Gluing local gauge conditions in BV quantum field theory

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Quantum field theory

The Batalin–Vilkovisky (BV) formalism is widely used by physicists in the study of quantum field theories with enhanced symmetry, for example, supersymmetric field theories or Chern–Simons field theory.

It is a refinement of the functional integral approach to quantum field theory, originally due to Feynman, which represents the amplitudes of the quantized theory as integrals over the space of fields.

When there are gauge symmetries, it is necessary to impose gauge conditions. Becchi–Rouet–Stora (BRS) formalism of ghost fields was developed in order to express the volume form along a gauge slice in the functional integral, writing it as a Berezin integral for an odd parity vector superbundle over the space of physical fields. This formalism may also be understood as a quantization of Lie algebra cohomology for the Lie algebra of the gauge group.

In models with supersymmetry, the BRS formalism often does not apply: the supergroup of symmetries may only act on the space of solutions of the classical field equations (the Euler-Lagrange locus) only. The BV formalism allows the extension of the symmetry away from this locus ("off-shell").

There are some parallels between the geometry of the BV formalism and symplectic geometry: for example, the Darboux and Weinstein–Darboux theorems hold in BV geometry. On the other hand, in BV geometry the Liouville theorem fails.

The replacement for the Liouville measure is additional data. According to Batalin and Vilkovisky, it consists of two parts:

- a solution of the "quantum master equation", which is obtained from the action of the original classical field theory by coupling to ghosts and extending to higher orders in $\hbar$ by performing renormalization;
- the choice of a Lagrangian in the space of fields, which may be interpreted as a choice of gauge for the field theory.

In the first part of this talk, we will explain Khudaverdian's formulation of the BV formalism, which is particularly geometric. In the second part, we describe a generalization of the BV formalism, which applies when there is no single suitable choice Lagrangian. We replace the Lagrangian by a "flexible Lagrangian", which is a coherent family of Lagrangians defined in different patches and glued together in a manner reminiscent of the proof of the de Rham theorem.

We hope that our approach will allow the treatment of topological terms in actions (nonlocal actions whose associated Euler-Lagrange equation is local), and of the Gribov ambiguity (the impossibility of making a global choice of gauge condition due to topological obstructions). Our aim is to apply this method to the quantization of the Green–Schwarz superstring.

All of the results of this talk are taken from

E. G. and Sean W. Pohorence, *Global gauge conditions in the Batalin-Vilkovisky formalism*. To appear. arXiv:1911.11269

Supermanifolds

I expect that most of the audience has passing familiarity with supermanifolds. If not, one of the best references is

D. A. Leites, *Introduction to the theory of supermanifolds*, Russian Math. Surveys, **35** (1980), no. 1, 1–64

Supergeometry is based on superspaces, which are $\mathbb{Z}/2$-graded vector spaces. (We work over $\mathbb{R}$.) Every finite superspace is isomorphic to $\mathbb{R}^{n|m}$, where n is the dimension of the even, or bosonic, subspace, and m is the dimension of the odd, or fermionic, subspace. We only consider bases of superspaces that are homogeneous. If v is a homogeneous element, denote by $|v| \in \mathbb{Z}/2$ the parity, or degree modulo 2, of v.

A commutative superalgebra is a superspace with an associative product, homogeneous of even degree in the sense that $|ab| = |a| + |b|$, and such that

$$ab = (-1)^{|a|\,|b|} ba.$$

This sign rule is familiar from de Rham theory.

Examples of commutative superalgebras are commutative algebras (concentrated in even degree) and exterior algebras $\Lambda\mathbb{R}^m$.

A supermanifold M is a ringed space $(M_0, \mathcal{O})$, where M_0 is an n-dimensional manifold (the body of M) and $\mathcal{O}$ is a sheaf of commutative superalgebras. We require that M_0 have a cover by charts U_α, equipped with an isomorphism of commutative superalgebras

$$\mathcal{O}(U_\alpha) \cong C^\infty(U_\alpha) \otimes \Lambda\mathbb{R}^m.$$

This isomorphism must induce the identity on the two sides after quotienting by nilpotent elements.

We may refine the cover so that each of the open sets U_α is a coordinate chart, with coordinates $(x^1, \ldots, x^n)$. We may also think of the generators of the exterior algebra $\Lambda\mathbb{R}^m$ as odd coordinates $(\xi^1, \ldots, \xi^m)$.

We say that the supermanifold has dimension $n|m$.

Much of differential geometry extends from manifolds to supermanifolds: differentiable maps, Taylor's theorem, the chain rule, the inverse function theorem, the theory of fiber bundles, and the definition of the tangent and cotangent bundles.

We can define the de Rham complex, but there is a big difference: the differentials $d\xi^a$ and $d\xi^b$ commute. In particular, the de Rham complex is unbounded. This means that a theory of integration must be based on a different line bundle, due to Berezin.

The definition of this line bundle is based on the geometry of the tangent bundle, which is spanned at a point of U_α by the vector space with basis

$$\left\{ \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial \xi^a} \right\} .$$

A remark on grading

In applications, all supermanifolds in the BV formalism are $\mathbb{Z}$-graded. In other words, the commutative superalgebras of functions have an auxilliary $\mathbb{Z}$-grading, which does not in general have anything to do with the parity of the functions. We will ignore this point today, since it does not affect the outlines of the formalism that we will present.

Lie supergroups

A Lie supergroup is a supermanifold with the structure of a group. We will only need linear supergroups: these are sub-supermanifolds [sic] of a general linear supergroup $\mathrm{GL}(n|m)$, which is itself an open supermanifold of the superspace of all linear maps from $\mathbb{R}^{n|m}$ to itself. We may represent such a matrix in block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

The soul of $\mathrm{GL}(n|m)$ is $\mathrm{GL}(n) \times \mathrm{GL}(m)$, parametrized by the blocks $A \in \mathrm{GL}(n)$ and $D \in \mathrm{GL}(m)$. The odd coordinates of $\mathrm{GL}(n|m)$ are the matrix entries of $B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ and $C \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$.

The Berezinian

Just as the tangent bundle of an n-dimensional manifold has structure group $\mathrm{GL}(n)$, the tangent bundle of an $n|m$-supermanifold has structure group $\mathrm{GL}(n|m)$. Given a rational representation of $\mathrm{GL}(n|m)$, we may form the associated vector bundle.

The supergroup $\mathrm{GL}(n|m)$ has a character

$$\mathrm{Ber} : \mathrm{GL}(n|m) \to \mathrm{GL}(1)$$

given by the explicit formula

$$\begin{aligned} \mathrm{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \mathrm{Ber} \begin{pmatrix} I_n & 0 \\ CA^{-1} & I_m \end{pmatrix} \mathrm{Ber} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \\ &= \frac{\det(A)}{\det(D - CA^{-1}B)} . \end{aligned}$$

Note that this character really is rational, and not polynomial.

Integration

The density bundle $|\Omega|$ on a supermanifold is the line bundle associated to the character $|\operatorname{Ber}|^{-1}$. Berezin showed that there is a covariantly defined integral

$$\int : \Gamma_c(M, |\Omega|) \to \mathbb{R}.$$

In local coordinates, a section of $|\Omega|$ has the form

$$f(x,\xi) \left| \frac{dx^1 \dots dx^n}{d\xi^1 \dots d\xi^m} \right|,$$

and its integral is obtained by integrating the function

$$\frac{\partial^m f(x,0)}{\partial\xi^1 \dots \partial\xi^m}$$

with respect to x.

Odd symplectic supermanifolds

A two-form ω determines a bundle map from TM to T^*M; it is non-degenerate if this map is invertible, and symplectic if it is non-degenerate and closed. The symplectic form is odd if its total parity is odd.

Darboux's theorem holds, with essentially the same proof as usual: there are coordinates $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ such that

$$\omega = \sum_{a=1}^{n} dx^a \wedge d\xi^a.$$

In particular, the dimension of an odd symplectic supermanifold is $n|n$.

The odd cotangent bundle $M = \Pi T^*X$, where X is a supermanifold, is an odd symplectic supermanifold. In physics, the coordinates on the base are called fields, and the coordinates along the fibres are called antifields.

The structure group of the tangent bundle of an odd symplectic supermanifold does not have a conventional notation: Manin calls it $\Pi\mathrm{Sp}(n|n)$.

The following observation was made by Khuadaverdian and Voronov. It is implicit in the work of Batalin and Vilkovisky.

Lemma

There is a character $\mathrm{Ber}^{1/2} : \Pi\mathrm{Sp}(n|n) \to \mathrm{GL}(1)$ *whose square is the restriction of the Berezinian along* $\Pi\mathrm{Sp}(n|n) \subset \mathrm{GL}(n|n)$.

In fact, we have the formula

$$\mathrm{Ber}^{1/2} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A).$$

The line bundle associated to the character $|\,\mathrm{Ber}\,|^{-1/2}$ is the half-density bundle $|\Omega|^{1/2}$. Its nontriviality is a measure of the failure of Liouville's theorem for odd symplectic supermanifolds. (By contrast, the symplectic group $\mathrm{Sp}(2n)$ is a subgroup of the special linear group $\mathrm{SL}(2n)$.)

Khudaverdian's differential operator

In terms of a Darboux coordinate system, we may write a half-density as

$$f(x,\xi)\,|dx^1 \dots dx^n|.$$

There is an operator defined on sections $\Gamma(M,|\Omega|^{1/2})$, given in a Darboux coordinate system by the formula

$$\Delta f = \sum_{a=1}^{n} \frac{\partial^2 f(x,\xi)}{\partial x^a \partial \xi^a}\,|dx^1 \dots dx^n|.$$

It is clear that $\Delta^2 = 0$ (since $\partial^2 f/\partial\xi^a\partial\xi^a = 0$).

Theorem (Khudaverdian)

The operator Δ is independent of the Darboux coordinate system.

Ševera's theorem

Ševera has found an elegant explanation for the origin of the sheaf $|\Omega|^{1/2}$ with its differential Δ, at least when M is an odd cotangent bundle ΠT^*X: he shows that it is quasi-isomorphic to the de Rham complex of M with respect to the differential

$$\alpha \mapsto d\alpha + \omega \wedge \alpha.$$

This is proved using homological perturbation theory: the contracting homotopy is quite explicit.

Pavol Ševera, *On the origin of the BV operator on odd symplectic supermanifolds,* Lett. Math. Phys., **78** (2006), no. 1, 55–59.

It is an important problem to generalize this result to more general odd symplectic supermanifolds.

The quantum master equation

A nowhere-vanishing section $s \in \Gamma(M, |\Omega|^{1/2})$ of the half-density bundle satisfying $\Delta s = 0$ is called an orientation of M. (This terminology is due to Behrend and Fantechi.) A supermanifold with a pair (ω, s) of an odd symplectic form and an orientation may be called a Batalin–Vilkovisky supermanifold. This encapsulates the basic structures of quantum field theory.

Suppose that in a Darboux coordinate system, the orientation s is given by the formula $e^S \, |dx^1 \dots dx^n|$ for some function S, called the action. Then the equation $\Delta s = 0$ becomes

$$\sum_{a=1}^{n} \left(\frac{\partial^2 S}{\partial x^a \partial \xi^a} + \frac{\partial S}{\partial x^a} \frac{\partial S}{\partial \xi^a} \right) = 0.$$

This is the quantum master equation of Batalin and Vilkovisky.

In quantum field theory, we may expand S in powers of $\hbar$ (the loop expansion):

$$S = \sum_{g=0}^{\infty} \hbar^{g-1} S_g .$$

The quantum master equation becomes

$$\sum_{a=1}^{n} \left(\frac{\partial^2 S_{g-1}}{\partial x^a \partial \xi^a} + \sum_{h=0}^{g} \frac{\partial S_h}{\partial x^a} \frac{\partial S_{g-h}}{\partial \xi^a} \right) = 0.$$

The leading term of this equation is the classical master equation

$$\sum_{a=1}^{n} \frac{\partial S_0}{\partial x^a} \frac{\partial S_0}{\partial \xi^a} = 0.$$

This equation turns out to encapsulate all generalized gauge symmetries of classical actions. . . .

Integrals on Batalin-Vilkovisky manifolds

In modern language, the problem that Batalin and Vilkovisky answer is: how do we construct linear forms $\int : \Gamma_c(M, |\Omega|^{1/2}) \to \mathbb{R}$ such that

$$\int \circ \Delta = 0.$$

We call such a linear form a trace; they are used by Batalin and Vilkovisky to define the functional integrals of quantum field theories.

Kontsevich and Schwarz (private communication) interpret traces as generalized sections $Z \in \Gamma^{-\infty}(M, |\Omega|^{1/2})$ satisfying $\Delta Z = 0$.

A Lagrangian submanifold of an odd symplectic manifold M is a supermanifold $\iota : L \hookrightarrow M$ properly embedded in M such that the symplectic form vanishes on restriction to L, and induces an isomorphism between the tangent and normal bundles of L.

It follows that the half-density bundle $|\Omega_M|_M^{1/2}$ restricts along L to the density bundle $|\Omega_L|$.

Batalin and Vilkovisky's main result is that for any $\sigma \in \Gamma_c(M, |\Omega|^{1/2})$,

$$\int_L \iota^* \Delta\sigma = 0.$$

That is, integration of half-forms on M along a Lagrangian L defines a trace.

Families of Lagrangians

In the remainder of this talk, we will give a more general construction of traces on the complex $\Gamma_c(M, |\Omega|^{1/2})$ of half-densities. The key ingredient is a formula of Mikhailov and Schwarz for the variation of $\int_{L_t}$ when L_t varies in a family of Lagrangians.

The tangent space at L to the space of Lagrangians in a symplectic manifold equals the space of closed 1-forms on L. The same result holds for Lagrangians in odd symplectic manifolds.

There is one small difference: the resulting 1-form has odd parity, and the odd de Rham cohomology vanishes, hence the 1-form is actually exact, and represented by an odd function on L. Furthermore, this function is uniquely determined.

If L_t is a one-parameter family of Lagrangians, let η_t be the corresponding family of functions. The 1-form $\eta_t \, dt$ has even total parity, since the 1-form dt has odd parity in the de Rham complex of the interval.

Let $\Delta^k = \{0 \leq t_1 \leq \cdots \leq t_k \leq 1\} \subset \mathbb{R}^k$ be the k-dimensional simplex.

If $\iota : L \times \Delta^k \to M$ is an k-dimensional family of Lagrangians, we may apply the above construction to obtain a 1-form η on $L \times \Delta^k$, of even total parity.

Let δ be the de Rham differential along Δ^k.

Theorem

If $\sigma \in \Gamma_c(M, |\Omega|^{1/2}) \otimes \Omega^*(\Delta^k)$, we have

$$\delta \int_L e^{-\eta} \iota^* \sigma = \int_L e^{-\eta} \iota^* (\delta + \Delta) \sigma.$$

The proof uses the Darboux–Weinstein theorem for Lagrangians in odd symplectic manifolds; this is proved in the same way as the usual Darboux-Weinstein theorem, using Moser's trick. The theorem of Mikhailov and Schwarz is an infinitesimal version of this formula.

Flexible Lagrangian submanifolds

Let M be an odd symplectic supermanifold, and let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open cover, with partition of unity $\{\phi_\alpha\}$.

By a flexible Lagrangian, we mean a collection of parametrized families of proper embeddings

$$\begin{array}{ccc} L_{\alpha_0\dots\alpha_k} \times \Delta^k & \longrightarrow & U_{\alpha_0\dots\alpha_k} \times \Delta^k \\ & \searrow \quad \Delta^k \quad \swarrow & \end{array}$$

one for each multi-index $\alpha_0 \dots \alpha_k$.

These families are required to glue together appropriately as we approach the faces. For example, as $t \to 0$, $\iota_{\alpha_0\alpha_1}(t)$ approaches the intersection of $\iota_{\alpha_0} : L_{\alpha_0} \to M_{\alpha_0}$ with U_{α_1}, while as as $t \to 1$, it approaches the intersection of $\iota_{\alpha_1} : L_{\alpha_1} \to M_{\alpha_1}$ with U_{α_0}.

The Thom–Whitney complex

We borrow an idea from Sullivan's rational homotopy theory. The Thom-Whitney complex $\operatorname{Tot}|\Omega|^{1/2}(\mathcal{U})$ is the subspace of

$$\prod_{k=0}^{\infty} \prod_{\alpha_0 \dots \alpha_k} \Gamma(U_{\alpha_0 \dots \alpha_k}, |\Omega|^{1/2}) \otimes \Omega^*(\Delta^k)$$

compatible with all of the simplicial maps. (This is formally similar to the definition of the geometric realization of a simplicial space: this similarity is no accident.)

The differential on this complex is $\delta + \Delta$, and the complex is quasi-isomorphic to the Čech complex of the sheaf $|\Omega|^{1/2}$, and hence, since these sheaves are fine, to the $\mathbb{Z}/2$-graded complex $\Gamma(M, |\Omega|^{1/2})$ with differential Δ.

The main theorem

Let $\Phi_{\alpha_0\dots\alpha_k}$ be the differential operator on $|\Omega|^{1/2}(U_{\alpha_0\dots\alpha_k})$ given by the formula

$$\Phi_{\alpha_0\dots\alpha_k} = \frac{1}{k+1}\sum_{i=0}^{k}[\Delta,\phi_{\alpha_0}]\dots[\Delta,\phi_{\alpha_{i-1}}]\phi_{\alpha_i}[\Delta,\phi_{\alpha_{i+1}}]\dots[\Delta,\phi_{\alpha_k}].$$

Theorem

Let $\sigma_\bullet$ be a compactly supported element of the Thom–Whitney complex $\operatorname{Tot}|\Omega|^{1/2}(\mathcal{U})$. *Define the linear form*

$$\mathrm{Z}(\sigma_\bullet) = \sum_{k=0}^{\infty}(-1)^k \sum_{\alpha_0\dots\alpha_k} \int_{\Delta^k}\int_{L_{\alpha_0\dots\alpha_k}} e^{-\eta_{\alpha_0\dots\alpha_k}}\, \iota^*_{\alpha_0\dots\alpha_k}\big(\Phi_{\alpha_0\dots\alpha_k}\sigma_{\alpha_0\dots\alpha_k}\big).$$

Then Z is a trace: $\mathrm{Z}\big((\delta+\Delta)\sigma_\bullet\big) = 0$.

A rank one BV model is one where S has linear dependence on the antifields. The applications in our paper are to the superparticle, which is a rank two theory, in which the action has a quadratic dependence on the antifields.

The BRS formalism is a special case of rank one models. In this setting, our main result is contained in the article
C. Becchi, G. Giusto and C. Imbimbo, *The functional measure of gauge theories in the presence of Gribov horizons*, arXiv:hep-th/9811018