Supercharacters of algebraic groups: the geometric approach.

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Historical Background

Beginning of Character Sheaves

1. Lusztig, George Character sheaves. I. Adv. in Math. 56 (1985), no. 3, 193–237.
2. Lusztig, George Character sheaves. II. Adv. in Math. 57 (1985), no. 3, 226–265.
3. Lusztig, George Character sheaves. III. Adv. in Math. 57 (1985), no. 3, 266–315.
4. Lusztig, George Character sheaves. IV. Adv. in Math. 59 (1986), no. 1, 1–63.
5. Lusztig, George Character sheaves. V. Adv. in Math. 61 (1986), no. 2, 103–155.

Lusztig Conjecture (2006)

There exists a theory of character sheaves for Unipotent Groups.

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Drinfeld and Boyarchenko Conjecture (2010)

For a unipotent group G there exists a collection $CS(G)$ of complexes of sheaves on G such that:

(a) The isomorphism classes in $CS(G)$ are invariant under all automorphisms of G.

(b) The complexes in $CS(G)$ are irreducible perverse sheaves.

(c) The trace functions of the complexes in $CS(G)$ are exactly the irreducible characters of $G(\mathbb{F}_q)$.

Luzstig Proved that (1985 and 2006)

1. When $G = GL_n$ there exists that collection $CS(G)$.
2. For some reductive groups it doesn't exist.
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(b) The complexes in $CS(G)$ are irreducible perverse sheaves.

(c) The trace functions of the complexes in $CS(G)$ form a basis for the space of class functions of $G(\mathbb{F}_q)$.

Lusztig (1985) and Drinfeld & Boyarchenko proved (2010)

The weaker conjecture holds in the following cases

1. Connected reductive groups.
2. Unipotent groups of nilpotence class lower than p.
3. Connected commutative groups.

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Drinfeld and Boyarchenko Conjecture (2010)

The weaker conjecture holds for all unipotent groups.

Classical Finite Groups Result

Given a finite group G, there exists a bijection between the central minimal idempotents of the group algebra and the irreducible characters of G.

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Character Sheaves (2010)

Let e be a minimal closed idempotent in $\mathcal{D}_G(G)$, a *Character Sheaf* $\mathcal{L}$ (associated to e) is a perverse indecomposable sheaf such that $e * \mathcal{L} \simeq \mathcal{L}$.

Monoidal Categories

Semigroupal Category

A *semigroupal category* is a triple $(\mathcal{M}, \otimes, \alpha)$ such that $\mathcal{M}$ is a category and

1. $\otimes$ is a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$,
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$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z)$$

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Monoidal Category

1. An object E on a semigroupal category $\mathcal{M}$ is *unital* if the functors: $X \mapsto X \otimes E$ and $X \mapsto e \otimes X$ are isomorphic to the identity functor.
2. If a semigroupal category has an unital object then it's called *monoidal*.

Idempotents

Let $\mathcal{M}$ be a monoidal category.

1. An object $e \in \mathcal{M}$ is a *weak idempotent* if $e \otimes e \simeq e$.
2. A morphism $\mathbb{1} \xrightarrow{\pi} e$ with $e \in \mathcal{M}$ is an *idempotent arrow* if after tensoring with e it becomes an isomorphism.
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Hecke Category

Let $\mathcal{M}$ be a monoidal category, and $e \in \mathcal{M}$ a weak idempotent we can define the following subcategories:

1. $e\mathcal{M} = \{X \in \mathcal{M} \mid e \otimes X \simeq X\}$.
2. $\mathcal{M}e = \{X \in \mathcal{M} \mid X \otimes e \simeq X\}$.
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Lemma

Let e be a weak idempotent in a monoidal category $\mathcal{M}$, then the Hecke subcategory is a semigroupal category. If e is closed idempotent then the Hecke subcategory is a monoidal category with e as unital object.

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Braided Categories

A *braided monoidal category* is a monoidal category with a commutative constrain: $\gamma_{X,Y} : X \otimes Y \xrightarrow{\simeq} Y \otimes X$.

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Minimal Idempotents

Let $\mathcal{M}$ a braided monoidal category with a zero object. Then an object $e \in \mathcal{M}$ is a *minimal closed* (respectively weak) idempotent if $e \neq 0$, and for all closed (respectively weak) idempotent e' we have either $e \otimes e' = 0$ or $e \otimes e' \simeq e$.

$\mathcal{D}_G(G)$ category

Notation

From now on G will always denote an unipotent group (a closed subgroup of the unitriangular matrices).

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The Category $\mathcal{D}(G)$

We shall denote $\mathcal{D}(G)$ the *derived category* of constructible complexes of $\overline{\mathbb{Q}}_\ell-$sheaves on G.

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The Equivariant Category $\mathcal{D}_G(G)$

We define the *equivariant derived category* $\mathcal{D}_G(G)$ as a colection of objects $M \in \mathcal{D}(G)$, such that $\alpha^* M \xrightarrow{\simeq} \pi^* M$.

Functors in the Equivariant Derived Category

For any morphism $f : X \to Y$, $G-$invariant we can consider the functors:

1. $f^* : \mathcal{D}_G(X) \to \mathcal{D}_G(Y)$
2. $f_! : \mathcal{D}_G(Y) \to \mathcal{D}_G(X)$.

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Monoidal structure in $\mathcal{D}_G(G)$

Consider the bifunctor $* : \mathcal{D}_G(G) \times \mathcal{D}_G(G) \to \mathcal{D}_G(G)$, defined by $M * N = \mu_!(M \boxtimes N)$, and the unit object $\mathbb{1} = 1_!(\overline{\mathbb{Q}}_\ell)$.

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Braided structure in $\mathcal{D}_G(G)$

The triple $(\mathcal{D}_G(G), *, \mathbb{1})$ is a monoidal braided category.

Character Sheaves

Let e be a minimal closed idempotent in $\mathcal{D}_G(G)$, and consider $\mathcal{M}_e^{perv}$ the subcategory of the Hecke subcategory $e\mathcal{D}_G(G)$, such that the complex is a perverse sheaf on G.

The *Lusztig packet of character sheaves* defined by e is the set of indecomposable objects in $\mathcal{M}_e^{perv}$, and we call an object in a Lusztig packet a *Character Sheaf*.

We define n_e as the integer (if exists) such that $e[-n_e]$ is perverse.

Characters vs Sheaves

Induction On Finite Groups

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- Define the action:

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Mackey Irreducibility criterion

If $\chi \in Irr(H)$ so $ind_{\boldsymbol{H}}^{\boldsymbol{G}}\chi$ is irreducible if and only if $\overline{\chi} * \delta_{\boldsymbol{x}} * \overline{\chi} = 0$ for all $x \in G \backslash H$.

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Admissible Pair for finite groups

Consider (H, χ) with $\chi \in Hom(H, \mathbb{C})$. Let G' be the stabilizer of the pair (H, χ) then the pair is admissible if:

(a) G'/H is commutative.

(b) The map:

$$B_\chi : G'/H \times G'/H \to \mathbb{C}^\times$$
$$(g_1, g_2) \mapsto \chi(C_{g_1}(g_2)g_2^{-1})$$

induces $G'/H \overset{\simeq}{\rightarrow} Hom(G'/H, \mathbb{C}^\times)$.

(c) For all $g \in G \backslash G'$ we have that $\chi_{|H \cap H^g} \neq \chi^g_{|H \cap H^g}$.

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Geometric Mackey Irreducibility criterion

Given a $M \in \mathcal{D}(G')$ we say that it satisfies the *Geometric Mackey Condition* (with respect to G), if for all $x \in G(k) \backslash G'(k)$ we have $\overline{M} * \delta_x * \overline{M} = 0$.

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Admissible Pair

Consider $(H, \mathcal{L})$ where $\mathcal{L}$ is a multiplicative local system on H such that:

(a) Let G' be the stabilizer of the pair $(H, \mathcal{L})$ and consider its neutral connected component G'^0. Then G'^0/H is commutative.

(b) The morphism $\varphi_{\mathcal{L}} : G'^0/H \to (G'^0/H)^*$ is an isogeny.

(c) For all $g \in G(k) \backslash G'(k)$ we have: $\mathcal{L}_{|(H \cap H^g)^\circ} \ncong \mathcal{L}^g_{|(H \cap H^g)^\circ}$.

Then we call the pair $(H, \mathcal{L})$ an admissible pair.

Main Results

Proposition (Drinfeld and Boyarchenko 2010)

Let G be a finite nilpotent group, then every irreducible character of G is induced from some linear character of some admissible pair.

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Heisenberg minimal idempotent

Consider $(H, \mathcal{L})$ an admissible pair, and let G' be its stabilizer. Let $e_{\mathcal{L}} = \mathcal{L} \otimes (\mathbb{K}_H)$ and denote by $e'_{\mathcal{L}}$ its extension by zero to G'.

We call $e'_{\mathcal{L}}$ the Heisenberg minimal idempotent on G' defined by the pair $(H, \mathcal{L})$.

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Lemma (Boyarchenko 2010)

The object $e'_{\mathcal{L}} \in \mathcal{D}_{G'}(G')$ is a closed idempotent, a minimal weak idempotent (so it's a minimal closed idempotent), and satisfies the *Geometric Mackey Condition* .

Lemma (Boyarchenko and Drinfeld 2011)

- If $M, N \in \mathcal{D}_{G'}(G')$ satisfies the *Geometric Mackey Condition* then $ind_{G'}^{G}(M) * ind_{G'}^{G}(N) \xrightarrow{\simeq} ind_{G'}^{G}(M * N)$
- If $e \in \mathcal{D}_{G'}(G')$ be a weak idempotent that satisfy the *Geometric Mackey Condition* then for all $M, N \in e\mathcal{D}_{G'}(G')$ we have $\overline{M} * \delta_x * \overline{N} = 0$ for all $x \in G(k)\backslash G'(k)$.

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Lemma (Boyarchenko 2010)

Let $e \in \mathcal{D}_{G'}(G')$ be a weak idempotent that satisfy *Geometric Mackey Condition* . Then:

- If $M \in e\mathcal{D}_{G'}(G')$ then $ind_{G'}^{G}(M) \in f\mathcal{D}_{G}(G)$ (where $f = ind_{G'}^{G} e$).
- If e and f are closed idempotents then $(ind_{G'}^{G})_{|e\mathcal{D}_{G'}(G')}$ is an equivalence of $e\mathcal{D}_{G'}(G')$ and $f\mathcal{D}_{G}(G)$.

Corollary (Boyarchenko and Drinfeld 2011)

Let $e \in \mathcal{D}_{G'}(G')$ be a closed idempotent that satisfy *Geometric Mackey Condition* and such that $f = ind_{G'}^{G} e$ is closed. Then:

- $e\mathcal{D}_{G'}(G')$ and $f\mathcal{D}_G(G)$ are monoidal categories.
- We have an monoidal equivalence:

$$(ind_{G'}^{G})_{|e\mathcal{D}_{G'}(G')} : e\mathcal{D}_{G'}(G') \to f\mathcal{D}_G(G).$$

Lemma (Boyarchenko 2010)

Let $e \in \mathcal{D}_{G'}(G')$ be a weak idempotent that satisfy the *Geometric Mackey Condition* . Then the object $f = ind_{G'}^{G} e \in \mathcal{D}_G(G)$ is a weak idempotent, and if e is minimal f is minimal as well.

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Proposition (Boyarchenko and Drinfeld 2011)

- If $e \in \mathcal{D}_G(G)$ is a closed minimal idempotent then it's a weak minimal idempotent.
- If $e \in \mathcal{D}_G(G)$ is a weak minimal idempotent then e it's closed.

Theorem (Boyarchenko and Drinfeld 2011)

Let $e \in \mathcal{D}_{G'}(G')$ be a closed minimal idempotent such that satisfy the *Geometric Mackey Condition* and let $f = ind_{G'}^{G} e$. Then:

- f is a closed minimal idempotent in $\mathcal{D}_G(G)$.
- The functor $ind_{G'}^{G}$ restricts to a monoidal equivalence $e\mathcal{D}_{G'}(G') \to f\mathcal{D}_G(G)$.
- If $M \in e\mathcal{D}_{G'}(G')$ is perverse then $ind_{G'}^{G}(M)[dim(G/G')]$ is perverse as well.
- $n_f = n_e - dim(G/G')$.

Proposition (Boyarchenko and Drinfeld 2011)

For all $N \in \mathcal{D}(G)$, non zero, There exists a minimal closed idempotent $f \in \mathcal{D}_G(G)$ such that $N * f \neq 0$, with f induced from some Heisenberg idempotent.

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Theorem (Boyarchenko and Drinfeld 2011)

If $e'_{\mathcal{L}} \in \mathcal{D}_{G'}(G')$ is an Heisenberg idempotent for some admissible pair $(H, \mathcal{L})$ then:

- $f = ind_{G'}^{G} e'_{\mathcal{L}}$ is a minimal closed idempotent.
- $n_{e'_{\mathcal{L}}} = dim(H)$ and $n_f = dim(H) - dim(G/G')$.
- Every $f \in \mathcal{D}_G(G)$ minimal closed idempotent comes from an admissible pair.

Theorem (Deshpande 2010 and Datta 2010)

If $e'_{\mathcal{L}} \in \mathcal{D}_{G'}(G')$ is an Heisenberg idempotent then:

- $\mathcal{M}^{perv}_{e'_{\mathcal{L}}}$ is an abelian category semisimple with a finite number of simple objects.
- There exists an unique integer $n_{e'_{\mathcal{L}}}$ such that $e'_{\mathcal{L}}[-n_{e'_{\mathcal{L}}}] \in \mathcal{M}^{perv}_{e'_{\mathcal{L}}}$ (Moreover we have $0 \leq n_{e'_{\mathcal{L}}} \leq dim(G)$). And $\mathcal{M}_{e'_{\mathcal{L}}} := \mathcal{M}^{perv}_{e'_{\mathcal{L}}}[n_{e'_{\mathcal{L}}}]$ is monoidal.

Theorem (Deshpande 2010 and Datta 2010)

If $e'_{\mathcal{L}} \in \mathcal{D}_{G'}(G')$ is an Heisenberg idempotent then:

- $\mathcal{M}^{perv}_{e'_{\mathcal{L}}}$ is an abelian category semisimple with a finite number of simple objects.
- There exists an unique integer $n_{e'_{\mathcal{L}}}$ such that $e'_{\mathcal{L}}[-n_{e'_{\mathcal{L}}}] \in \mathcal{M}^{perv}_{e'_{\mathcal{L}}}$ (Moreover we have $0 \leq n_{e'_{\mathcal{L}}} \leq dim(G)$). And $\mathcal{M}_{e'_{\mathcal{L}}} := \mathcal{M}^{perv}_{e'_{\mathcal{L}}}[n_{e'_{\mathcal{L}}}]$ is monoidal.

Theorem (Boyarchenko and Drinfeld 2011)

If $e \in \mathcal{D}_G(G)$ is a closed minimal idempotent. Then:

- $\mathcal{M}^{perv}_{e}$ is an abelian category semisimple with a finite number of simple objects.
- There exists an unique integer n_e such that $e[-n_e] \in \mathcal{M}^{perv}_{e}$ (Moreover we have $0 \leq n_e \leq dim(G)$). And $\mathcal{M}_e := \mathcal{M}^{perv}_{e}[n_e]$ is monoidal.

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