Linear quasi-categories as templicial modules

joint work with Arne Mertens

Wendy Lowen, University of Antwerp

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Linear categories

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- The category $\mathrm{Mod}(A)$ of (left) A-modules
- Grothendieck abelian categories, by the Gabriel-Popescu theorem these are precisely localisations of module categories

Quasi-categories

A *quasi-category* is a simplicial set satisfying the weak Kan condition, that is, inner horns can be filled. Originally due to Boardman-Vogt, the theory was later developed by Joyal, Lurie in the context of higher categories.

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Future: establish linear ∞-topoi recovering Grothendieck categories as truncations

Simplicial objects

Let $\mathbf{\Delta}$ be the *simplex category*:

- objects: the posets $[n] = \{0, \ldots, n\}$ with $n \geq 0$
- order morphisms $f : [n] \to [m]$ $(i \leq j \implies f(i) \leq f(j))$

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The category $\mathbf{\Delta}$ is generated by

- coface maps $\delta_j : [n-1] \to [n]$ (which "misses j") ($0 \leq j \leq n$)
- codegeneracy maps $\sigma_i : [n+1] \longrightarrow [n]$ (which "doubles i") ($0 \leq i \leq n$)

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Let $\mathcal{V}$ be a category. The category of *simplicial $\mathcal{V}$-objects* is

$$S\mathcal{V} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{V}).$$

An important special case is the category of simplicial sets $\mathrm{SSet} = S\,\mathrm{Set}$.

Simplicial objects

Consider $S\mathcal{V} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{V})$ for a category $\mathcal{V}$ as above.
A simplicial $\mathcal{V}$-object $X \in S\mathcal{V}$ with $X_n = X([n])$ is uniquely determined by

- face maps $d_j = X(\delta_j) : X_n \longrightarrow X_{n-1}$ $(0 \leq j \leq n)$
- degeneracy maps $s_i = X(\sigma_i) : X_n \longrightarrow X_{n+1}$ $(0 \leq j \leq n)$

satisfying the simplicial identities

$$d_i d_j = d_{j-1} d_i \quad i < j$$

$$s_i s_j = s_{j+1} s_i \quad i \leq j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{id} & i = j \text{ or } i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

Nerve functors

Consider the Yoneda embedding

$$Y : \mathbf{\Delta} \to \mathrm{SSet} : [n] \mapsto \Delta^n = \mathbf{\Delta}(-, [n]).$$

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The importance of SSet in algebraic topology and homological algebra stems from realisations of $\mathbf{\Delta}$ inside other categories $\mathcal{C}$ of interest through functors

$$\rho_{\mathcal{C}} : \mathbf{\Delta} \to \mathcal{C} : [n] \mapsto \Delta^n_{\mathcal{C}}$$

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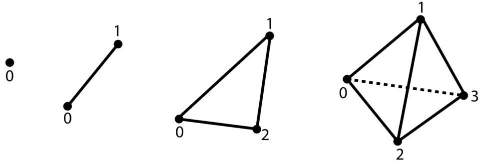
Such a cosimplicial $\mathcal{C}$-object $\rho_{\mathcal{C}}$ gives rise to a *nerve functor*

$$N_{\mathcal{C}} : \mathcal{C} \to \mathrm{SSet} : C \mapsto N_{\mathcal{C}}(C) = \mathcal{C}(\rho_{\mathcal{C}}(-), C)$$

with $N_{\mathcal{C}}(C)_n = \mathcal{C}(\Delta^n_{\mathcal{C}}, C)$.

Topological nerve

In $\mathcal{C} = \mathsf{Top}$, consider the *standard topological n-simplices* Δ^n_{Top}:



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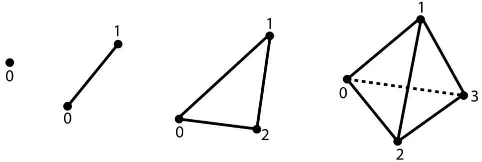

The corresponding $\rho_{\mathsf{Top}} : \mathbf{\Delta} \to \mathsf{Top} : [n] \mapsto \Delta^n_{\mathsf{Top}}$ gives rise to the *singular simplicial set* functor $\mathsf{Sing} = N_{\mathsf{Top}}$ with

$$\mathsf{Sing}(X)_n = \mathsf{Top}(\Delta^n_{\mathsf{Top}}, X).$$

Categorical nerve

In $\mathcal{C} = \mathsf{Cat}$ the category Δ^n_{Cat} has $\mathrm{Ob}(\Delta^n_{\mathsf{Cat}}) = [n]$ and

$$\mathrm{Hom}_{\Delta^n_{\mathsf{Cat}}}(i,j) = \begin{cases} * & i \leq j \\ \varnothing & \text{else} \end{cases}$$

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The corresponding $\rho_{\mathsf{Cat}} : \mathbf{\Delta} \to \mathsf{Cat} : [n] \mapsto \Delta^n_{\mathsf{Cat}}$ gives rise to the *categorical nerve* functor $N = N_{\mathsf{Cat}}$ with

$$N(\mathcal{A})_n = \coprod_{A_0,\dots,A_n \in \mathrm{Ob}(\mathcal{A})} \mathcal{A}(A_0, A_1) \times \dots \times \mathcal{A}(A_{n-1}, A_n)$$

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$$u = (\ A_0 \xrightarrow[f_1]{} A_1 \longrightarrow \ldots \longrightarrow A_{n-1} \xrightarrow[f_n]{} A_n\ ) \in N(\mathcal{A})_n$$

- $d_i(u) = (f_1, \ldots, f_{i+1}f_i, \ldots, f_n)$ for $1 \leq i \leq n-1$
- $d_0(u) = (f_2, \ldots, f_n), \qquad d_n(u) = (f_1, \ldots, f_{n-1})$

Dg and A_∞-nerve

Let k be a commutative ground ring. For $\mathcal{C} = A_\infty\text{-}\,\mathsf{Cat}$, consider the composition $\rho_{A_\infty\text{-}\,\mathsf{Cat}}$:

$$\mathbf{\Delta} \xrightarrow[\rho_{\mathsf{Cat}}]{} \mathsf{Cat} \xrightarrow[k(-)]{} \mathsf{Cat}(k) \longrightarrow A_\infty\text{-}\,\mathsf{Cat} \;\; : [n] \mapsto \Delta^n_\infty$$

This gives rise to the A_∞-nerve and its restriction, the dg nerve:

$$N_{dg} : dg\,\mathsf{Cat}(k) \to A_\infty\text{-}\,\mathsf{Cat} \to \mathsf{SSet} : \mathcal{A} \mapsto A_\infty\text{-}\,\mathsf{Fun}(\Delta^n_\infty, \mathcal{A}).$$

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Question

For k-linear categories, can we define a nerve taking values in k-modules rather than sets?

Linear nerve?

Let $\mathcal{A}$ be a k-linear category. Consider the k-modules

$$N_k(\mathcal{A})_n = \bigoplus_{A_0,\ldots,A_n \in \mathrm{Ob}(\mathcal{A})} \mathcal{A}(A_0, A_1) \otimes \ldots \otimes \mathcal{A}(A_{n-1}, A_n)$$

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Problem

The $N_k(\mathcal{A})_n$ do not constitute a simplicial k-module!

The finite interval category

Let $\mathbf{\Delta}_f$ be the *finite interval category*:

- objects: the posets $[n] = \{0, \ldots, n\}$ with $n \geq 0$
- order morphisms $f : [n] \to [m]$ with $f(0) = 0$ and $f(n) = m$

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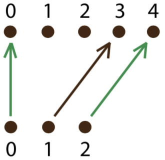

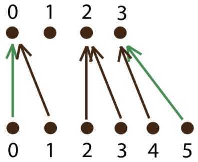

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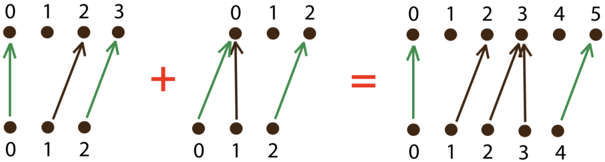
The finite interval category

The category $\mathbf{\Delta}_f$ is strict monoidal with $[n] + [m] = [n + m]$. The tensor unit is $[0]$. The sum of morphisms looks like this:



Colax monoidal functors

Let $H : \mathcal{U} \longrightarrow \mathcal{V}$ be a functor between monoidal categories. A *colax monoidal structure* on H consists of a natural transformation

$$\mu : H(- \otimes_{\mathcal{U}} -) \to H(-) \otimes_{\mathcal{V}} H(-)$$

and a morphism $\epsilon : H(I_{\mathcal{U}}) \to I_{\mathcal{V}}$ satisfying the natural coassociativity and counitality axioms. The structure is *strong monoidal* if μ is an isomorphism and *stongly unital* if ϵ is an isomorphism.

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Proposition (Leinster)

Let $(\mathcal{V}, \times, 1)$ *be a cartesian monoidal category. There is an isomorphism of categories*

$$\mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}) \simeq S\mathcal{V}.$$

In particular, we have $\mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathrm{Set}) \simeq \mathrm{SSet}$.

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For a colax monoidal functor $(X : \mathbf{\Delta}_f^{op} \to \mathcal{V}, \mu, \epsilon)$ we obtain outer face maps d_0 and d_n respectively as:

$$X_{n+1} \xrightarrow[\mu_{1,n}]{} X_1 \times X_n \xrightarrow[p_2]{} X_n$$

and

$$X_{n+1} \xrightarrow[\mu_{n,1}]{} X_n \times X_1 \xrightarrow[p_1]{} X_n$$

In general, the comultiplication μ of a colax monoidal functor is a stand-in for the outer face maps.

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Inspiration: Leinster's homotopy monoids

- generalised Deligne conjecture (Shoikhet)
- Segalic approach to enriched higher categories (Bacard)

Templicial objects

Let $(\mathcal{V}, \otimes, I)$ be a (nice) monoidal category and S a set. A $\mathcal{V}$-quiver on vertex set S consists of $\mathcal{V}$-objects $Q(a, b)$ for $a, b \in S$. The category $\mathrm{Quiv}_S(\mathcal{V})$ of $\mathcal{V}$-quivers on S is monoidal with

$$(Q \otimes_S P)(a, b) = \coprod_{c \in S} Q(a, c) \otimes P(c, b) \quad \text{and} \quad I_S(a, b) = \begin{cases} I & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

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Definition

A *templicial $\mathcal{V}$-object with base S* is a strongly unital colax monoidal functor $X : \mathbf{\Delta}_f^{op} \to \mathrm{Quiv}_S(\mathcal{V})$.

Templicial $\mathcal{V}$-objects (with varying base) form a category $S_{\otimes}\mathcal{V}$.

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We have $\mathrm{SSet} \simeq S_{\times}\, \mathrm{Set}$.

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Concretely, a templicial $\mathcal{V}$-object $X \in S_{\otimes}\mathcal{V}$ is given by

$$X_n(a, b) \in \mathcal{V}$$

for $n \in \mathbb{N}$, $a, b \in S$ with $X_0(a, b) \simeq \begin{cases} I & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$

and comultiplications

$$\mu_{n,m} : X_{n+m}(a, b) \to \coprod_c X_n(a, c) \otimes X_m(c, b).$$

Enriched nerve

Proposition

Let $\mathcal{C}$ be a small $\mathcal{V}$-category.

1. *The $\mathcal{V}$-quivers*
$$N_{\mathcal{V}}(\mathcal{C})_n = \mathcal{C}^{\otimes n}$$
can be naturally endowed with the structure of a templicial $\mathcal{V}$-object with base $\mathrm{Ob}(\mathcal{C})$ *and*
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the canonical isomorphism.
2. *There is a resulting fully faithful enriched nerve functor* $N_{\mathcal{V}} : \mathrm{Cat}(\mathcal{V}) \to S_{\otimes}\mathcal{V}$.
3. *A templicial $\mathcal{V}$-object X is isomorphic to the nerve of a small $\mathcal{V}$-category if and only if X is strong monoidal.*

Underlying simplicial set

The free-forget adjunction $F : \mathrm{Set} \leftrightarrows \mathcal{V} : U$ with $F(S) = \coprod_{a \in S} I$ and $U(V) = \mathcal{V}(I, V)$ gives rise to a free-forget adjunction $\tilde{F} : \mathrm{SSet} \leftrightarrows S_{\otimes}\mathcal{V} : \tilde{U}$ with $\tilde{U}(X)_n = S_{\otimes}\mathcal{V}(\tilde{F}(\Delta^n), X)$.

Proposition

Consider $(X, S) \in S_{\otimes}\mathcal{V}$ with underlying simplicial set $\tilde{U}(X)$. An n-simplex of $\tilde{U}(X)$ is equivalent to a pair

$$\Big((\alpha_i \in S)_{0 \leq i \leq n}, \big(\alpha_{i,j} \in U\big(X_{j-i}(\alpha_i, \alpha_j)\big)\big)_{0 \leq i < j \leq n} \Big)$$

such that for all $0 \leq i < k < j \leq n$, we have

$$\mu_{k-i, j-k}(\alpha_{i,j}) = \alpha_{i,k} \otimes \alpha_{k,j}$$

In particular, we have $\tilde{U}(X)_0 \simeq S$.

Underlying simplicial set

Proposition

Consider $(X, S) \in S_{\otimes}\mathcal{V}$ *with* $\tilde{U}(X)_n = S_{\otimes}\mathcal{V}(\tilde{F}(\Delta^n), X)$. *An n-simplex of* $\tilde{U}(X)$ *is equivalent to a pair*

$$\left((\alpha_i \in S)_{0\leq i\leq n}, \left(\alpha_{i,j} \in U\big(X_{j-i}(\alpha_i, \alpha_j)\big)\right)_{0\leq i<j\leq n}\right)$$

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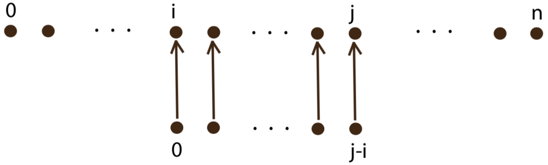
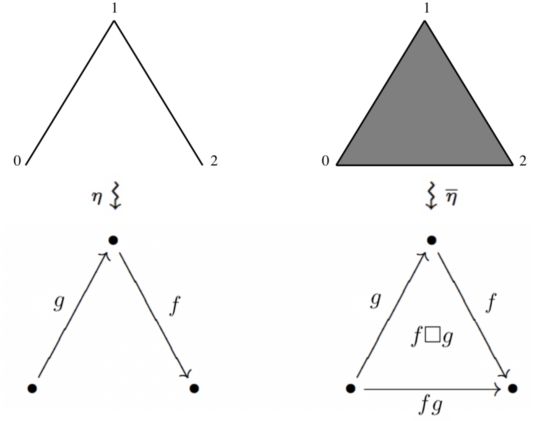


Quasi-categories

A *quasi-category* X is a simplicial set satisfying the weak Kan property, that is, inner horns can be filled. To illustrate the idea, consider $\Lambda^2_1 \subseteq \Delta^2$ and a filling $\bar{\eta} : \Delta^2 \to X$ of $\eta : \Lambda^2_1 \to X$.



Enriched quasi-categories

Definition

A templicial $\mathcal{V}$-object (X, S) is a *$\mathcal{V}$-quasi-category* if for all $0 < k < n$, every diagram of solid arrows in $S_{\otimes}\mathcal{V}$

$$\begin{tikzcd} \tilde{F}(\Lambda^n_k) \arrow[r] \arrow[d, hook] & X \\ \tilde{F}(\Delta^n) \arrow[ur, dashed] & \end{tikzcd}$$

has a lift represented by the dotted arrow. The full subcategory of $S_{\otimes}\mathcal{V}$ spanned by the $\mathcal{V}$-quasi-categories is denoted by $\mathrm{QCat}(\mathcal{V})$.

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Remark

A templicial $\mathcal{V}$-object X is a $\mathcal{V}$-quasi-category if and only if its underlying simplicial set $\tilde{U}(X)$ is an ordinary quasi-category.

Enriched quasi-categories

Proposition

Let $\mathcal{C}$ be a small $\mathcal{V}$-category with underlying category $\bar{\mathcal{C}}$. We have

$$\tilde{U}(N_{\mathcal{V}}(\mathcal{C})) = N(\bar{\mathcal{C}}).$$

In particular, $N_{\mathcal{V}}(\mathcal{C})$ is a $\mathcal{V}$-quasi-category.

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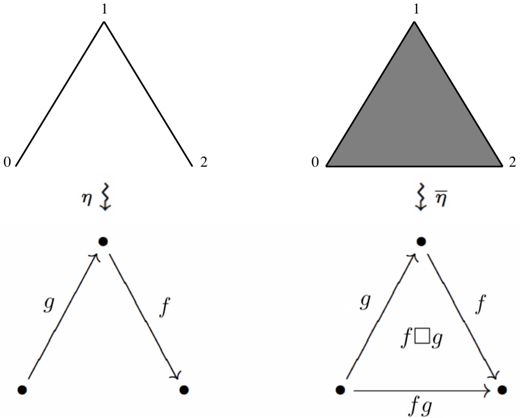
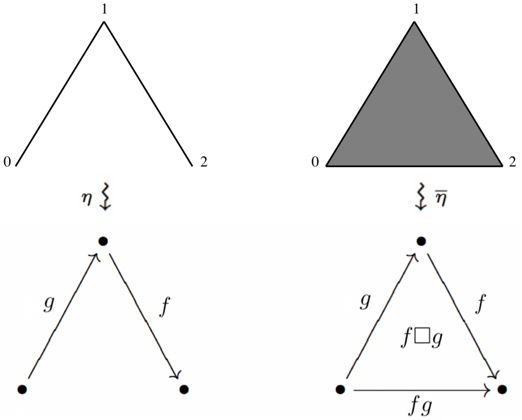
Question

Let $\mathcal{A}$ be a quasi-category. Is $\tilde{F}(\mathcal{A})$ a $\mathcal{V}$-quasi-category?

Enriched quasi-categories



Enriched quasi-categories



Observation: the 2-simplex $f \square g$ with $d_1(f \square g) = fg$ can be extracted from a morphism

$$Z : X_1 \otimes X_1 \to X_2 : f \otimes g \mapsto f \square g$$

Nonassociative Frobenius (naF) structures

Let $(H : \mathcal{U} \to \mathcal{V}, \mu, \epsilon)$ be a strongly unital colax monoidal functor. A *nonassociative Frobenius (naF) structure* on H is a natural transformation

$$Z : H(-) \otimes_{\mathcal{V}} H(-) \to H(- \otimes_{\mathcal{U}} -)$$

with unit ϵ^{-1} and such that the Frobenius identities hold:

$$\begin{tikzcd}
H(A \otimes B) \otimes H(C) \arrow[r, "\mu_{A,B} \otimes \mathrm{id}"] \arrow[d, "Z_{A \otimes B, C}"'] & H(A) \otimes H(B) \otimes H(C) \arrow[d, "\mathrm{id} \otimes Z_{B,C}"] \\
H(A \otimes B \otimes C) \arrow[r, "\mu_{A, B \otimes C}"'] & H(A) \otimes H(B \otimes C)
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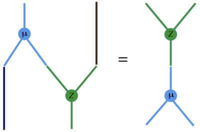

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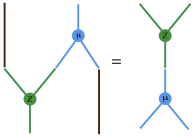

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with unit $\epsilon^{-1} : H(I_{\mathcal{U}}) \longrightarrow I_{\mathcal{V}}$ and such that the Frobenius identities hold.
If Z is moreover associative, then H is a Frobenius monoidal functor in the sense of Day-Pastro.

naF templicial objects

Definition

A *naF-templicial object* is a templicial object equipped with a naF-structure.

Concretely, this means that X is endowed with additional (nonassociative) multiplications

$$Z^{p,q} : X_p(a, c) \otimes X_q(c, b) \to X_{p+q}(a, b)$$

such that

$$\mu_{k,l} Z^{p,q} = \begin{cases} (Z^{p,k-p} \otimes \mathrm{id}_{X_l})(\mathrm{id}_{X_p} \otimes \mu_{k-p,l}) & \text{if } p \leq k \\ (\mathrm{id}_{X_k} \otimes Z^{p-k,q})(\mu_{k,p-k} \otimes \mathrm{id}_{X_q}) & \text{if } p \geq k \end{cases}$$

for all $k, l, p, q \geq 0$ such that $k + l = p + q$.

naF templicial objects (2)

Example

Let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category. Its nerve $N_{\mathcal{V}}(\mathcal{C})$ is a strong monoidal functor $\mathbf{\Delta}_f^{op} \to \mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}(\mathcal{V})$. In particular, $N_{\mathcal{V}}(\mathcal{C})$ is a naF-templicial object whose multiplication is given by the inverses of the comultiplication maps $\mu_{k,l} : \mathcal{C}^{\otimes k+l} \xrightarrow{\sim} \mathcal{C}^{\otimes k} \otimes \mathcal{C}^{\otimes l}$.

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For example, let A be a k-algebra considered as a one-object k-linear category. Then $N_k(A)_n = A^{\otimes n}$ for $n \geq 0$ and

$$T(A) = \oplus_{n\geq 0} N_k(A)_n.$$

- comultiplication: separating tensors
- multiplication: concatenating tensors

naF templicial modules

Proposition

Let X be an ordinary quasi-category. Then, X has a naF-structure.

naF templicial modules

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From now on we put $\mathcal{V} = \mathsf{Mod}(k)$. We show:

Proposition

A naF templicial module X is a linear quasi-category.

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Corollary

The free templicial module functor $\tilde{F} : \mathrm{SSet} \to S_{\otimes} \mathrm{Mod}(k)$ restricts to $\tilde{F} : \mathrm{QCat} \to \mathrm{QCat}(k)$.

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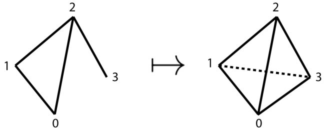
Sketch:

- The multiplications $Z^{p,q}$ fill simplices joined in a vertex, eg. $Z^{2,1} : X_2(0,2) \otimes X_1(2,3) \to X_3(0,3)$:



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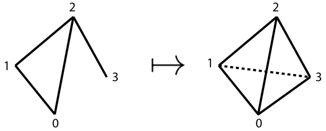
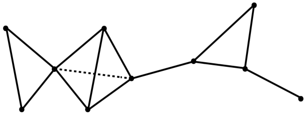
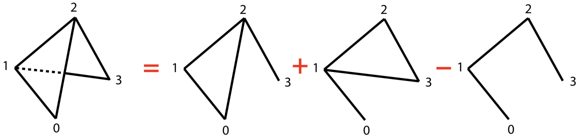


- Multiplications $Z^{p_1,\dots,p_l}$ inductively fill "necklaces"



naF templicial modules

- Consider the wedge $W^n = \partial_0\Delta^n \cup \partial_n\Delta^n \subseteq \Delta^n$. Wedges can be filled by decomposing them into necklaces, eg:



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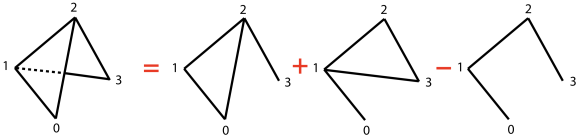
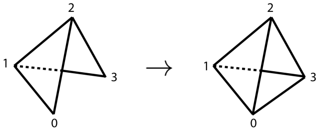


- In the linear case, filling horns is equivalent to filling wedges. Note that $W^n \subseteq \Delta^n_k$ for $0 < k < n$.



Linear quasi-categories

Theorem

We have a diagram of functors

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, shift right, "\mathcal{F}"'] \arrow[d, shift right, "N"'] & \mathrm{Cat}(k) \arrow[l, shift right, "\mathcal{U}"'] \arrow[d, shift right, "N_k"'] \\ \mathrm{QCat} \arrow[r, shift right, "\tilde{F}"'] \arrow[u, shift right, "h"'] & \mathrm{QCat}(k) \arrow[l, shift right, "\tilde{U}"'] \arrow[u, shift right, "h_k"'] \end{tikzcd}$$

which commutes in the sense that

$$N_k \circ \mathcal{F} \simeq \tilde{F} \circ N \qquad \tilde{U} \circ N_k \simeq N \circ \mathcal{U}$$

$$\mathcal{F} \circ h \simeq h_k \circ \tilde{F} \qquad h \circ \tilde{U} \simeq \mathcal{U} \circ h_k$$

Moreover, we have the following adjunctions:

$$h \dashv N, \quad h_k \dashv N_k, \quad \mathcal{F} \dashv \mathcal{U}, \quad \tilde{F} \dashv \tilde{U}$$

The linear dg nerve

Let $S_{\otimes}^{Frob}\,\mathrm{Mod}(k)$ denote the category of templicial modules with an *associative* Frobenius structure.

Theorem

There is a linear dg nerve functor

$$N_k^{dg} : dg\,\mathrm{Cat}(k) \to \mathrm{QCat}(k)$$

which gives rise to an equivalence of categories

$$dg\,\mathrm{Cat}_{\geq 0}(k) \simeq S_{\otimes}^{Frob}\,\mathrm{Mod}(k).$$

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Proof.

$dg\,\mathrm{Cat}_{\geq 0}(k)_S \simeq \mathrm{Cat}(S_+\,\mathrm{Mod}(k))_S \simeq \mathrm{Lax}(\mathbf{\Delta}_+^{op}, \mathrm{Quiv}_S(k)) \simeq \mathrm{Frob}_{su}(\mathbf{\Delta}_f^{op}, \mathrm{Quiv}_S(k)) \simeq S_{\otimes}^{Frob}\,\mathrm{Mod}(k)_S$ □

Relation with homotopy categories and dg nerve

$$\begin{tikzcd}
\mathrm{QCat}(k) \arrow[dr, shift left, "h_k"] & & dg\,\mathrm{Cat}_{\geq 0}(k) \arrow[ll, "N_k^{dg}"'] \arrow[dl, shift right, "H_0"'] \\
& \mathrm{Cat}(k) \arrow[ul, shift left, "N_k"] \arrow[ur, shift right, "\iota"'] &
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\mathrm{QCat}(k) \arrow[dr, "\tilde{U}"'] & & dg\,\mathrm{Cat}_{\geq 0}(k) \arrow[ll, "N_k^{dg}"'] \arrow[dl, "N_{dg}"] \\
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$$

Future directions

- basic theory, relations with other approaches
- homotopy theory (model structures, derived categories)
- relation with A_∞-categories
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THANK YOU!