A brief introduction to groupoids

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A roapmap of this talk:

- How groupoids describe symmetry (Symmetry beyond groups).
- Groupoids in topology:
  - Fundamental groupoid.
  - Van Kampen's theorem.
  - Applications.

Motivation

Symmetry Groupoids

Many objects which we recognize as symmetric admit few or no non-trivial symmetries. Groupoids allow to fix this.

Recall that a group is a set G together with a multiplication

$$G \times G \to G :: (g_1, g_2) \mapsto g_1 g_2$$

satisfying:

- Associativity: For all $g_i \in G$, $i \in \{1, 2, 3\}$:

$$(g_1 g_2) g_3 = g_1 (g_2 g_3)$$

- Identity: There exists an element $e \in G$:

$$ge = eg = g$$

- Inverse: For all $g \in G$ there exists $g^{-1} \in G$

$$gg^{-1} = g^{-1} g = e$$

Main example: Group of isometries of $\mathbb{R}^n$

If $x, y \in \mathbb{R}^n$

$$d(x,y) = \|x - y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)}$$

The Euclidean group is:

$$E(n) = \{\phi : \mathbb{R}^n \to \mathbb{R}^n : d(\phi(x), \phi(y)) = d(x,y), \forall x, y \in \mathbb{R}^n\}$$

with multiplication: $E(n) \times E(n) \to E(n) :: (\phi_1, \phi_2) \mapsto \phi_1 \circ \phi_2$.

Every isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is of the form:

$$\phi(x) = Ax + b$$

where $b \in \mathbb{R}^n$ and A is an orthogonal matrix:

$$AA^t = A^t A = I$$

$$isometry = orthogonal\ \ transformation + translation.$$

A proper isometry is an isometry which preserves orientation iff $\phi(x) = Ax + b$ with $det(A) = 1$.

The Euclidean group has some familiar subgroups:

- The group of translations:

$$\mathbb{R}^n = \{\phi \in E(n) : \phi \text{ is a translation}\} \cong \{b \in \mathbb{R}^n\}$$

- The orthogonal group:

$$\begin{aligned} O(n) &= \{\phi \in E(n) : \phi \text{ is an orthogonal transf.}\} \\ &\cong \{A : AA^t = I = A^t A\} \end{aligned}$$

- The special orthogonal group "rotations":

$$\begin{aligned} SO(n) &= \{\phi \in E(n) : \phi \text{ is proper}\} \\ &\cong \{A : AA^t = I = A^t A, det(A) = 1\} \end{aligned}$$

If $\Omega \subset \mathbb{R}^n$, the group of symmetries of Ω is:

$$G_\Omega = \{\phi \in E(n) : \phi(\Omega) = \Omega\}$$

the group of proper symmetries:

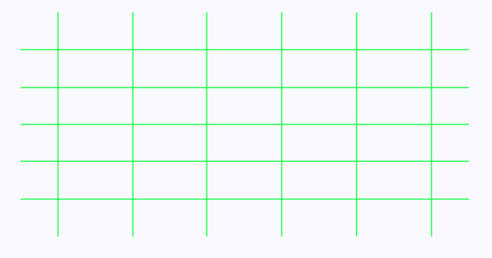
$$\tilde{G}_\Omega = \{\phi \in E(n) : \phi(\Omega) = \Omega,\ \phi \text{ is proper}\}$$

Example

$$\tilde{G}_{S^1} = SO(2) = \{\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} :\in \theta \in \mathbb{R}\}.$$

Tiling by rectangles of $\mathbb{R}^2$

Take $\Omega \subset \mathbb{R}^2$ the tiling of $\mathbb{R}^2$ by 2×1 rectangles



What is the group of symmetries of G_Ω?

Tiling by rectangles of $\mathbb{R}^2$ (cont...)

The group G_Ω consists of:

- Translations by elements of $\Gamma = 2\mathbb{Z} \times \mathbb{Z}$:

$$(x, y) \mapsto (x, y) + (2n, m) \qquad n, m \in \mathbb{Z}$$

- Reflections through points in $\frac{1}{2}\Gamma = \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$:

$$(x, y) \mapsto (n - x, m/2 - y), \qquad n, m \in \mathbb{Z}$$

- Reflections through horizontal and vertical lines:

$$\begin{aligned} (x, y) &\mapsto (x, m/2 - y) \\ (x, y) &\mapsto (n - x, y), \qquad n, m \in \mathbb{Z}. \end{aligned}$$

This tiling has a lot of symmetry!

Instead of tiling, take B a real bathroom floor:

The group of symmetries shrinks drastically:

$$G_B = \mathbb{Z}_2 \times \mathbb{Z}_2 \quad \text{and} \quad |G_B| = 4$$

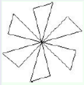
However, we can still recognize a repetitive pattern...

Not surprising! There are very few symmetry groups:

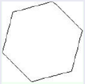
Theorem

The possible finite proper symmetry groups of a bounded region $\Omega \subset \mathbb{R}^3$ are:

- *The group C_n of rotations by $2\pi/n$ around an axis.*



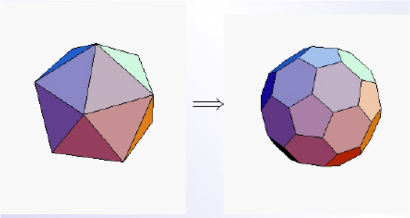
- *The group D_n of symmetries of a regular n-side polyhedron.*



- *The 3 groups of symmetries of platonic solids.*

Symmetry Groupoids

To distinguish the football ball from the icosahedron, to describe the symmetry of a bathroom floor, and in many other problems, we need groupoids



(They have the same symmetry group)

Symmetry Groupoids (cont...)

Look at the tiling Ω: Define

$$\mathcal{G}_\Omega = \{(x, \phi, y) : x, y \in \mathbb{R}^2, \phi \in G_\Omega, x = \phi(y)\}$$

with the partially defined multiplication:

$$(x, \phi, y)(y, \psi, z) = (x, \phi \circ \psi, z)$$

We can see every $g = (x, \phi, y) \in \mathcal{G}_\Omega$ as an arrow: $\cdot_x \xleftarrow{\ g\ } \cdot_y$

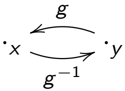
Then, we have:

- Source and target maps: $s, t : \mathcal{G}_\Omega \to \mathbb{R}^2$:

$$s(x, \phi, y) = y, \quad t(x, \phi, y) = x.$$

Symmetry Groupoids (cont...)

- Identity arrows, $1_x = (x, I, x)$: 

- Inverse arrows, $g^{-1} = (y, \phi^{-1}, x)$: 

They satisfy the group like properties:

1. Multip: $(g, h) \mapsto gh$, defined iff $s(g) = t(h)$.
2. Associa: $(gh)k = g(hk)$ whenever is defined.
3. Iden: $1_x g = g = g 1_y$ if $t(g) = x$, $s(g) = y$.
4. Inv: $gg^{-1} = 1_x$ and $g^{-1}g = 1_y$.

Definition (Groupoids from an algebraic vision)

A groupoid with base B is a set $\mathcal{G}$ with map $s, t : \mathcal{G} \to B$ and a partially defined operation satisfying $1 - 4$.

Symmetry Groupoids (cont...)

We can restrict the symmetry groupoid $\mathcal{G}_\Omega$ of the tiling to the real bathroom floor $B \subset \mathbb{R}^2$:

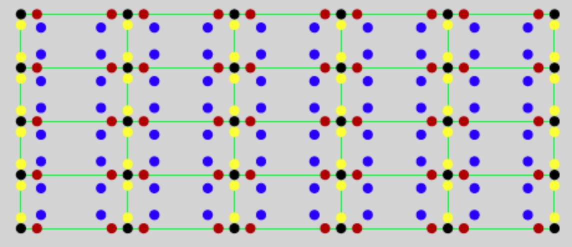
$$\mathcal{G}_B = \{(x, \phi, y) : x, y \in B, \phi \in G_\Omega, x = \phi(y)\}$$

The groupoid $\mathcal{G}_B$ captures the symmetry of the real bathroom floor. We need two elementary concepts from groupoid theory:

- Two elements $x, y \in B$ belong to the same orbit of $\mathcal{G}$ if they can be connected by an arrow: $\cdot x \xleftarrow[g]{} \cdot y$
- The isotropy group of $x \in B$ is the set of arrows $g \in \mathcal{G}$ from x to x.

Symmetry Groupoids (cont...)

For the symmetry groupoid $\mathcal{G}_B$ of the real bathroom floor: the orbits consist of points similarly place within their tile, or within the grout.



The only points with non-trivial isotropy are those in $(\mathbb{Z} \times \frac{1}{2}\mathbb{Z}) \cap B$. For these, the isotropy group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Groupoids in Topology

Groupoids play an important role in many other contexts, not related symmetry...

Definition (Groupoids from a categorical vision)

A groupoid is a category where every morphism is invertible. A category $\mathcal{C}$ consists of a collection of objects $(Obj(\mathcal{C}))$, and also for any two objects X and Y a collection of morphism $(Hom_{\mathcal{C}}(X,Y))$ $f : X \to Y$ such that for every X, There exists $id_X : X \to X$ and for every $f : X \to Y$ and $g : Y \to Z$, there is a $g \circ f : X \to Z$ such that $(fg)h = f(gh)$, $id \circ f = f$ and $g \circ id = g$.

Notice that the two definitions of groupoid given are equivalent! Categories and morphisms are as follows: set (functions), group (homomorphisms), vector space (linear transformation), topological spaces (continuous functions). A group is a 1-object groupoid!

Definition (Functor)

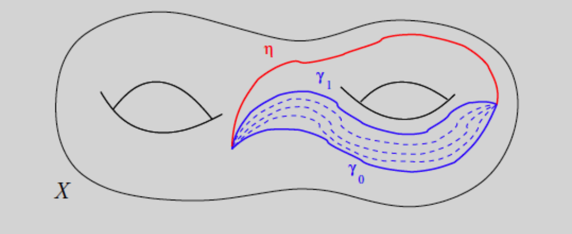
For $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are categories, a functor consists of $F : Obj(\mathcal{C}) \to Obj(\mathcal{D}) :: C \mapsto F(C)$ and for all $C_1, C_2 \in Obj(\mathcal{C})$, $F : Hom_{\mathcal{C}}(C_1, C_2) \to Hom_{\mathcal{D}}(F(C_1), F(C_2))$: $F(id_C) = id_{F(C)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Example

The power set functor $P : Set \to Set$ maps each set to its power set and each function $f : X \to Y$ to the map which sends $U \subseteq X$ to its image $f(U) \subseteq Y$. One can also consider the contravariant power set functor which sends $f : X \to Y$ to the map which sends $V \subseteq Y$ to its inverse image $f^{-1}(V) \subseteq X$.

Fundamental Groupoid of a space

Let X be a topological space. Look at continuous curves $\gamma : [0,1] \to X$ (*paths*)



$[\gamma]$ =homotopy class of γ ($[\gamma_0] = [\gamma_1]$ but $[\gamma_0] \neq [\eta]$)

Fundamental Groupoid of a space (cont...)

The fundamental groupoid of X with base $B = X$ (points of the space) is

$$\Pi(X) = \{[\gamma] : \gamma : [0,1] \to X\}$$

the structure maps are:

- source and target give initial and final points: $s([\gamma]) = \gamma(0), \qquad t([\gamma]) = \gamma(1)$
- product is concatenation of curves: $[\gamma] \cdot [\eta] = [\gamma \cdot \eta]$.
- units are the constant curves: $1_x = [\gamma]$ where $\gamma(t) = x$.
- inverse is the opposite curve: $[\gamma]^{-1} = [\bar{\gamma}]$ where $\bar{\gamma}(t) = \gamma(1-t)$.

Fundamental Groupoid of a space (cont...)

The fundamental groupoid $\Pi(X) = \{[\gamma] : \gamma : [0,1] \to X\}$ one has:

- One orbit for each connected component of X.
- Isotropy group of $x \in X$ is the fundamental group

$$\Pi(X,x) = \{[\gamma] : \gamma \quad \text{is a loop based at } x\}.$$

Remark

Notice that $\Pi : Top \to Grpd$ is a functor: For any continuous map $f : X \to Y$ we have the corresponding map $f_* : \Pi(X) \to \Pi(Y)$ such that $x \mapsto f(x)$ and $[\alpha] \mapsto [f \circ \alpha]$

Van Kampen's Theorem

Theorem (Van Kampen)

If $X = U \cup V$ is a space with U and V open sets. Then

$$\begin{array}{ccc} \Pi(U \cap V) & \longrightarrow & \Pi(U) \\ \downarrow & & \downarrow \\ \Pi(V) & \longrightarrow & \Pi(X) \end{array}$$

is a pushout square.

Warning!

Recall the classical VKT for fundamental groups:

Theorem (VKT for fundamental groups)

If $X = U \cup V$, with U and V are open and path-connected, and $U \cap V$ path-connected, then the induced homomorphism $\phi : \Pi(U, x) *_{\Pi(U \cap V, x)} \Pi(V, x) \to \Pi(X, x)$ is an isomorphism.

Remark

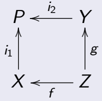
The classical VKT can't be used to compute $\Pi(S^1, x)$!. Notice that in a non-trivial decomposition of S^1 into two connected open sets, the intersection is not path connected.

Again, Groupoids allow us to fix this!

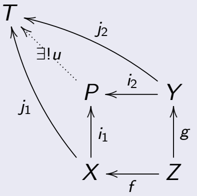
Pushout squares are special commutative diagrams:

Definition (Pushout on a category $\mathcal{C}$)

The pushout of f and g consists of an object P and two morphism $i_1 : X \to P$ and $i_2 : Y \to P$ for which the following diagram commutes:



Moreover, (P, i_1, i_2) must be universal with respect to this diagram:



Some examples:

Example

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle g}\downarrow & & \downarrow \\
E & \longrightarrow & D
\end{array}
$$

- *In $\mathcal{C} = Sets$: the pushout is $D = B \amalg E/\sim$, where $\sim$ is the equivalence relation generated by $f(a) \sim g(a)$ for all $a \in A$.*
- *In $\mathcal{C} = Grp$: the pushout $D = B *_A E$ is called amalgamated product. This can be describe as $(B * E)/\langle f(a)^{-1}g(a) : a \in A\rangle$.*

Sketch of the proof of VKT:

Let's show directly that $\Pi(X)$ satisfies the universal property.
Consider a commutative square of groupoids

$$\begin{array}{ccc} \Pi(U \cap V) & \longrightarrow & \Pi(U) \\ \downarrow & & \downarrow \Gamma \\ \Pi(V) & \xrightarrow{\Lambda} & G \end{array}$$

where G is an arbitrary groupoid. Goal: $\exists! \ \Phi : \Pi(X) \to G$.

- An object in $\Pi(X)$ is a point $x \in X$. Then either $x \in U$ or $x \in V$ (or both). If $x \in U$ then $\Phi(x) = \Gamma(x)$. Similarly if $x \in V$, $\Phi(x) = \Lambda(x)$. If $x \in U \cap V$, these definitions agree by the commutative square above.

Sketch of the proof of VKT: (cont.)

- A morphism in $\Pi(X)$ is $[\alpha]$ (homotopy class) of some path α in X. If α lay solely in U, we set $\Phi(\alpha) = \Lambda(\alpha)$. Similarly if α were in V. In general, we can always split up α into a composition $\alpha_1 \circ ... \circ \alpha_n$ of a bunch of paths, each of which lies completely in U or in V, then we can set $\Phi(\alpha) = F_1(\alpha_1) \circ ... \circ F_n(\alpha_n)$ where each F_i is either Γ or Λ.

It suffices to show that Φ is independent of the choice of decomposition, and that it really defines a functor. It is clear that it is functorial if well-defined.

Sketch of the proof of VKT: (cont.)

Say that α and β are two homotopic paths between the same pair of points in X and let $H : [0,1] \times [0,1] \to X$ be a homotopy between them.

Lemma (Lebesgue covering lemma)

Let $K \subset \mathbb{R}^n$ be compact, and $K = \bigcup_i U_i$ be an open cover. Then, there exist $\epsilon > 0$ such that $B_\epsilon(x) \cap K \subset U_j$ for some j and $x \in K$.

By the Lemma, we can subdivide the square into little squares so that each one is sent by H to either U or V. For each little square, we get an equality in the fundamental groupoid of either U or V between composites of the paths obtained by restricting H to the sides: $H|_{right} \circ H|_{top} = H|_{bottom} \circ H|_{left}$. Applying either Γ or Λ we get an equality in G. Adding them all together proves that $\Phi(\alpha) = \Phi(\beta)$. □

Actually we can define $\Pi(X, A)$ where $A \subseteq X$, which is a full subcategory of $\Pi(X)$ with objects in A: $\Pi(X, A)$ has objects just the points of A (not all X), but the morphisms are the same as before.

Theorem (Van Kampen)

If $X = U \cup V$ is a space with U and V open sets, and $A \subset X$ contains at least one point in each component of $U \cap V$, U and V. Then

$$\begin{array}{ccc} \Pi(U \cap V, A) & \longrightarrow & \Pi(U, A) \\ \downarrow & & \downarrow \\ \Pi(V, A) & \longrightarrow & \Pi(X, A) \end{array}$$

is a pushout square.

Some applications

We can use the van Kampen theorem to compute the fundamental groupoids of most basic spaces.

Example (The fundamental group of a circle)

Take U and V be open arcs, intersecting at two points $A = \{p, q\}$. Since each of U and V are contractible (homotopy type of a point). $\Pi(U, A)$ and $\Pi(V, A)$ are the groupoids with two objects p and q and a single isomorphism. $\Pi(U \cap V, A)$ is just the discrete groupoid on two objects. The pushout $\Pi(X, A)$ is therefore a groupoid on two objects and p and q with two isomorphisms $u, v : p \to q$.

Example (The fundamental group of a circle (cont.))

If we have an upper path u *and a lower path* v*, we can give a full description of the groupoid:*

- $p \to p : (v^{-1}u)^n$
- $p \to q : u(v^{-1}u)^n$
- $q \to q : (vu^{-1})^n$
- $q \to p : (v^{-1}u)^n u^{-1}$.

Therefore, $\Pi(S^1, p) = \{(v^{-1}u)^n : n \in \mathbb{Z}\} \cong \mathbb{Z}$

Other applications...

- Four dimensional manifolds can have arbitrary finitely generated fundamental group.
- The Jordan curve theorem.
- Covering spaces.
- etc...

Some references

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