Symmetric functions in noncommuting variables and supercharacters of unitriangular groups

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Outline

- Symmetric functions in noncommuting variables (**NCSym**) and set partitions
- Group representations and characters
- Supercharacter theories
- A supercharacter theory for the unitriangular group
- The algebra of superclass functions **SC**
- The isomorphism **SC** $\simeq$ **NCSym**

Symmetric functions in noncommuting variables

We denote by $\mathbb{C}\langle\langle X\rangle\rangle$ the $\mathbb{C}$-algebra of formal power series in the noncommuting variables $X = \{x_1, x_2, \ldots\}$.

The symmetric group S_n acts on $\mathbb{C}\langle\langle X\rangle\rangle$ by permutation of the variables:

$$\sigma \cdot f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots), \quad \sigma \in S_n.$$

Definition

The *algebra of symmetric function in noncommuting variables* is the subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ formed by all formal power series of bounded degree that are invariant for the action of S_n, for all $n \in \mathbb{N}$.
We denote this algebra by **NCSym**.

Set partitions

Definition

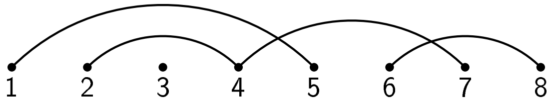
A *set partition* of $[n] = \{1, \ldots, n\}$ is a family $\pi = \{B_1, \ldots, B_\ell\}$ of nonempty and mutually disjoint subsets of $[n]$ whose union is $[n]$. We write $\pi = B_1/B_2/\cdots/B_\ell \vdash [n]$.

The set of *arcs* of π is

$$\mathcal{D}(\pi) = \{(i,j) \in [n] \times [n] : i < j, i \text{ and } j \text{ are in the same block } B_s \text{ and}$$
$$\text{there is no } k \in B_s \text{ such that } i < k < j\}$$

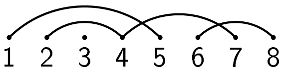
Example

Let $\pi = 15/247/3/68 \vdash [8]$. We can represent π by the following diagram:



Set partitions

From the diagram representation of $\pi = 15/247/3/68 \vdash [8]$ we can get a matrix representation:



$$e_\pi = \sum_{(i,j)\in\mathcal{D}(\pi)} e_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives a bijection between set partitions of $[n]$ and $n \times n$ matrices with entries equal to 0 or 1 and at most one 1 in each row and column.

Some operations on set partitions

"Concatenation" of set partitions:
Let $\pi = B_1/B_2/\cdots/B_s \vdash [m], \sigma = C_1/C_2/\cdots/C_r \vdash [n]$. We define $\pi/\sigma \vdash [m+n]$ as

$$\pi/\sigma = B_1/B_2/\cdots/B_s/C_1+m/C_2+m/\cdots/C_r+m.$$

(We place the diagram of σ after the diagram of π and relabel the vertices.)

"Reduction" of a set partition:
Let $\pi \vdash [n]$ and let $I \subseteq [n]$ such that $|I| = m$. We denote by π_I the set partition of $[m]$ that is obtained in the following way:

- in the arc diagram of π, delete points that are not in I and arcs that connect to at least one such point;
- relabel the remaining points in increasing order with the elements of $[m]$.

The monomial basis of **NCSym**

Let $\pi \vdash [n]$. A *monomial of shape* π is a product of indeterminates

$$x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $i_r = i_s$ if and only if r and s are in the same block of π.
The *monomial symmetric function* $m_\pi \in \mathbf{NCSym}$ is the sum of all monomials of shape π.

We denote by $\mathbf{NCSym}_n$ the vector space of symmetric functions of degree n.

Theorem

The set $\{m_\pi : \pi \vdash [n]\}$ *is a basis for the vector space* $\mathbf{NCSym}_n$.

The Hopf algebra **NCSym**

Theorem

Let $\pi \vdash [m]$ *and* $\sigma \vdash [n]$. *We have*

$$m_\pi \cdot m_\sigma = \sum_{\substack{\tau \vdash [m+n] \\ \tau \wedge (\hat{1}_m / \hat{1}_n) = \pi/\sigma}} m_\tau .$$

We can also define a *coproduct* $\Delta : \mathbf{NCSym} \to \mathbf{NCSym} \otimes \mathbf{NCSym}$ on **NCSym** as follows: if $\pi \vdash [n]$, then

$$\Delta(m_\pi) = \sum_{I \subseteq [n]} m_{\pi_I} \otimes m_{\pi_{I^c}} .$$

Theorem

NCSym *is a Hopf algebra.*

Group representations

Definition

A *representation* of a group G is a group homomorphism

$$\rho : G \to GL(n, \mathbb{C}).$$

We are especially interested in *irreducible* representations: these are, in some sense, the simplest representations of G and the building blocks for all the other representations of G.

Characters

Definition

Let $\rho : G \to GL(n, \mathbb{C})$ be a representation of a group G. The *character* of G afforded by ρ is the function

$$\chi : G \to \mathbb{C}, \qquad \chi(g) = \mathrm{tr}(\rho(g)).$$

A character is said to be *irreducible* if it is afforded by an irreducible representation of G. We denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G.

- Characters are *class functions*: a character χ takes a constant value on any conjugacy class K of G.
- The number of irreducible characters of G is equal to the number of conjugacy classes of G.

Characters

- Given a character χ, we have

$$\chi = \sum_{\psi \in \mathrm{Irr}(G)} n_\psi \psi,$$

  where $n_\psi \in \mathbb{N}_0$ are not all zero. (If $n_\psi \neq 0$, we say that ψ is a *constituent* of χ).

We can define an inner product on the vector space of all complex-valued functions on a group G:

$$\langle \varphi, \psi \rangle = \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

- Irreducible characters are an orthonormal basis for the vector space of class functions on G.

Supercharacter theories

Definition

A *supercharacter theory* of a finite group G is a pair $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ is a set partition of G and $\mathcal{Y}$ is a set of mutually orthogonal characters of G, such that:

(i) $|\mathcal{X}| = |\mathcal{Y}|$;

(ii) every character $\chi \in \mathcal{Y}$ takes a constant value on each set $K \in \mathcal{X}$;

(iii) each irreducible character of G is a constituent of one of the characters $\chi \in \mathcal{Y}$.

The sets $K \in \mathcal{X}$ are called *superclasses* and the characters $\chi \in \mathcal{Y}$ are called *supercharacters*.

Unitriangular group

Let $\mathbb{F}$ be a finite field of order q and $n \in \mathbb{N}$.

The *unitriangular group* U_n is the group of unitriangular matrices with entries from $\mathbb{F}$, that is, matrices of the form

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

We have $U_n = 1 + \mathfrak{u}_n$, where $\mathfrak{u}_n$ is the $\mathbb{F}$-algebra of strictly upper triangular matrices with entries from $\mathbb{F}$.

$\mathbb{F}^\times$-coloured set partitions

Definition

Let $\mathbb{F}$ be a finite field. An $\mathbb{F}^\times$-*colouring* of a set partition $\pi \vdash [n]$ is a map $\phi : \mathcal{D}(\pi) \to \mathbb{F}^\times$.

We denote:

$$\mathcal{S}_n(\mathbb{F}) = \{(\pi, \phi) : \pi \vdash [n], \phi \text{ is a } \mathbb{F}^\times\text{-colouring of } \pi\}.$$

To every $\mathbb{F}^\times$-coloured partition $(\pi, \phi) \in \mathcal{S}_n(\mathbb{F})$ we associate the matrix

$$e_{\pi,\phi} = \sum_{(i,j)\in\mathcal{D}(\pi)} \phi(i,j) e_{ij}.$$

Superclasses of U_n

The group U_n acts on $\mathfrak{u}_n$ by left and right multiplication:

$$\begin{aligned}(U_n \times U_n) \times \mathfrak{u}_n &\to \mathfrak{u}_n \\ ((g,h),a) &\mapsto gah^{-1}\end{aligned}$$

In each orbit $U_n a U_n$, $a \in \mathfrak{u}_n$, there is exactly one matrix of the form $e_{\pi,\phi}$, for some $(\pi,\phi) \in \mathcal{S}_n(\mathbb{F})$.

The superclasses of U_n are the sets

$$\mathcal{K}_\pi = \prod_\phi (1 + U_n e_{\pi,\phi} U_n),$$

for $\pi \vdash [n]$.

Supercharacters of U_n

The unitriangular group U_n acts on $\mathfrak{u}_n^\circ := \mathrm{Irr}(\mathfrak{u}_n^+)$:

$$(U_n \times U_n) \times \mathfrak{u}_n^\circ \to \mathfrak{u}_n^\circ$$
$$((g,h),\theta) \mapsto g^{-1}\theta h$$

where $g\theta h(a) = \theta(gah^{-1})$ for all $a \in \mathfrak{u}_n$.

There is an isomorphism $\mathfrak{u}_n^+ \simeq \mathfrak{u}_n^\circ$. In particular, we can consider elements $\theta^{\pi,\phi} \in \mathfrak{u}_n^\circ$ corresponding by this isomorphism to $e_{\pi,\phi} \in \mathfrak{u}_n$.

For each $\theta \in \mathfrak{u}_n^\circ$, we define

$$\chi^\theta(g) = \frac{|U_n\theta|}{|U_n\theta U_n|} \sum_{\tau \in U_n\theta U_n} \tau(g-1), \quad g \in U_n.$$

Lemma

For any $\theta, \theta' \in \mathfrak{u}_n^\circ$, we have

$$\chi^\theta = \chi^{\theta'} \quad \text{if and only if} \quad U_n\theta U_n = U_n\theta' U_n.$$

Supercharacters of U_n

We denote by $\chi^{\pi,\phi}$ the character corresponding to the two-sided orbit of $\theta^{\pi,\phi}$.

The supercharacters of U_n are

$$\chi^{\pi} = \sum_{\phi} \chi^{\pi,\phi},$$

for $\pi \vdash [n]$.

Theorem

The pair $(\mathcal{X}, \mathcal{Y})$, where

$$\mathcal{X} = \{\mathcal{K}_{\pi} : \pi \vdash [n]\} \quad \textit{and} \quad \mathcal{Y} = \{\chi^{\pi} : \pi \vdash [n]\}$$

is a supercharacter theory for U_n.

The algebra $\mathbf{SC}$

For each $n \in \mathbb{N}$, we define $\mathbf{SC}_n$ to be the complex vector space spanned by the supercharacters χ^π of U_n.
We define $\mathbf{SC}_0 = \mathbb{C}$.

For each n, $\mathbf{SC}_n$ consists of superclass functions for U_n; that is, complex-valued functions of G that are constant on each superclass $\mathcal{K}_\pi$.

The set $\{\kappa_\pi : \pi \vdash [n]\}$, where κ_π is the characteristic function of $\mathcal{K}_\pi$, is a basis for $\mathbf{SC}_n$.

The *space of superclass functions of U_n* is defined to be

$$\mathbf{SC} = \bigoplus_{n \geq 0} \mathbf{SC}_n.$$

The algebra **SC**

We define a product on **SC** as follows: given supercharacters χ^π and χ^σ,

$$\chi^\pi \cdot \chi^\sigma = \chi^{\pi/\sigma}.$$

Theorem

Let $\pi \vdash [m]$ and $\sigma \vdash [n]$. We have

$$\kappa_\pi \cdot \kappa_\sigma = \sum_{\substack{\tau \vdash [m+n] \\ \tau \wedge (\hat{1}_m/\hat{1}_n) = \pi/\sigma}} \kappa_\tau .$$

Theorem

There is an isomorphism of $\mathbb{C}$-algebras

$$ch : \mathbf{SC} \to \mathbf{NCSym}$$

defined by $ch(\kappa_\pi) = m_\pi$ for all $\pi \vdash [n]$ and all $n \in \mathbb{N}$.

The Hopf algebra **SC**

Recall the coproduct in **NCSym**:

$$\Delta(m_\pi) = \sum_{I \subseteq [n]} m_{\pi_I} \otimes m_{\pi_{I^c}}\,.$$

Using the isomorphism ch, we can transport this coproduct to **SC**:

$$\Delta(\kappa_\pi) = \sum_{I \subseteq [n]} \kappa_{\pi_I} \otimes \kappa_{\pi_{I^c}}\,.$$

Theorem

There is an isomorphism of Hopf algebras

$$ch : \mathbf{SC} \to \mathbf{NCSym}$$

defined by $ch(\kappa_\pi) = m_\pi$ *for all* $\pi \vdash [n]$ *and all* $n \in \mathbb{N}$*.*

The Hopf algebra SC

We can give a representation theoretical interpretation of the product and coproduct in **SC**.

Product:

$$\phi \cdot \psi = \mathrm{Inf}_{U_m \times U_n}^{U_{m+n}}(\phi \otimes \psi), \quad \phi \in \mathbf{SC}_m, \psi \in \mathbf{SC}_n$$

Coproduct:

$$\Delta(\phi) = \sum_{J \subseteq [n]} {}^{J}\mathrm{Res}_{U_{|J|} \times U_{|J^c|}}^{U_n}(\phi), \quad \phi \in \mathbf{SC}_n$$

References

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