PHASE TRANSITIONS, RANDOM MATRICES AND $T\bar{T}$ DEFORMATION

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Based on joint work with Leonardo Santilli (Lisboa), Jorge Russo (Barcelona) and Richard J. Szabo (Edinburgh).

ORGANIZATION OF THE TALK

- Introduction to random matrix theory
  - Some definitions and kernel
  - Relationship with structured matrices. Schur insertions

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  - Effects of deformation on Douglas-Kazakov phase transition
  - Comments on the q-deformed theory

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  - Comments on the q-deformed theory
- Outlook and references

INTRODUCTION TO RANDOM MATRIX THEORY

- Random matrix theory roughly aims to understand of the properties (such as the statistics of matrix eigenvalues) of matrices with entries drawn randomly from various probability distributions.
- Let $M = (M_{jk})_{j,k=1}^{N}$ be a square $N \times N$ matrix with randomly distributed elements M_{jk}. This is a random matrix with respect to a probability distribution, defined by

$$P_\beta^N(M)dM = \exp\left(-\beta \mathrm{Tr} V(M)\right) dM.$$

- Most studied ensembles are the Gaussian ensembles, $V(M) = M^2$. It can be shown that the previous expression is automatically restricted to the form

$$P(M) = \exp(-a\mathrm{Tr}M^2 + b\mathrm{Tr}M + c), \quad a > 0.$$

  if one postulates statistical independence of the matrix elements M_{jk}. There are three different ensembles defined, depending on the values of the parameter $\beta = 1, 2$ or 4.

INTRODUCTION TO RANDOM MATRIX THEORY

- Ensembles of random $N \times N$ matrices M are defined by the following demands:

1. The probability $P(M)d[M]$ is invariant under any transformation $M \to U^{-1}MU$, where U is either an orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$) matrix. That is to say, if $M' = U^{-1}MU$ where U belongs to the unitary group $U(N;\beta)$, then $P\,(M')\,d\,[M']$ =$P\,(M)\,d\,[M]$.
2. The matrix elements which are not related by the symmetry of the matrix are statistically independent (Gaussian ensembles).

RMT: INTEGRATION OVER EIGENVALUES

- Diagonalization: for each matrix M there is a matrix U that maps it onto its eigenvalues. The Jacobian of the transformation is $J_\beta(\{x_i\}) = \prod_{i<j} |x_i - x_j|^\beta$. The resulting expression is

$$P(x_1, \dots, x_N) = C_N \prod_{i<j} |x_i - x_j|^\beta \exp\left[-\sum_{i=1}^{N} V(x_i)\right],$$

- The main relevant quantities are m-partial integrations over the previous N-dimensional probability density function.
- The simplest case to treat analytically: Hermitian ($\beta = 2$) ensemble.
- Weyl integration formula: G Lie group, T maximal torus. The Haar integral of a class function computed as an integral over the torus. Ex: $G = U(N)$

$$\int_G f(g)dg = \frac{1}{n!}\int_T f\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \prod_{i<j} |t_i - t_j|^2 \, dt$$

REPRODUCING KERNEL

- Let $p_N(x) = a_N x^N + \ldots$ be the N-th orthogonal polynomial associated to $\mathrm{e}^{-V(x)}$. k-point correlation function can be computed from the two-point kernel as follows (for $\beta = 2$)

$$R_k(x_1, x_2, \ldots, x_k) = \det \left[K_N\left(x_i, x_j\right)\right]_{1 \le i,j \le k}.$$

- Orthogonal polynomials method $\Rightarrow$ explicit expressions for $K_N\left(x_i, x_j\right)$. The two-point kernel (using also the Christoffel-Darboux identity) is

$$\begin{aligned} K_N\left(x, y\right) &= \mathrm{e}^{-\frac{(V(x)+V(y))}{2}} \sum_{k=0}^{N-1} p_k\left(x\right) p_k(y) \\ &= \frac{c_{N-1}}{c_N} \frac{p_N(x) p_{N-1}(y) - p_{N-1}\left(x\right) p_N\left(y\right)}{x-y} \mathrm{e}^{-\frac{(V(x)+V(y))}{2}}. \end{aligned}$$

- This kernel is important also in statistical and machine learning (e.g. support vector machines) and in statistics in general since, without any scaling can be understood as a delta sequence.

TOEPLITZ DETERMINANTS

$$\det_{N\times N}(d_{j-k}) = \det_{N\times N}\begin{pmatrix} d_0 & d_{-1} & d_{-2} & d_{-3} & \\ d_1 & d_0 & d_{-1} & d_{-2} & \ddots \\ d_2 & d_1 & d_0 & d_{-1} & \ddots \\ d_3 & d_2 & d_1 & d_0 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

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- Heine-Szegő identity

$$D_N(f) = \frac{1}{N!}\frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{1\le j<k\le N}|e^{i\theta_j} - e^{i\theta_k}|^2\prod_{k=1}^{N} f(e^{i\theta_k})d\theta_k.$$

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- This expression is of random matrix type. The integrand can be interpreted as the joint p.d.f. of the eigenvalues of a unitary random matrix ensemble.

MINORS OF TOEPLITZ MATRICES

$$D_N(f) = \det_{N\times N}(d_{j-k}) = \det_{N\times N} \begin{pmatrix} d_0 & d_{-1} & d_{-2} & d_{-3} & d_{-4} & d_{-5} & \\ d_1 & d_0 & d_{-1} & d_{-2} & d_{-3} & d_{-4} & \ddots \\ d_2 & d_1 & d_0 & d_{-1} & d_{-2} & d_{-3} & \ddots \\ d_3 & d_2 & d_1 & d_0 & d_{-1} & d_{-2} & \ddots \\ d_4 & d_3 & d_2 & d_1 & d_0 & d_{-1} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

MINORS OF TOEPLITZ MATRICES

$$D_N^{\lambda,\mu}(f) = \det_{N\times N}(d_{j-\lambda_j-k+\mu_k}) = \det_{N\times N}\begin{pmatrix} \cancel{d_0} & \cancel{d_{-1}} & \cancel{d_{-2}} & \cancel{d_{-3}} & \cancel{d_{-4}} & \cancel{d_{-5}} & \\ d_1 & \cancel{d_0} & d_{-1} & \cancel{d_{-2}} & d_{-3} & d_{-4} & \ddots \\ \cancel{d_2} & \cancel{d_1} & \cancel{d_0} & \cancel{d_{-1}} & \cancel{d_{-2}} & \cancel{d_{-3}} & \ddots \\ d_3 & \cancel{d_2} & d_1 & \cancel{d_0} & d_{-1} & d_{-2} & \ddots \\ d_4 & \cancel{d_3} & d_2 & \cancel{d_1} & d_0 & d_{-1} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The striking pattern is encoded in partitions. Above, $\lambda = (1)$, $\mu = (2,1)$.
We always assume $l(\lambda), l(\mu) \le N$.

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The striking pattern is encoded in partitions. Above, $\lambda = (1)$, $\mu = (2,1)$.
We always assume $l(\lambda), l(\mu) \leq N$. Generalized Heine-Szegő identity

$$D_N^{\lambda,\mu}(f) = \int_{U(N)} \overline{s_\lambda(M)} s_\mu(M) f(M) dM =$$

$$\frac{1}{N!}\frac{1}{(2\pi)^N}\int_0^{2\pi} \dots \int_0^{2\pi} s_\lambda(e^{-i\theta}) s_\mu(e^{i\theta}) \prod_{1\leq j<k\leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{k=1}^{N} f(e^{i\theta_k}) d\theta_k.$$

Unitary and discrete matrix models: phases and a 2d YM motivation

UNITARY VS DISCRETE MATRIX MODELS

$2d$ YM: Wilson and heat-kernel lattice actions

- Lattice theory with Wilson action $\Rightarrow$ integration over gauge group $SU(N)$ or $U(N)$.

UNITARY VS DISCRETE MATRIX MODELS

2d YM: Wilson and heat-kernel lattice actions

- Lattice theory with Wilson action $\Rightarrow$ integration over gauge group $SU(N)$ or $U(N)$. This gives the Gross-Witten-Wadia matrix model. Integration variables on the unit circle.

$$\mathcal{Z}_{\mathrm{GWW}} = \int_{[-\pi,\pi]^N} d^N\theta e^{\frac{2}{g}\sum_{j=1}^{N}\cos\theta_j} \prod_{i<j}\left(2\sin\frac{\theta_i-\theta_j}{2}\right)^2$$

  Large N limit with $\lambda = gN$ fixed. Third order phase transition at $\lambda = 2$.

- With heat-kernel lattice action: Migdal formula. For the sphere S^2

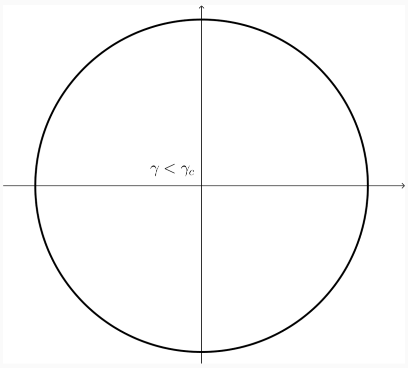
$$\mathcal{Z}_{\mathrm{YM}} = \sum_R (\dim R)^2 \exp\left\{-\frac{a g_{\mathrm{YM}}^2}{2} C_2(R)\right\}. \tag{1}$$

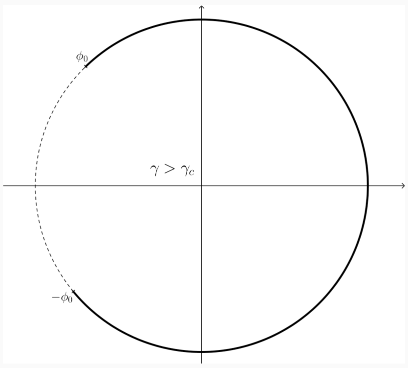
This can be written as a discrete Gaussian ensemble (discrete GUE). Both models are studied with standard methods in Random Matrix Theory, showing a third order phase transition at large N (GWW and Douglas-Kazakov transition).

UNITARY MATRIX MODELS AND PHASE TRANSITIONS

However, the underlying mechanism is different, from the point of view of the random matrix theory.

Beyond GWW: unitary matrix model with generic potential. Typically, one finds third order transitions of the GWW type between a phase in which the eigenvalues are spread all over the unit circle and a phase in which the eigenvalues are placed only on an arc of the circle. The two phases are separated by a critical point ($\gamma_c = \lambda_c^{-1}$ in the figure).



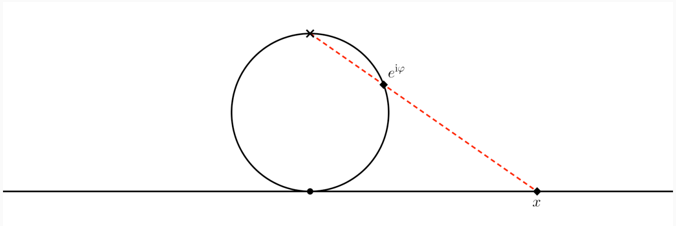


Characterization of phase structure with *infinitely many* interactions [Baik math/0001022, ST 1902.06649]. We again find a third order phase transition of GWW type.

MAPPING TO THE REAL LINE

Stereographic projection: send the eigenvalues from the unit circle to the real line with

$$e^{i\theta} = \frac{1 + ix}{1 - ix}, \qquad -\pi < \theta < \pi,\ x \in \mathbb{R},$$



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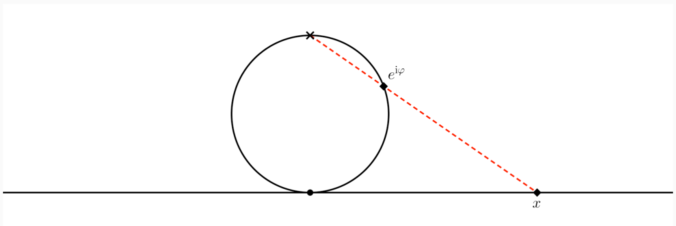

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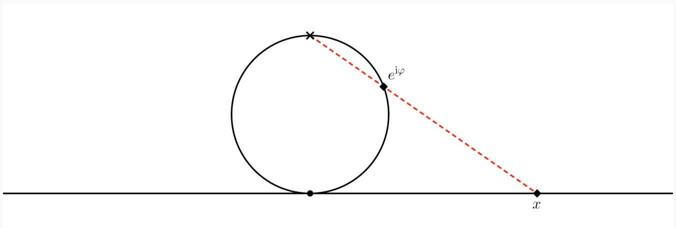

Puncture on the circle: only one phase will be reproduced at large N. Study of the partition function on the real line matches the analysis on the circle [ST 2003.10475] in one phase, but the unitary matrix model will show a richer phase structure.

The same happens in [RT 2007.08515], the generalized GWW model has a richer phase structure than the deformed Cauchy ensemble [J. Russo, 2006.00672] to which it maps.

ANOTHER RELATIONSHIP BETWEEN UNITARY AND DISCRETE MATRIX MODELS

In [ST 2003.10475] we established a relation between a unitary matrix model of the type discussed above (with infinitely many interactions) and a discrete matrix model.

$$\mathcal{Z}_{\text{unitary}} = \int_{[-\pi,\pi]^N} d^N\theta \prod_{j=1}^{N} [(1+te^{i\theta_j})^{M_1}(1+te^{-i\theta_j})^{M_2} \prod_{i<j}\left(2\sin\frac{\theta_i-\theta_j}{2}\right)^2$$

$$\mathcal{Z}_{\text{discrete}} = \sum_{h_1=0}^{M_1+N-1} \cdots \sum_{h_{M_1}=0}^{M_1+N-1} \prod_{1\le j<k\le M_1} (h_j-h_k)^2 \prod_{j=1}^{M_1} \binom{M_2-M_1+h_j}{h_j} t^{2h_j}$$

Discrete model is the classical Meixner ensemble, but with a hard wall. It is then a probability (when normalized) (K. Johansson ”Discrete orthogonal polynomial ensembles and the Plancherel measure”, Ann. of Math. 153 (2001), 259). Discussion of hard wall and six vertex model [Colomo-Pronko 1306.6207].

Important aspect: the connection is *not a direct map*, but uses intermediate steps and theory of random partitions.

UNITARY VS DISCRETE MATRIX MODEL: PHASE DISCREPANCY

While for $M_1 = M_2$ we find third order transition and agreement between observables in the two models, we find a second order transition in the unitary model when $M_1 \neq M_2$, opposed to the third order transition in the discrete ensemble.

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We learn then that the discrete and the unitary matrix model always agree at finite N, but only agree at large N and with scaling if both are real-valued.

$T\overline{T}$-deformation of $2d$ YM

DEFORMATION BY OPERATOR $T\bar{T}$

In recent years, the study of irrelevant deformations of 2d CFTs by operator $T\bar{T}$ [Smirnov-Zamolodchikov '16, Cavaglià-Negro-Szécsényi-Tateo '16] attracted considerable attention.

$T\bar{T}$-deformation of CFT has a geometric interpretation: it is equivalent to

- undeformed CFT coupled to JT gravity [Dubovsky et al. '17, '18].
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$T\bar{T}$-deformed theories have been analysed from different points of view. Recent work on the topic include the study of such deformation in the context of:

- AdS/CFT correspondence;
- SCFTs;
- generalization of $T\bar{T}$ and conserved currents;
- entanglement entropy;
- etc.

$T\overline{T}$-DEFORMATION OF BOSONIC MODEL

We will not consider any of those topics here. Instead, we follow a result by [Conti et al. '18]. Starting with a free bosonic Lagrangian

$$\mathcal{L}(\tau = 0)[\phi] = |\nabla\phi|^2$$

and turning on a $T\overline{T}$-deformation, $\mathcal{L}(\tau)[\phi]$ is the bosonic Born-Infeld model.

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Starting with an interacting Lagrangian

$$\mathcal{L}(\tau = 0)[\phi] = |\nabla\phi|^2 + V(\phi),$$

a general expression for its $T\overline{T}$-deformation was obtained.

For Abelian Yang-Mills on a Riemann surface, the deformed Lagrangian looks complicated, but the Hamiltonian is:

$$\mathcal{H}_{\mathrm{YM}}(\tau) = \frac{\mathcal{H}_{\mathrm{YM}}(\tau = 0)}{1 - \tau\mathcal{H}_{\mathrm{YM}}(\tau = 0)}.$$

They propose the same formula for the non-Abelian case.

YANG-MILLS ON A RIEMANN SURFACE

Quick reminder of 2d Yang-Mills on a closed Riemann surface of genus h.
Migdal formula for $SU(N)$: partition function is a sum over representations
[Migdal '75]

$$\mathcal{Z}_{\mathrm{YM}} = \sum_R (\dim R)^{2-2h} \exp\left\{ -\frac{a g_{\mathrm{YM}}^2}{2} C_2(R) \right\} . \tag{2}$$

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Irreducible $SU(N)$ representations are in one-to-one correspondence with partitions of N. We consider instead the $U(N)$ theory, whose irreducible representations are in one-to-one correspondence with N-tuples:

$$+\infty > r_1 \geq r_2 \geq \cdots \geq r_N > -\infty.$$

The $T\overline{T}$-deformation of the theory is of the form: [Conti et al, ST ', Ireland-Shyam, SST]

$$C_2(R) \mapsto \frac{C_2(R)}{1 - \tau C_2(R)}.$$

2D YANG-MILLS THEORY ON THE SPHERE

We focus on the study of two-dimensional Yang-Mills on the sphere S^2. Thus we set $h = 0$ in the Migdal formula (2). Properties of interest are:

- large N phase transition [Douglas-Kazakov '93];
- phase transition is triggered by instantons [Gross-Matytsin '94].

We consider the partition function of $T\overline{T}$-deformed $U(N)$ Yang-Mills:

$$\mathcal{Z}_N(a,\tau) = \sum_R (\dim R)^2 \exp\left(-\frac{a g_{\mathrm{YM}}^2}{2}\left(\frac{C_2(R)}{1-\frac{\tau}{N^3}C_2(R)}\right)\right), \tag{3}$$

and take the 't Hooft limit $N \to \infty$ with

$$A \equiv a g_{\mathrm{YM}}^2 N \quad \text{fixed.}$$

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$$A \equiv a g_{\rm YM}^2 N \quad \text{fixed.}$$

As the rank N grows, the leading contribution comes from the saddle point configuration of the action.

We follow standard techniques and introduce the eigenvalue density $\rho(h)$.

SADDLE POINT EQUATION

This leads to the saddle point equation:

$$\mathrm{P}\int du \frac{\rho(u)}{h-u} = \frac{Ah}{2}\sum_{j=0}^{\infty}(j+1)\,\tau^j\left[\int du\rho(u)u^2 - \frac{1}{12}\right]^j, \tag{4}$$

where we expanded the geometric series. We must take into account the discrete nature of the ensemble: there exists a minimum distance between two eigenvalues.

This translates into the condition

$$\rho(u) \leq 1$$

on the eigenvalue density. We studied in [ST 1810.05404] the saddle point equation (4) perturbatively in the $T\bar{T}$-deformation parameter τ.

CRITICAL POINT

The Douglas-Kazakov weak coupling phase extends up to the critical point

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In the $T\overline{T}$-deformed case, we obtained in JHEP 1901 (2019) 054, [1810.05404]:

$$A_{cr}(\tau) = \pi^2 \left(1 - \tau \left(\frac{1}{\pi^2} - \frac{1}{12}\right)\right)^2 ,$$

as long as $\tau < \frac{12\pi^2}{12-\pi^2}$, and no positive solution for τ bigger than this value.

When $A > A_{cr}(\tau)$, one has to find another solution in order to fulfill the condition $\rho(u) \leq 1$.

Ansatz: two-cut solution (again following Douglas-Kazakov)

THIRD ORDER PHASE TRANSITION

The strong coupling phase is more difficult to analyze but looking at the neighborhood of the critical area, we could show that:

$$\left.\frac{\partial^2 \log \mathcal{Z}_{\mathrm{YM}}}{\partial A^2}\right|_{A>A_{cr}} - \left.\frac{\partial^2 \log \mathcal{Z}_{\mathrm{YM}}}{\partial A^2}\right|_{A<A_{cr}} \to 0$$

$\Longrightarrow$ The phase transition is third order (as in Douglas-Kazakov). See the technical details in a dedicated talk by Leo (or slides).

Thus, the $T\overline{T}$-deformation introduced a nontrivial dependence on (A, τ), but in such a way that it does not affect the order of the phase transition.

It is known that the DK phase transition is induced by instantons, we can also reproduce that analysis in the deformed theory. For example, looking at the suppression factor of the one-instanton contribution, we will see that it decays faster with the area, explaining why the critical area is lowered.

INSTANTON ANALYSIS IN 2D YM

Partition function of 2dYM represented as sum over instanton contributions [Witten '92]:

$$\mathcal{Z}_{\mathrm{2dYM}} = \sum_{\ell \in \mathbb{Z}^N} w_{\mathrm{inst.}}(\ell) \exp\left(-\frac{2\pi^2 N}{A} \sum_{j=1}^{N} \ell_j^2 \right) .$$

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The weight $w_{\mathrm{inst.}}(\ell)$ can be computed via Poisson resummation [Minahan-Polychronakos '93]. Instanton contributions are exponentially suppressed at large N, e.g. first instanton suppressed as:

$$\frac{w_{\mathrm{inst.}}(1,0,\ldots,0)e^{-S_{\mathrm{inst.}}(1,0,\ldots,0)}}{w_{\mathrm{inst.}}(0,0,\ldots,0)e^{-S_{\mathrm{inst.}}(0,0,\ldots,0)}} = \text{const.} \times e^{-\frac{2\pi^2}{A} N \gamma\left(\frac{A}{\pi^2}\right)},$$

$$\gamma(x) := \sqrt{1-x} - \frac{x}{2} \log\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} \right).$$

Phase transition is induced by instantons [Gross-Matytsin '94].

INSTANTONS IN THE $T\overline{T}$-DEFORMED THEORY

Full Poisson resummation is hard in the $T\overline{T}$-deformed case, instead we can focus on the first instanton correction $\ell = (1, 0, \ldots, 0)$.

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We obtained that the first instanton correction is suppressed as:

$$\frac{Z_{(1,0,\ldots,0)}}{Z_{(0,\ldots,0)}} = \mathcal{C}' \exp\left[-N\frac{2\pi^2}{Ab_\infty}\gamma\left(\frac{Ab_\infty}{\pi^2}\right)\right], \tag{5}$$

where the function is the same as in the undeformed case. ($b_\infty > 1$ if $\tau > 0$).
In particular, *not suppressed* if $\tau > \frac{12\pi^2}{12-\pi^2}$.

Let us see two plots showing that the same suppression occurs at a lower area, hence the same instanton contribution happens earlier when increasing the area from zero, explaining the lowering of the critical area.

EXPONENTIAL SUPPRESSION OF INSTANTONS, $\tau = 0.1$

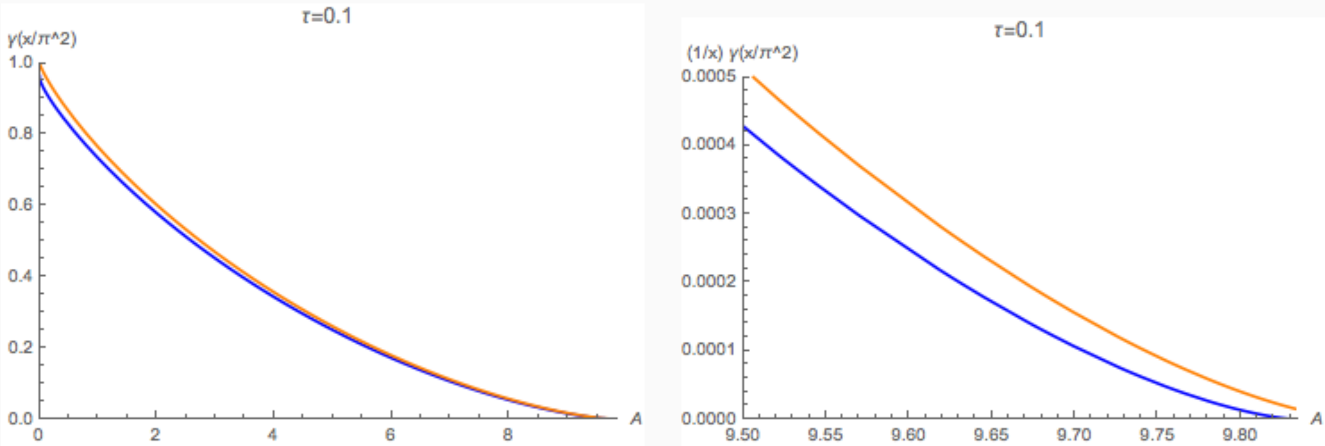

Figure 1: $\gamma\left(\frac{x}{\pi^2}\right)$: deformed (blue) and undeformed (orange) case. On the right: a zoom on the tail of $\frac{1}{x}\gamma\left(\frac{x}{\pi^2}\right)$. The plots are at $\tau = 0.1$

EXPONENTIAL SUPPRESSION OF INSTANTONS, $\tau = 0.5$

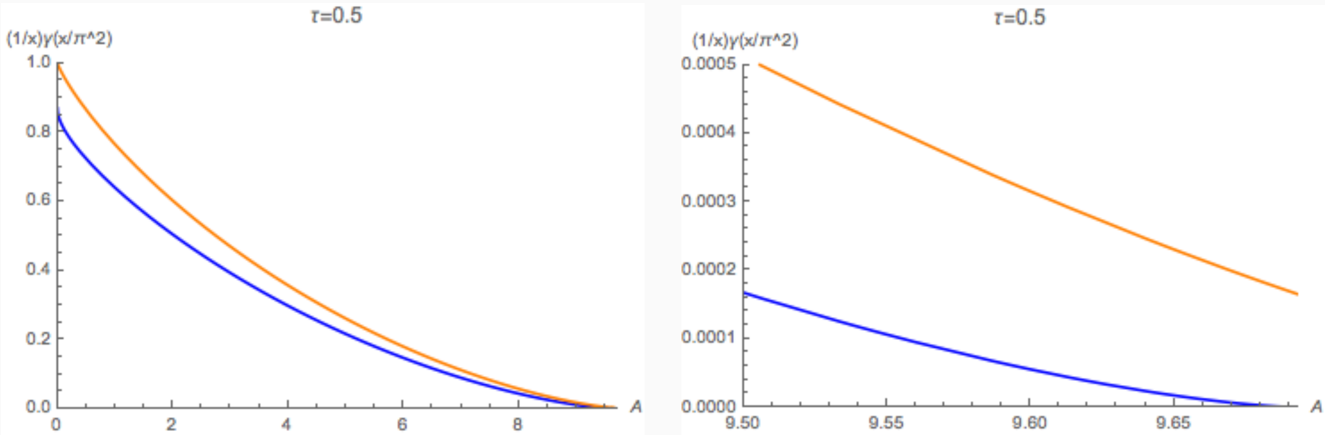

Figure 2: $\gamma\left(\frac{x}{\pi^2}\right)$: deformed (blue) and undeformed (orange) case. On the right: a zoom on the tail of $\frac{1}{x}\gamma\left(\frac{x}{\pi^2}\right)$. The plots are at $\tau = 0.5$. We see how the suppression factor vanishes at earlier values of A as τ is increased.

SUMMARY

Summary of the 2*d* YM results:

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- Yang-Mills on S^2 undergoes a large N phase transition. So does its $T\overline{T}$-deformed version $(*)$;
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Summary of the $2d$ YM results:

- Yang-Mills on S^2 undergoes a large N phase transition. So does its $T\overline{T}$-deformed version $(*)$;
- the phase transition is induced by instantons. The same happens in the $T\overline{T}$-deformed version $(*)$;
- the theory is well-defined for all $A > 0$.The same holds true for the $T\overline{T}$-deformed version.

$*$: the critical value is lowered, and eventually only one phase exists for large values of the deformation parameter τ. This is because, at a certain value of τ, instanton sectors cease to be suppressed.

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- Among many other features the matrix model is useful to establish and study phase transitions in gauge theory, in a double-scaling limit.

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