apply Euclid's algorithm to find the gcd of ℓ and $2m$:

$$\begin{aligned} \ell &= a_0\, 2m + r_0\,, \\ 2m &= a_1\, r_0 + r_1\,, \\ r_0 &= a_2\, r_1 + r_2\,, \\ r_1 &= a_3\, r_2 + r_3\,, \\ &\vdots \\ r_{n-3} &= a_{n-1}\, r_{n-2} + r_{n-1}\,, \\ r_{n-2} &= a_n\, r_{n-1}\,, \end{aligned}$$

The set of quotients $\{a_0, a_1, a_2, \ldots, a_n\}$ is elegantly encoded in the finite continued fraction representation of $\ell/2m$:

$$\frac{\ell}{2m} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

and determines the matrices

$$\gamma_* = \begin{pmatrix} \ell/g & q \\ 2m/g & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix} \quad n \text{ odd}$$

$$\gamma_* = \begin{pmatrix} q & \ell/g \\ s & 2m/g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \quad n \text{ even}$$

Determines the sequence $W(m, n, \ell)$

Result for $N = 1$

$\Delta < 0$

Given m, n, ℓ with $\Delta = 4mn - \ell^2 < 0$ and $0 \leq \ell \leq m$, compute

$$\ell/2m = [a_0, a_1, \ldots, a_r].$$

This defines

$$W(m, n, \ell)$$

and then in the $\mathscr{R}$-chamber,

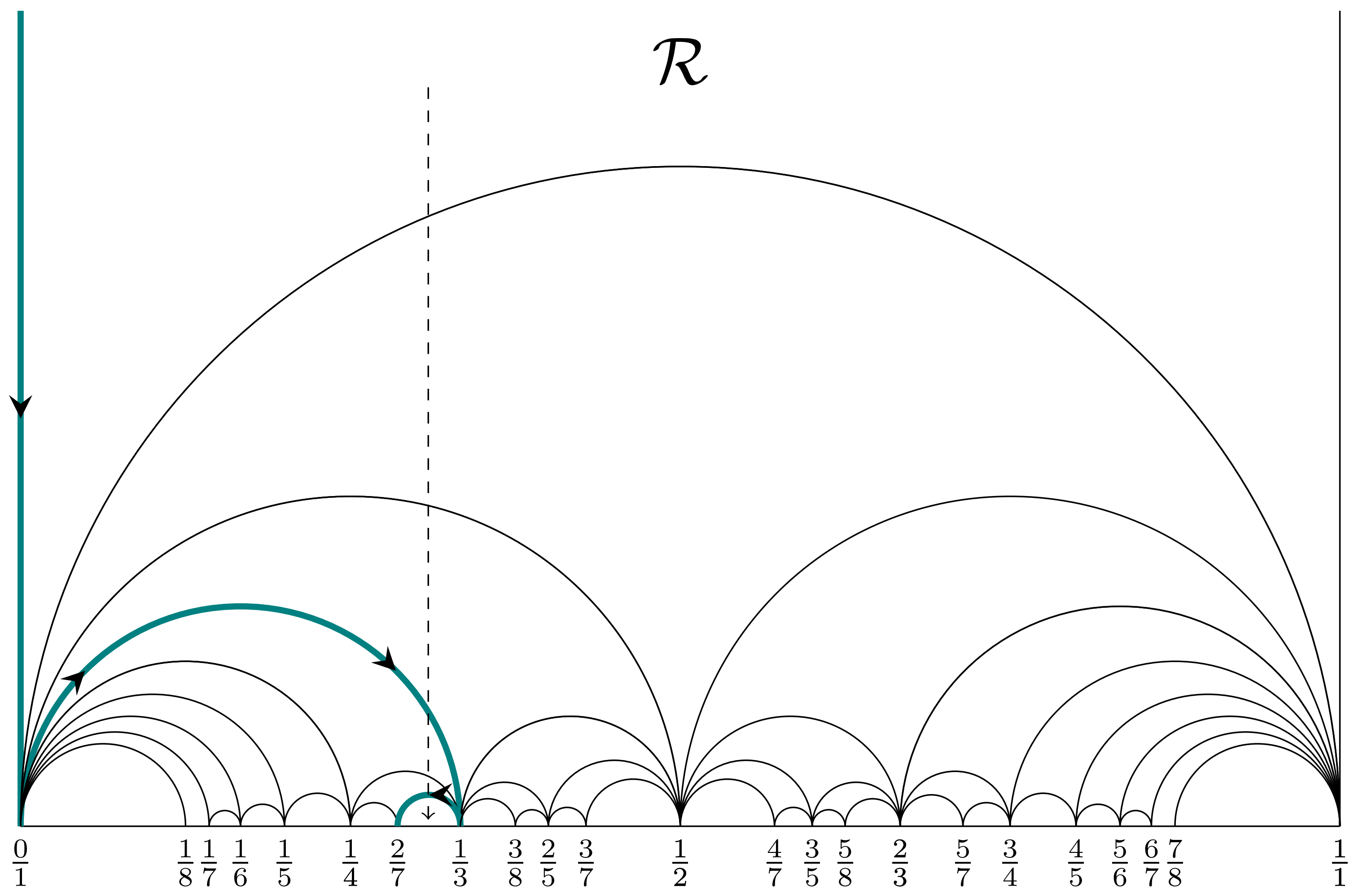
$$\boxed{d(m, n, \ell) = d_* + \sum_{i=1}^{k} \Delta_i = d_* + (-1)^{\ell+1} \sum_{\substack{i=1 \\ \gamma_i \in W(m,n,\ell)}}^{k} |\ell_{\gamma_i}|\, d_1(m_{\gamma_i})\, d_2(n_{\gamma_i})}$$

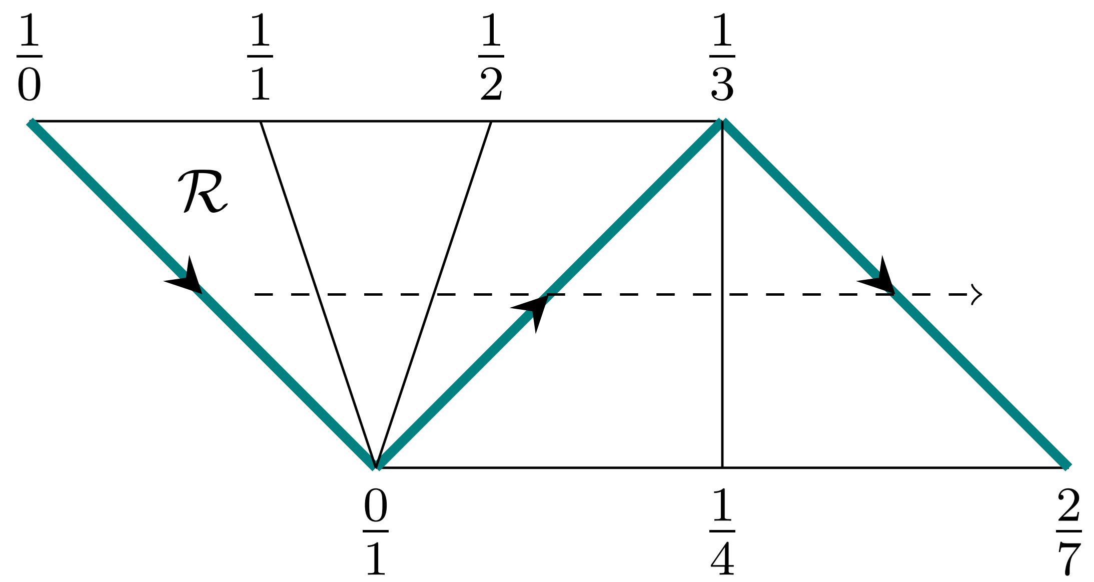
The set $W(m, n, \ell)$ is determined by the the ratio of the two numbers ℓ and $2m$

Diagrammatic representation $N=1 \; \Delta<0$

Take $\ell/2m = 2/7 = [0;a_1,a_2] = [0;3,2] \;\; = \cfrac{1}{3+\cfrac{1}{2}}$

$$\gamma_* = \begin{pmatrix}1 & 0\\ a_1 & 1\end{pmatrix}\begin{pmatrix}1 & a_2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2\\ 3 & 7\end{pmatrix}$$





$$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}, \begin{pmatrix}1 & 2\\ 3 & 7\end{pmatrix}$$

In general, from the $\mathscr{R}-$chamber to the chamber below $\ell/2m$, one crosses

$$\sum_{i=1}^{n} a_i \text{ walls}$$

Endpoint degeneracy $N = 1\ \Delta < 0$

$$m_{\gamma_*} = m\,\Delta/g^2 < 0$$

Two options:

- $m_{\gamma_*} < -1 \quad \Rightarrow \quad d_* = 0$
- $m_{\gamma_*} = -1 \quad \Rightarrow \quad d_* \neq 0$ but can be computed:

Arbitrarily negative

$$(m, n, \ell)_{\gamma_*} = (-1, n_*, \ell_*) \xrightarrow{T^j \ (j>0)} (-1, n_* - j^2 - j\ell_* = n_j, \ell_* + 2j = \ell_j)$$

$$d_* = \sum_{\mu \in \{T, T^2, \ldots, T^{j_0}\}} \ell_{\gamma_*\mu}\, d_1(-1)\, d_1(n_{\gamma_*\mu}) = \sum_{j=1}^{j_0} (\ell_* + 2j)\, d_1(-1)\, d_1(n_* - j^2 - j\ell_*)\,.$$

Equivalent to:

Extend continued fraction to $[0; a_1, a_2, \ldots, a_n, j_0]$

$\Delta = 0, N = 1$

Same logic, but now have

$$m_{\gamma_*} = m\,\Delta/g^2 = 0,$$
$$\ell_{\gamma_*} = -\,s\Delta/g = 0.$$

New relevant discrete invariant: $\gcd(m, n, \ell)$

Since $PSL(2,\mathbb{Z})$ action preserves $\tilde{g} = \gcd(m, n, \ell)$,

$$n_{\gamma_*} = q^2 m + s^2 n - qs\ell = \tilde{g}$$

We have a sequence of decay walls given by continued fraction of $\ell/2m$ and last wall yielding an immortal dyon with charge bilinears

$$(m_{\gamma_*}, n_{\gamma_*}, \ell_{\gamma_*}) = (0,\tilde{g},0) \text{ or } (\tilde{g},0,0).$$

$$d(0,\tilde{g},0)\,?$$

$\Delta = 0, N = 1$

Expand the inverse Igusa cusp form,

[Dabholkar, Murthy, Zagier '12]

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \psi_{-1} e^{-2\pi i \rho} + \sum_{m=0}^{\infty} \left(\psi_m^F(\sigma,v) + \psi_m^P(\sigma,v)\right) e^{2\pi i m \rho} \,,$$

$$\psi_0^F(\sigma) = 2\,\frac{E_2(\sigma)}{\eta^{24}(\sigma)} = -2 \sum_{n \geq -1} n\, d_1(n)\, q^n$$

$d(0,\tilde{g},0)$

Therefore

$$d(m,n,\ell) = 2\,\tilde{g}\, d_1(\tilde{g}) - \sum_{\gamma \in W(m,n,\ell)} |\ell_\gamma|\, d_1(m_\gamma)\, d_1(n_\gamma)$$

Note For $\Delta = 0$ the immortal degeneracy is only a function of $\tilde{g}$: $d_{immortal}(m,n,\ell) = 2\tilde{g} d_1(\tilde{g})$

$N > 1$

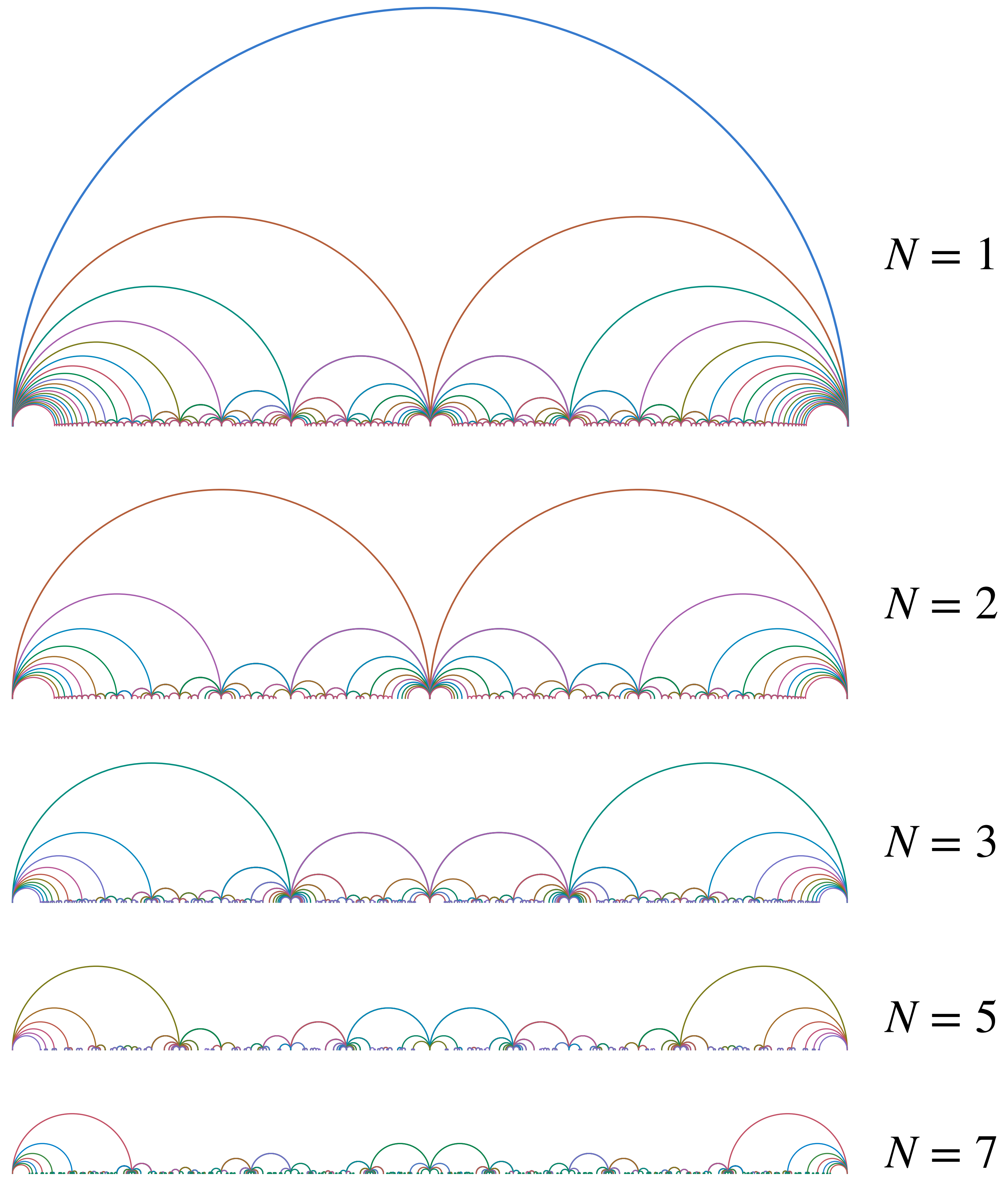
$\Delta \geq 0$

The logic is the same, but the details more intricate.

Proceed as earlier, build set $W(m, n, \ell)$ from the continued fraction of $\ell/2m$ but now select the matrices in $\Gamma_0(N)$.

For $\Delta = 0$ immortal counting function different,

$$\psi^F_{-k,0}(\sigma) = \frac{k+2}{12\,(N-1)} \frac{E_2(\sigma/N) - E_2(\sigma)}{\eta^{k+2}(\sigma/N)\,\eta^{k+2}(\sigma)} = \sum_{nN \in \mathbb{N}_0} d_N(n)\, q^n$$



[Bossard, Cosnier-Horeau, Pioline '18]

Summary

We use continued fractions to set up an arithmetic of decay walls which we used to explicitly compute all the polar coefficients of

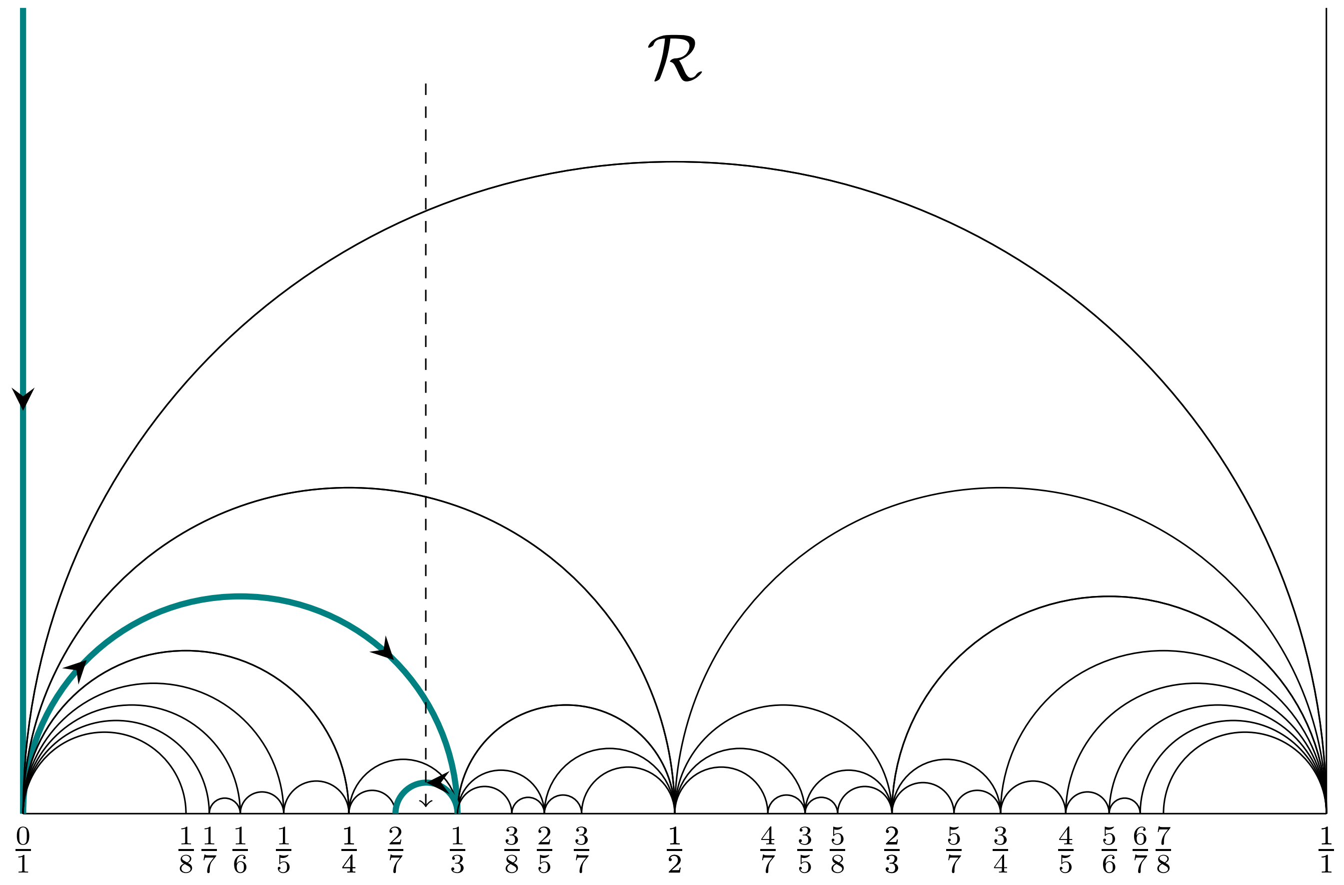
$$\frac{1}{\Phi_k}$$

The appearance of continued fractions is naturally explained by the theory of Binary Quadratic Forms $(m, n, \ell) \leftrightarrow mx^2 - \ell xy + ny^2$.

Consistent with [Moore '98]

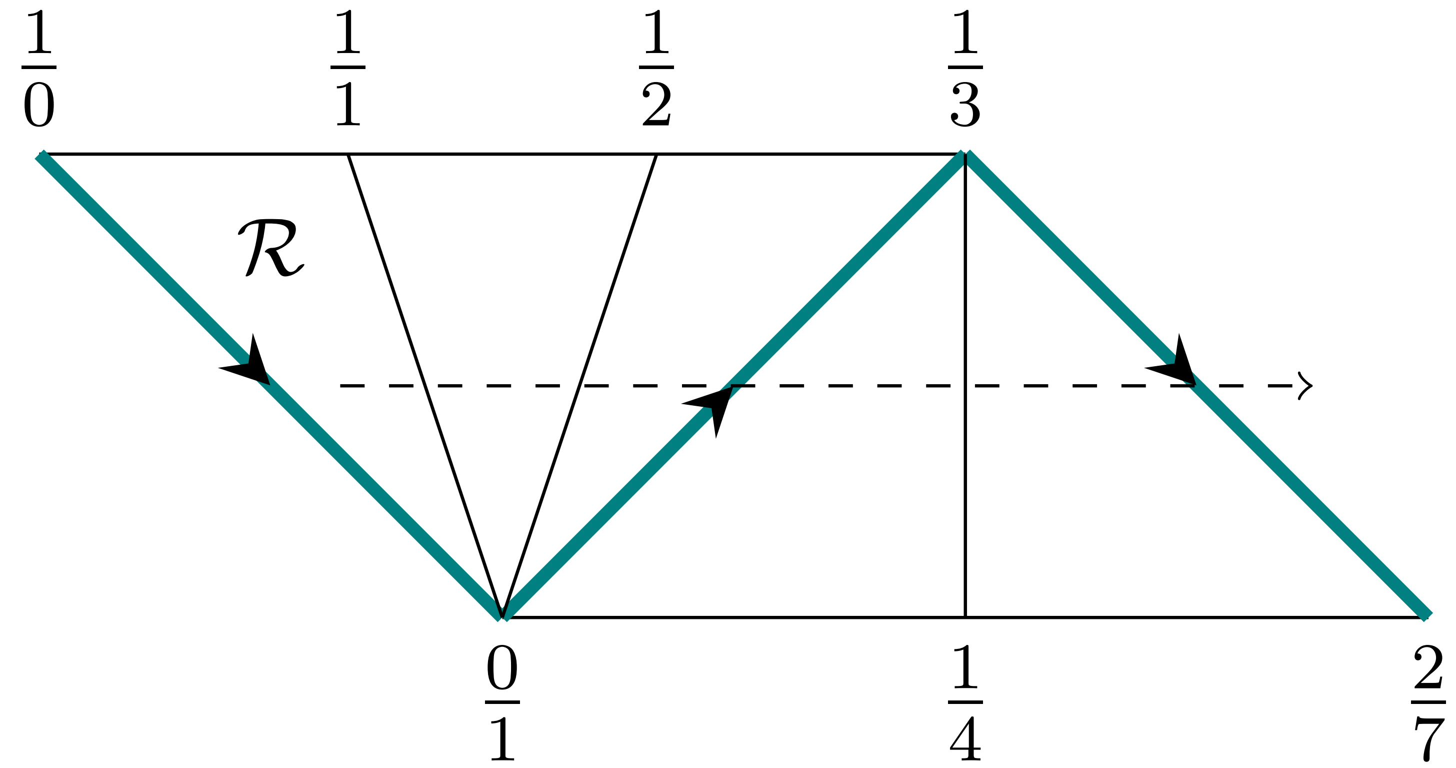
[Benjamin, Kachru, Ono, Rolen '18], [Banerjee, Bhand, Dutta, Sen, Singh '20], [Borsten, Duff, Marrani '20] ...

Thank you



Example

$N = 1$



$\ell/2m = 2/7 = [0;3,2]$ with walls $\begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix}, \begin{pmatrix}1&0\\3&1\end{pmatrix}, \begin{pmatrix}1&1\\3&4\end{pmatrix}, \begin{pmatrix}1&2\\3&7\end{pmatrix}$

$(m,n,\ell) = (49,4,28)$, $\Delta = 0$ $\longleftrightarrow$ $d_*(49,4,28) = d(1,0,0) = 2d_1(1) = 648$

$$(25,4,20), (9,4,12), (1,4,4), (1,1,2), (1,0,0)\,.$$

$$d(49,4,28) = 648 - \big(20d_1(25)d_1(4) + 12d_1(9)d_1(4) + 4d_1(1)d_1(4) + 2d_1(1)d_1(1)\big)$$
$$= -\,459\,542\,242\,945\,399\,203\,613\,080.$$

```
c[49, 4, 28] = 459542242945399203613080
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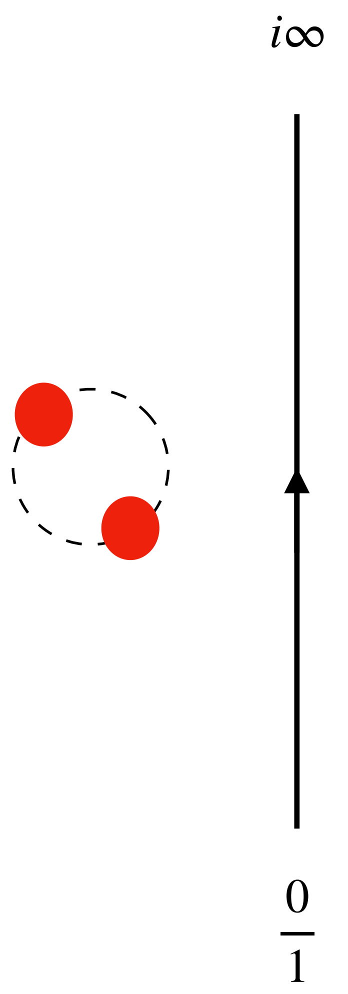
Orienting the walls

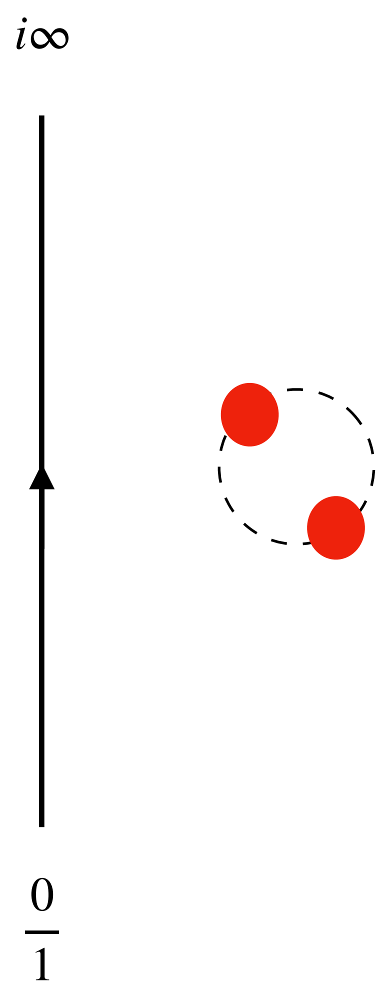
We give an orientation to the walls: from q/s to p/r. Then,

[Sen; 1104.1498]

For $\ell > 0$ the bound states exists to the left of the elementary T-wall



For $\ell < 0$ the bound states exists to the right of the elementary T-wall



This is extended to the other walls by mapping a generic dyon decay to the elementary T-wall

$$\gamma^{-1}\begin{pmatrix}Q\\P\end{pmatrix} = \begin{pmatrix}Q_\gamma\\P_\gamma\end{pmatrix} \to \begin{pmatrix}sQ-qP\\0\end{pmatrix}+\begin{pmatrix}0\\-rQ+pP\end{pmatrix} = \begin{pmatrix}Q_\gamma\\0\end{pmatrix}+\begin{pmatrix}0\\P_\gamma\end{pmatrix}.$$

Wall distinction

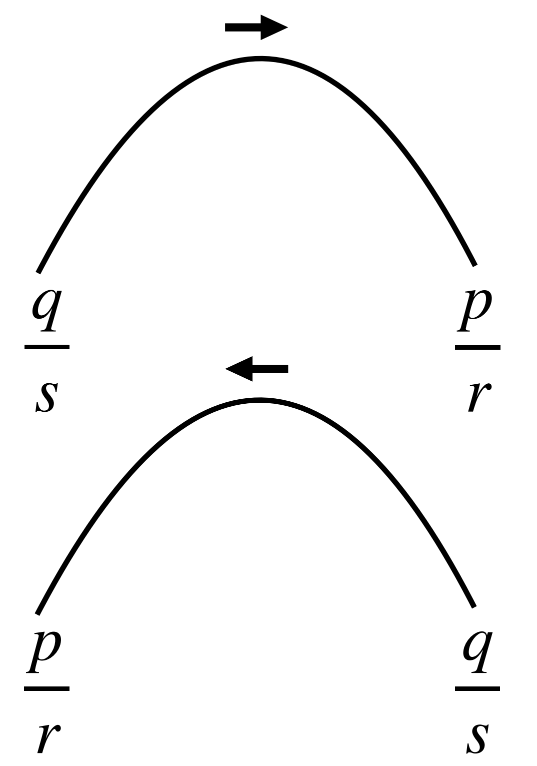
$$\frac{p}{r} - \frac{q}{s} = \frac{1}{rs}$$

$$ps - qr = 1$$

Define sets

$$\Gamma_+(N) = \left\{ \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma_0(N) \,|\, rs > 0 \right\}$$

$$\Gamma_-(N) = \left\{ \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma_0(N) \,|\, rs < 0 \right\}$$



Note that, for $N = 1$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma_0(1)$ and $\Gamma_+(1) = \Gamma_-(1)S$

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} S = \begin{pmatrix} -q & p \\ -s & r \end{pmatrix}$$

Want: $\gamma \in \Gamma_+(N)$ with $\ell_\gamma > 0$
or $\gamma \in \Gamma_-(N)$ with $\ell_\gamma < 0$

Explicit formula

$N = 1$ $\Delta < 0$

Continued fractions give the following explicit formula for (m, n, ℓ) with $4mn - \ell^2 < 0$ and $0 \leq \ell \leq m$:

Compute $\ell/2m = [0; a_1, \ldots, a_r]$. Define from these r numbers $m_{ij}, n_{ij}, \ell_{ij}$

$$d(m, n, \ell) = d_* + (-1)^{\ell+1} \sum_{i=1}^{r} \sum_{j=1}^{a_i} |\ell_{ij}| \, d_1(m_{ij}) d_1(n_{ij}) \; .$$

When $d_* \neq 0$, formula is actually simpler: it imposes $\ell = m$ and $n = \dfrac{1}{4}(m-1)$

$$d(m, n, \ell) = \left(\sum_{q=1}^{\left\lfloor \sqrt{\frac{m}{4}+1} - \frac{1}{2} \right\rfloor} (2q+1) \, d_1(n - q^2 - q) \right) + \tfrac{1}{2}(m+1) \, \left(d_1(n)\right)^2 + d_1(n) \; .$$

$N > 1, \Delta = 0$

For $\Delta = 0$ dyons, need to compute $d_* = d(0,\hat{g},0)$

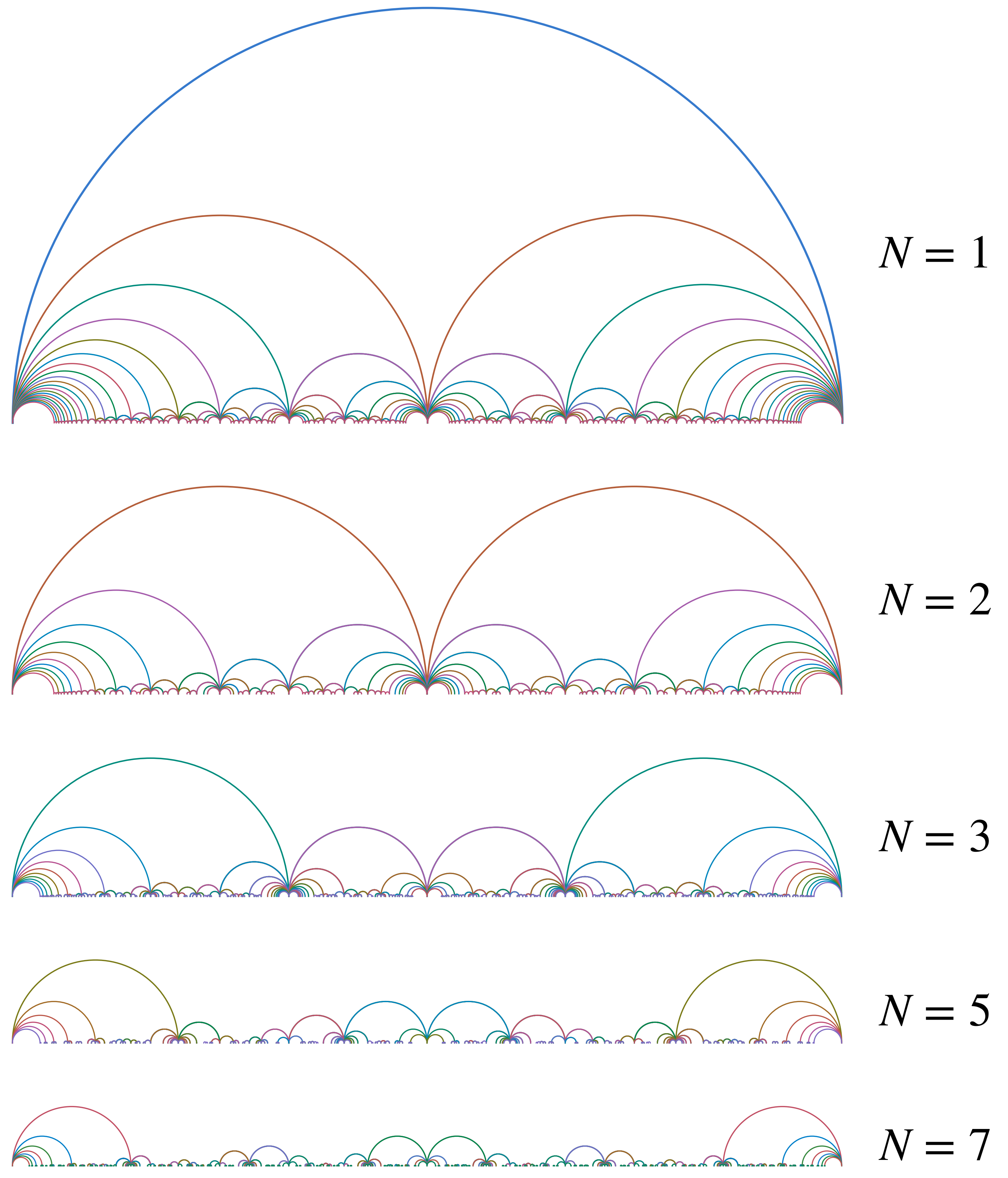
where $\hat{g} = \dfrac{\gcd(m, nN, \ell)}{N}$. Expand $1/\Phi_k$ and find

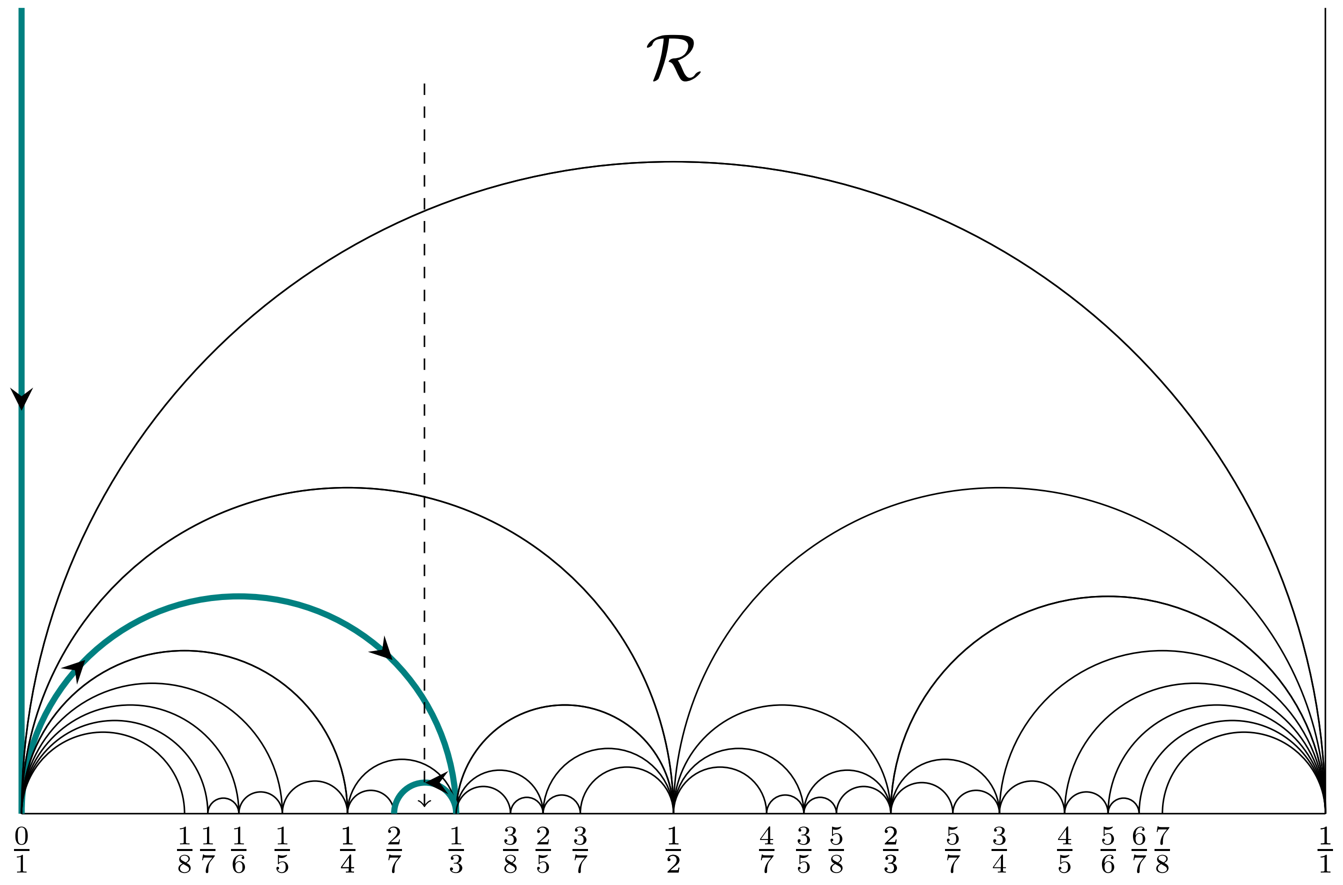
$$\psi^F_{-k,0}(\sigma) = \frac{k+2}{12\,(N-1)}\,\frac{E_2(\sigma/N) - E_2(\sigma)}{\eta^{k+2}(\sigma/N)\,\eta^{k+2}(\sigma)} = \sum_{nN \in \mathbb{N}_0} d_N(n)\, q^n$$

Immortal part

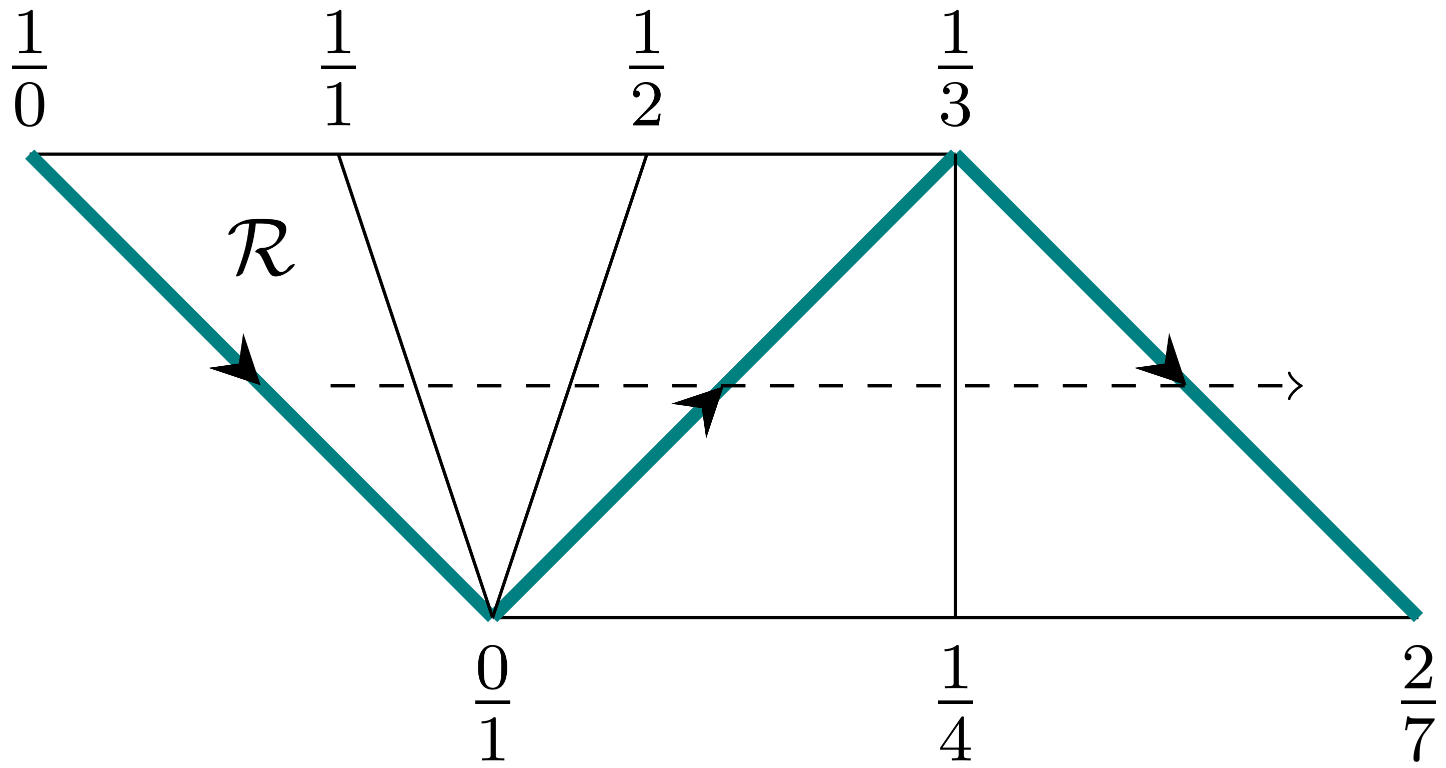
giving the final formula

$$d(m,n,\ell) = -\left(d_N(\hat{g}) + \sum_{\gamma \in W_N(m,n,\ell)} |\ell_\gamma|\, d_1(m_\gamma)\, d_2(n_\gamma) \right)$$





Examples



$N = 1$

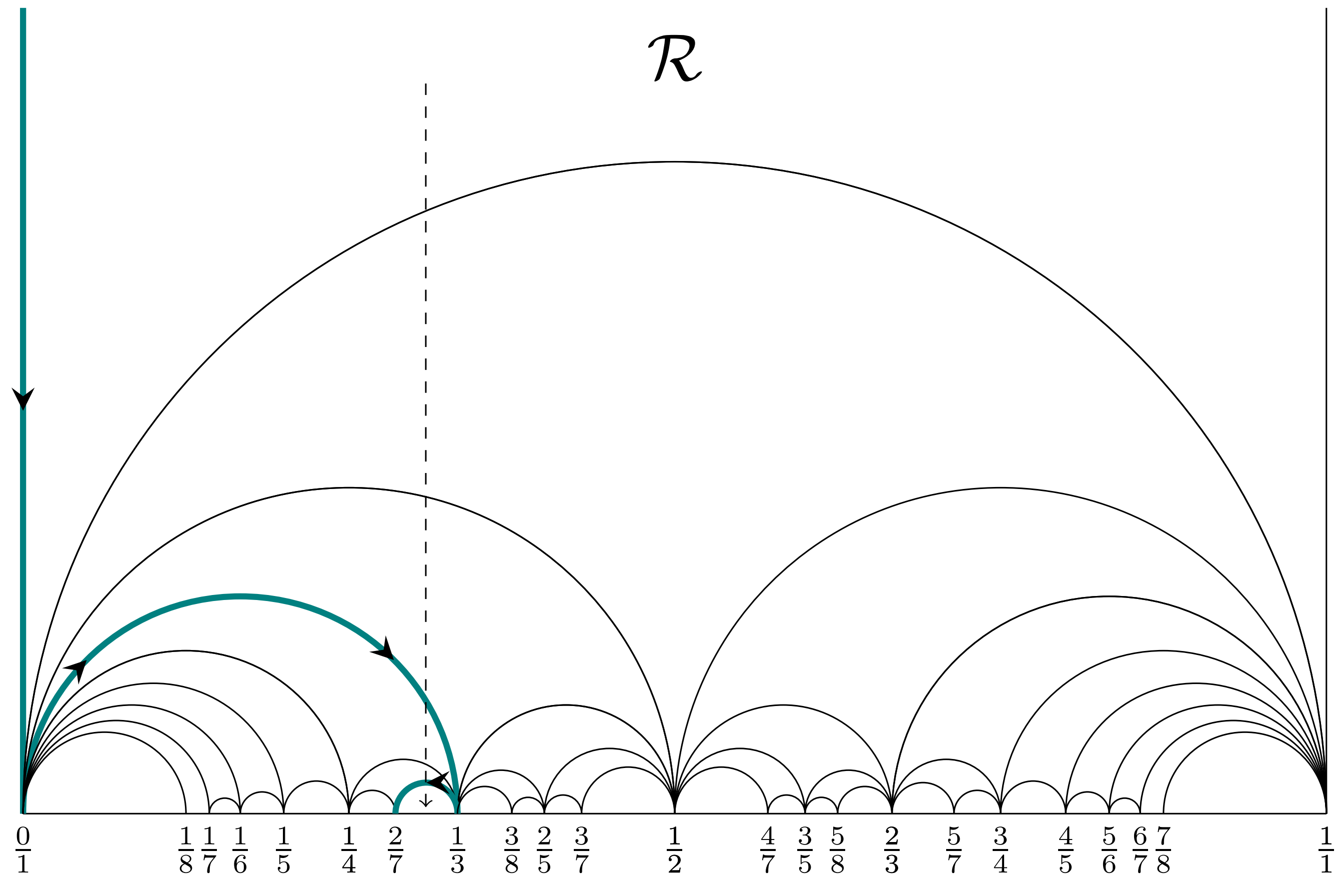
$\ell/2m = 2/7 = [0; 3,2]$ with walls $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$

1. a) $(m, n, \ell) = (14,1,8)$, $\Delta = -8$

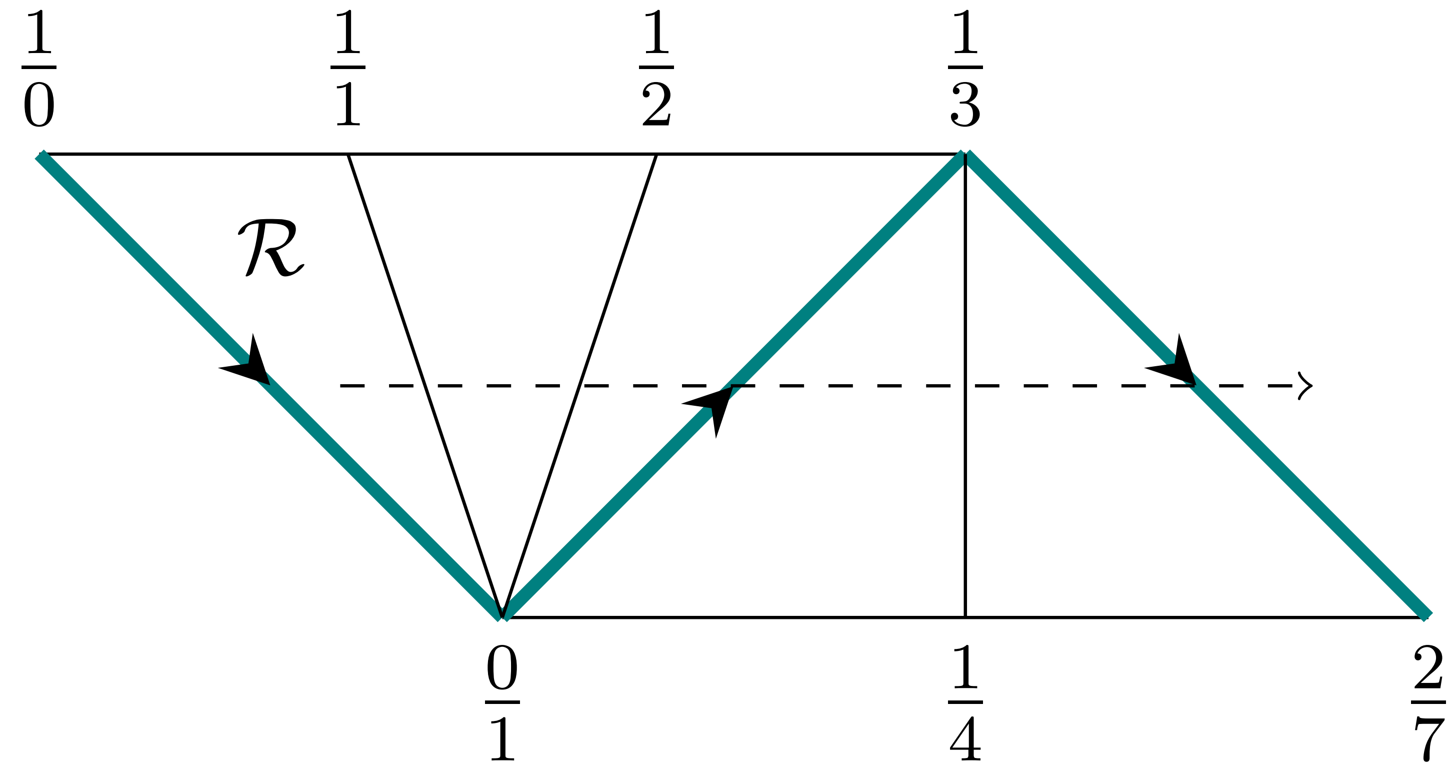
$$(7,1,6), (2,1,4), (-1,1,2), \cancel{(-1,-2,4)}, \cancel{(-1,-7,6)}\,.$$

$$d(14,1,8) = (-1)\Big(6d_1(7)d_1(1) + 4d_1(2)d_1(1) + 2d_1(-1)d_1(1)\Big) = -58\,671\,297\,648.$$

```
c[14, 1, 8] = 58 671 297 648
```



Examples



$N = 1$

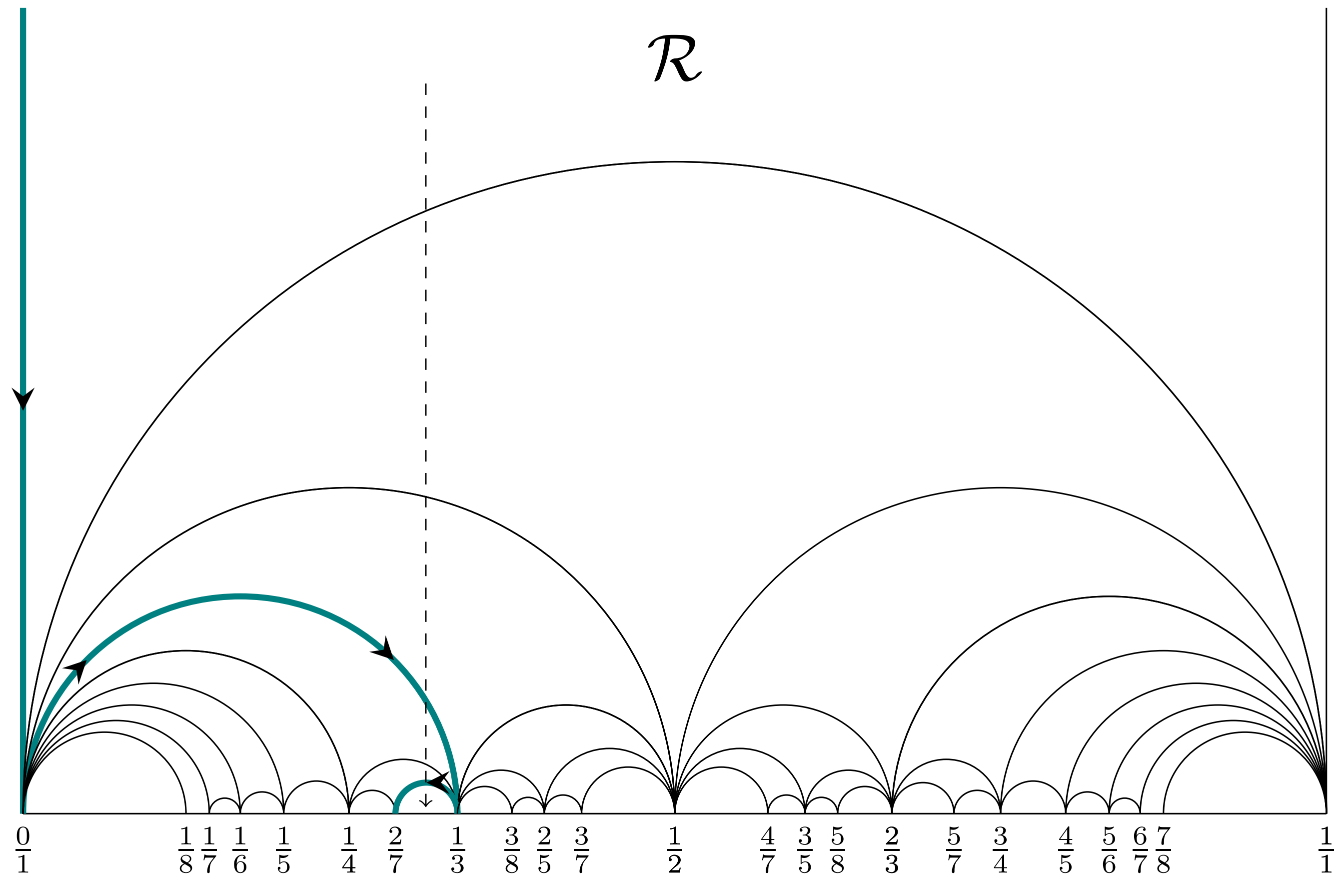
$\ell/2m = 2/7 = [0;3,2]$ with walls $\begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix}, \begin{pmatrix}1&0\\3&1\end{pmatrix}, \begin{pmatrix}1&1\\3&4\end{pmatrix}, \begin{pmatrix}1&2\\3&7\end{pmatrix}$

1. b) $(m,n,\ell) = (49,4,28)$, $\Delta = 0$ $\longleftrightarrow$ $d_*(49,4,28) = d(1,0,0) = 2d_1(1) = 648$

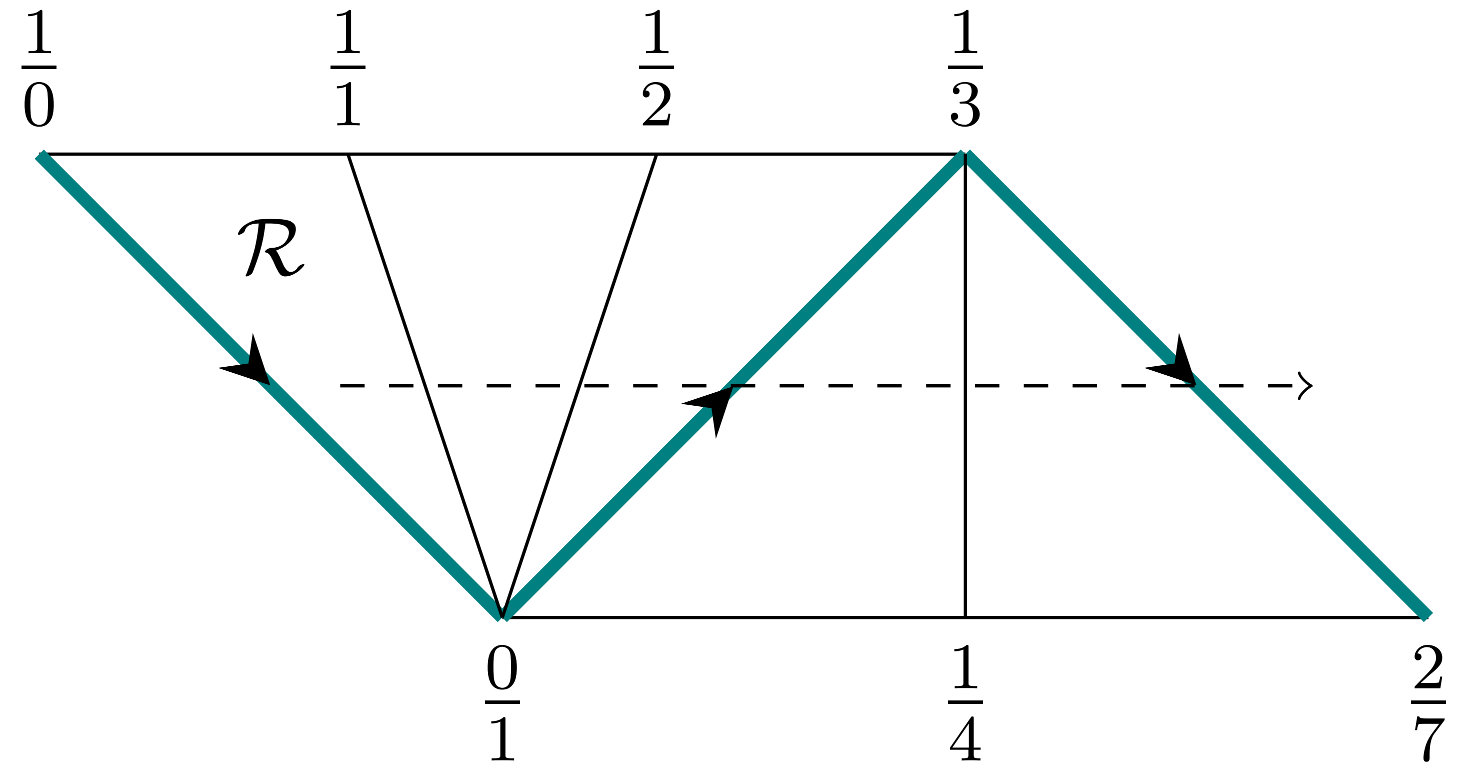
$$(25,4,20), (9,4,12), (1,4,4), (1,1,2), (1,0,0)\,.$$

$$d(49,4,28) = 648 - \big(20d_1(25)d_1(4) + 12d_1(9)d_1(4) + 4d_1(1)d_1(4) + 2d_1(1)d_1(1)\big)$$
$$= -\,459\,542\,242\,945\,399\,203\,613\,080.$$

```
c[49, 4, 28] = 459 542 242 945 399 203 613 080
```



Examples



$N = 2$

$\ell/2m = 2/7 = [0; 3,2]$ with walls $\cancel{\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}}, \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, \cancel{\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}}, \begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}, \cancel{\begin{pmatrix}1 & 2\\ 3 & 7\end{pmatrix}}$

$\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, \begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix}$

2. a) $(m, n, \ell) = (7, \frac{1}{2}, 4)$, $\Delta = -2$

$$(1, \frac{1}{2}, 2), (-1, -\frac{1}{2}, -2)\,.$$

```
c[7, 1/2, 4] = -5410
```

$$d(7, \frac{1}{2}, 4) = -2d_1(1)d_2\left(\frac{1}{2}\right) - 2d_1(-1)d_2\left(-\frac{1}{2}\right) = -5410$$

Discussion $\quad N = 1$

Consider

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{m=-1}^{\infty} \psi_m(\sigma,v)\, e^{2\pi i m \rho}$$

For $m \geq 0$ we can decompose

$$\psi_m = \psi_m^F + \psi_m^P$$

F: Finite

P: Polar

[DMZ, '11]

$$\psi_m^P(\sigma,v) = \frac{d(m)}{\eta^{24}(\sigma)} \mathscr{A}_{2,m}(\sigma,v) \qquad \mathscr{A}_{2,m}(\sigma,v) = \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s}\, y^{2ms+1}}{(1-q^s\, y)^2}$$

Appel-Lerch

Wall-crossing

Discussion $\quad N = 1$

The finite part is a mock Jacobi form which captures the single centre dyonic degeneracy

[DMZ, '11]

$$\psi_m^F(\sigma, v) = \sum_{n,\ell \in \mathbb{Z}} c_m^F(n, \ell)\, q^n\, y^\ell \quad , \quad q = e^{2\pi i \sigma} \quad , \quad y = e^{2\pi i v}$$

$$d_{immortal}(m, n, \ell) = (-1)^{\ell+1} c_m^F(n, \ell)$$

for $\Delta > 0, m > 0$.

In the $\mathscr{R}$-chamber $c(m, n, \ell) = c_m^F(n, \ell)$ for $0 \leq \ell \leq m$

[CKMRW, '19]

Therefore, we have computed $c_m^F(n, \ell)$ for $\Delta \leq 0$.

Discussion $N = 1$

The mixed Rademacher expansion

$$
\begin{aligned}
c_m^{\mathrm{F}}(n,\ell) \;=\; & 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}\\ 4m\widetilde{n}-\widetilde{\ell}^2<0}} c_m^{\mathrm{F}}(\widetilde{n},\widetilde{\ell})\, \frac{Kl\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m};k,\psi\big)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\Big(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\Big) \\
& + \sqrt{2m} \sum_{k=1}^{\infty} \frac{Kl\big(\frac{\Delta}{4m},-1;k,\psi\big)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\Big(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\Big) \qquad\qquad (\mathrm{A}.12) \\
& - \frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z}\\ g\in\mathbb{Z}/2mk\mathbb{Z}\\ g\equiv j (\mathrm{mod}\ 2m)}} \frac{Kl\big(\frac{\Delta}{4m},-1-\frac{g^2}{4m};k,\psi\big)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\
& \qquad \times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\Big(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\Big)(1-mu^2)^{25/4}\,\mathrm{d}u\,,
\end{aligned}
$$

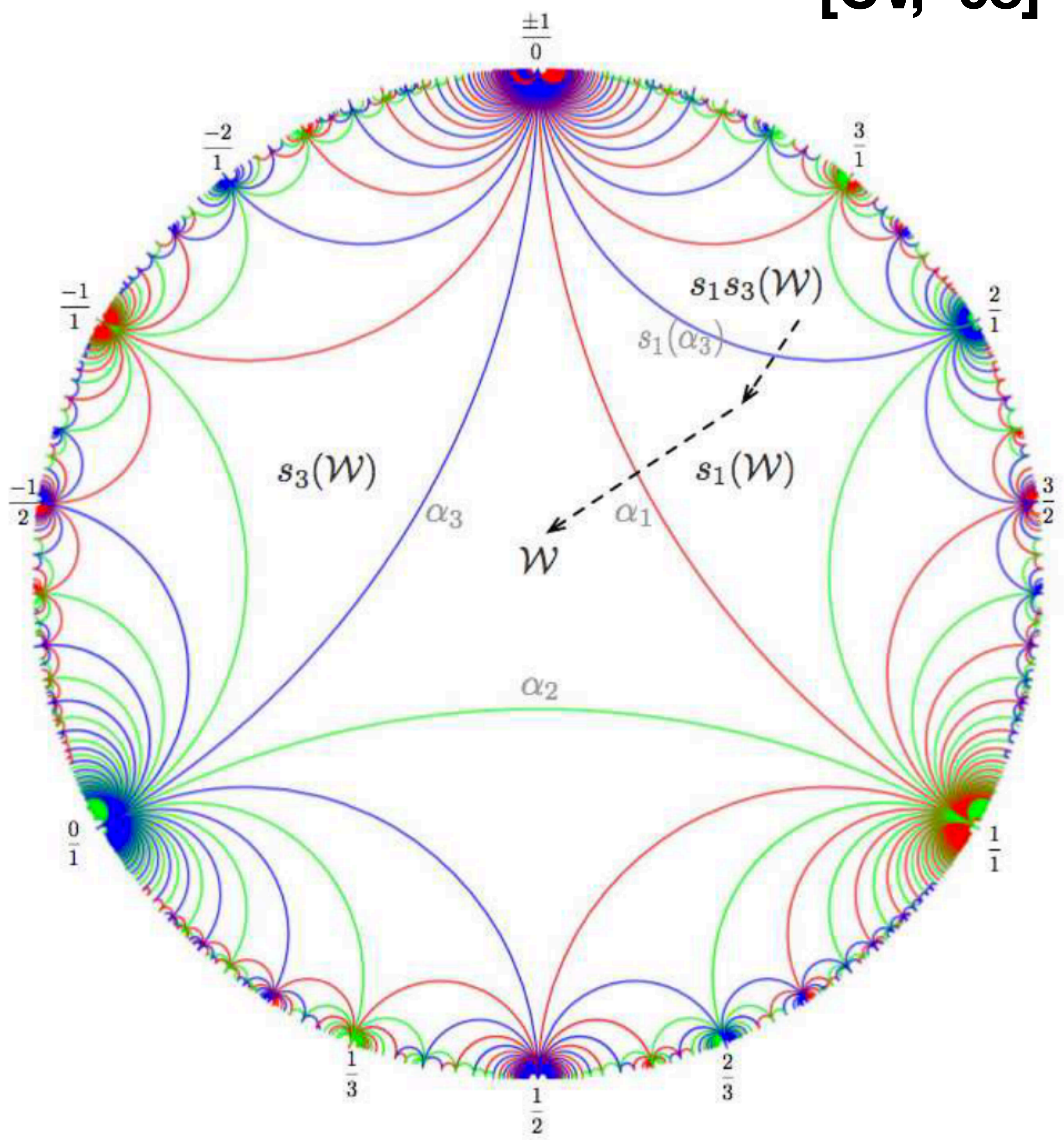
[FR, '17]

computes the coefficients $c_m^F(n,\ell)$ with $\Delta > 0$ in terms of $c_m^F(n',\ell')$ with $\Delta < 0$.

Single centre $1/4-$BPS black hole degeneracies with $I = 1$ are determined in terms of the continued fraction of the rational number $\ell/2m$ (and some extra input for the case $d_* = -1$)

Extra: Discrete attractor flow

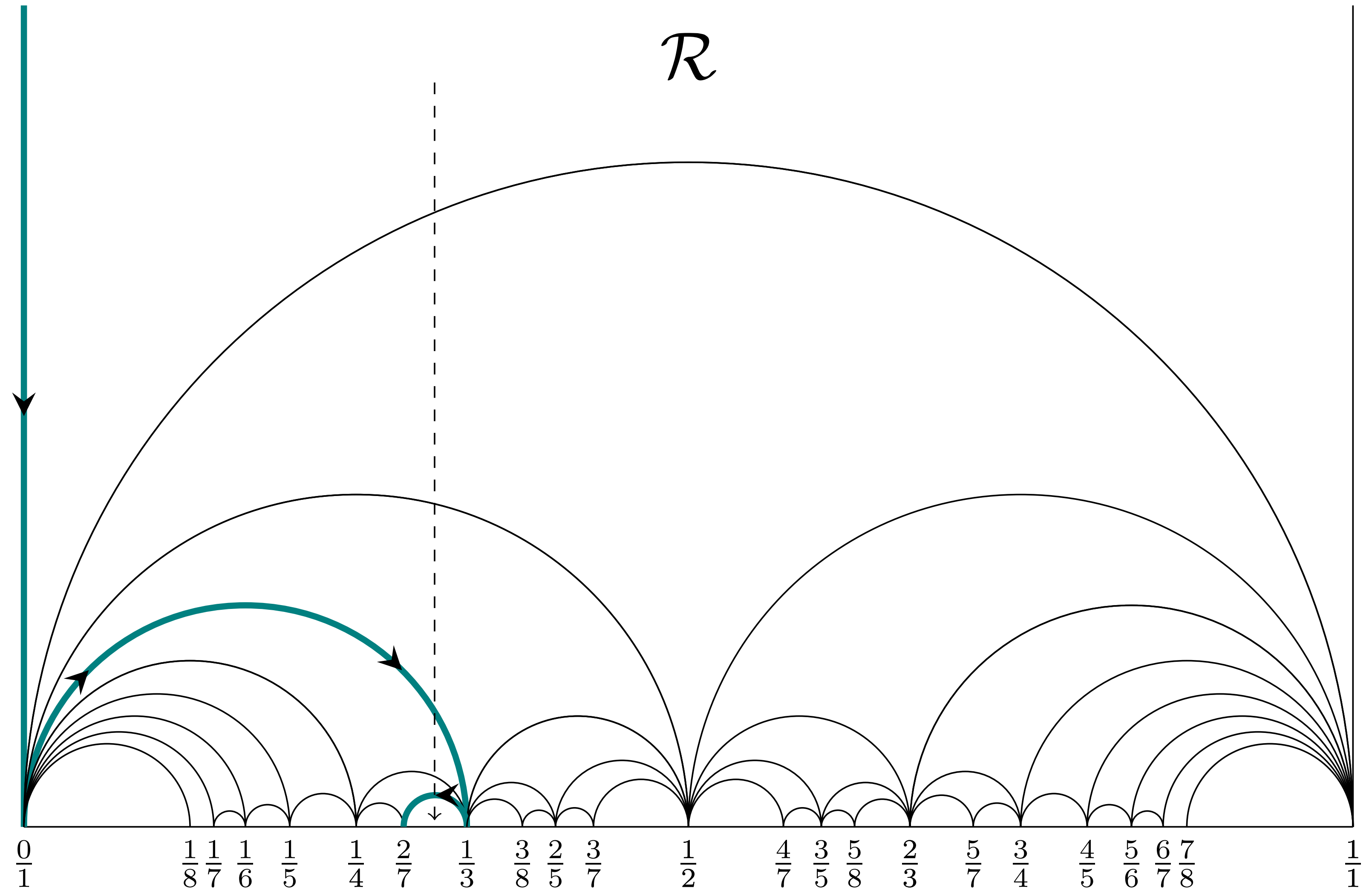
[CV, '08]



Discrete attractor flow related to Stern-Brocot tree

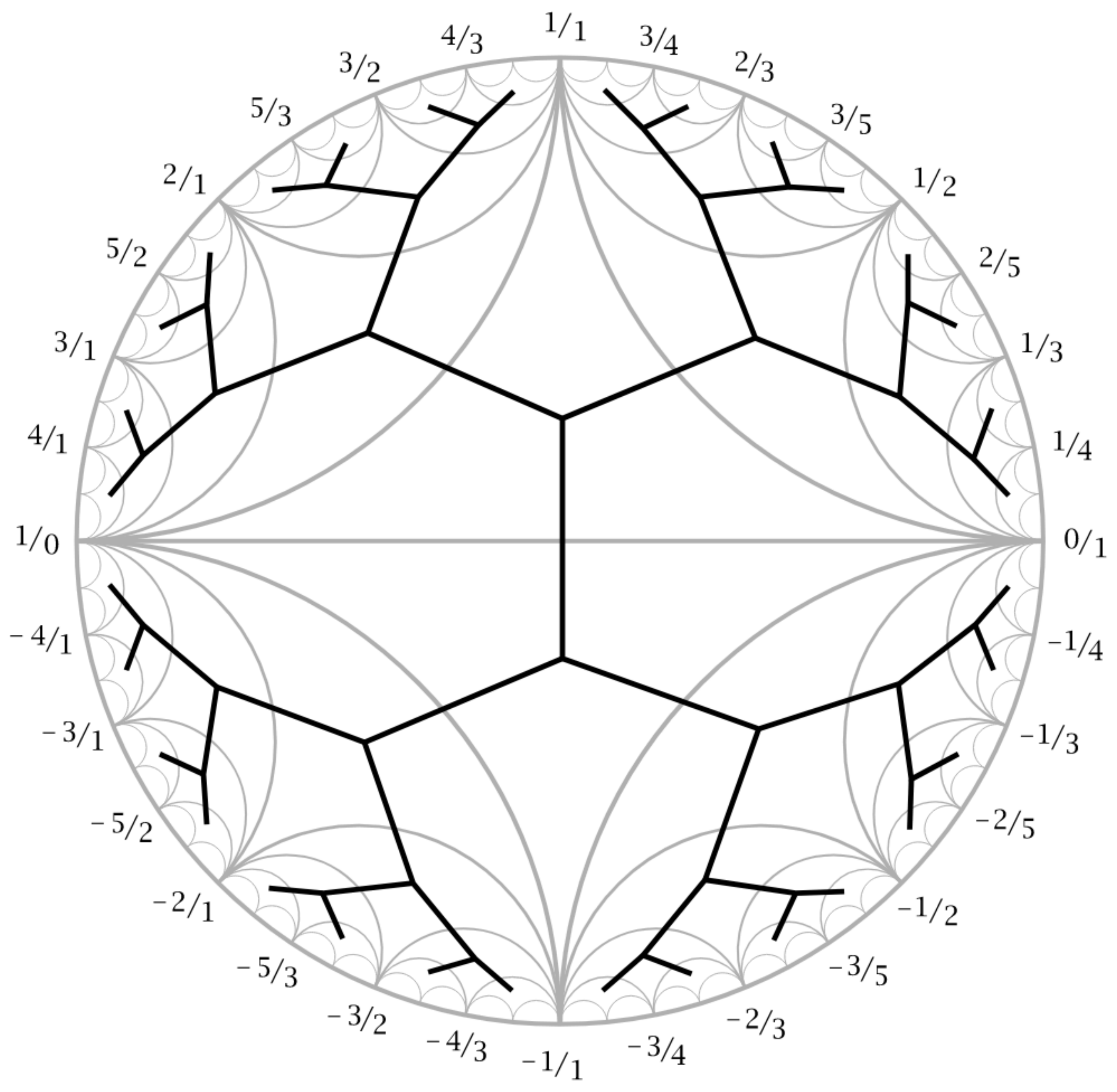
Stern-Brocot tree related to continued fractions

'Inverse discrete attractor flow'

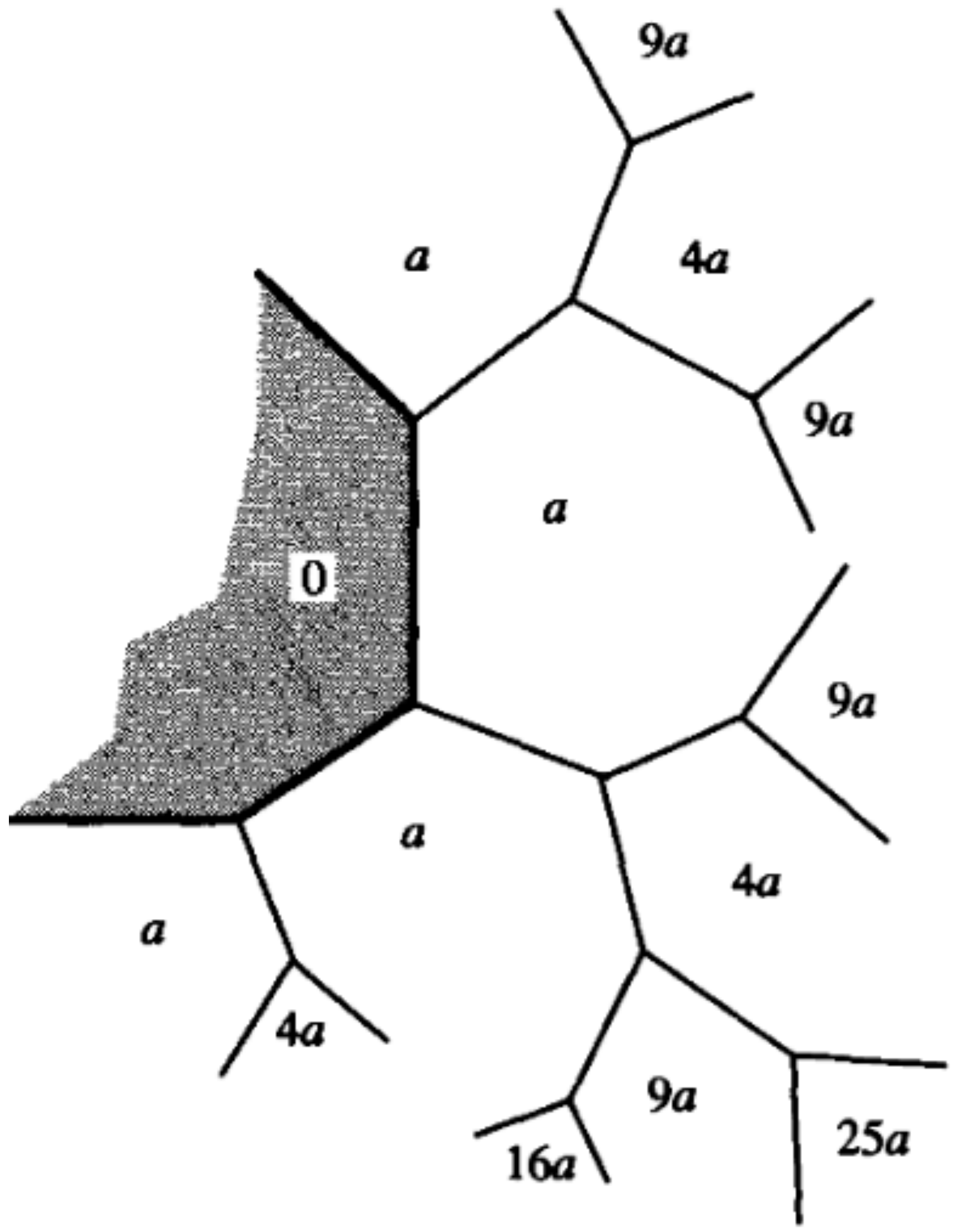


Extra: BQF

Binary quadratic forms



$$x^2 - |\Delta| y^2 = 4$$



(b)