"Problems" in Inverse Semigroups

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First results on the word problem

Anisimov's theorem for groups

Anisimov's theorem first generalization to inverse semigroups

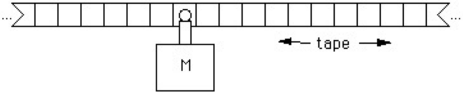
Anisimov's theorem second generalization to inverse semigroups

Description of the word problem in different structures

In the first section we are going to study the word problem and its decidability in different structures. To do that, first we need to introduce some ideas and definitions.

Description of the word problem in different structures

Turing Machine



Definition (X^+)

For a set X (alphabet), X^+ denotes the set of finite sequences (words) of elements of X.

If ε denotes the *empty word*, then $X^* = X^+ \cup \{\varepsilon\}$.

Example

If $X = \{x_1, x_2\}$, then $X^+ = \{x_1, x_2, {x_1}^2, x_1x_2, x_2x_1, {x_2}^2, \ldots\}$.

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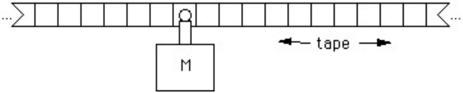

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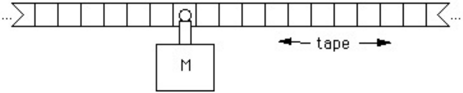

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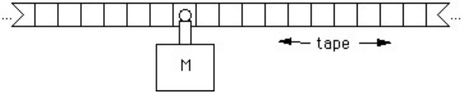

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Description of the word problem in different structures

Definition (Recursively Enumerable set)

A subset $E \subseteq X^+$ is *recursively enumerable* if there is a Turing Machine T whose alphabet contains X and T accepts w iff $w \in E$.

Definition (Recursive set)

A set $R \subseteq X^+$ is *recursive* if both R and $X^+ \setminus R$ are recursively enumerable subsets of X^+.

Idea (Word problem)

A semigroup/group S with generators $X = \{x_1, \ldots, x_n, \ldots\}$ has a *solvable word problem* if there is a "decision process" to determine, for an arbitrary pair of words $w, z \in X^+$, whether they are equal in S.

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For example,

$$(\mathbb{Z}_n, \cdot) \;=\; \langle x : x^n = 1\rangle \;=\; \{1, x, x^2, \ldots, x^{n-1}\} \;=\; \frac{\{x\}^+}{\langle (x^n, 1)\rangle},$$

$$x^{n+1} \equiv x$$

$$T \;=\; \langle x, y : x^2 = x, yx = y\rangle \;=\; \frac{\{x, y\}^+}{\langle (x^2, x), (yx, y)\rangle}$$

$$x^n yxy^2 \equiv xy^3 \equiv x^2 yxyxy$$

Notation

- If $X = \{x_1, \ldots, x_n, \ldots\}$, X^{-1} denotes the set $\{x_1^{-1}, \ldots, x_n^{-1}, \ldots\}$ with $X \cap X^{-1} = \emptyset$.

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Decidability of the word problem in different structures

Definition (Word problem for a semigroup)

A semigroup S generated by $X = \{x_1, \ldots, x_n, \ldots\}$ has a *solvable word problem* if the set

$$\{(w, z) \in X^+ \times X^+ : w = z \text{ in } S\} \text{ is recursive.}$$

Definition (Word problem for a group)

If G is a group with generators $X = \{x_1, \ldots, x_n, \ldots\}$, then G has a *solvable word problem* if the set

$$\{w \in (X \cup X^{-1})^+ : w = 1 \text{ in } G\} \text{ is recursive.}$$

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Decidability of the word problem in different structures

Theorem (Markov-Post, 1947)

There exists a semigroup (finitely presented) with an unsolvable word problem.

The proof relies on a theorem of Kleene which asserts that exists a recursively enumerable subset of the natural numbers that is not recursive and

$$\text{Turing Machine} \longrightarrow \text{Semigroup (finitely presented)}$$

Theorem (Novikov-Boone-Britton, 1954-1958)

There exists a group (finitely presented) with an unsolvable word problem.

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Word problem for an inverse semigroup

Definition (Inverse Semigroup)

A semigroup S is *inverse* if

$$\forall x \in S, \exists! x^{-1} \in S, \quad xx^{-1}x = x \quad \wedge \quad x^{-1}xx^{-1} = x^{-1}.$$

Groups $\hookrightarrow Sym(X)$ (Cayley's theorem)

Inverse semigroups $\hookrightarrow I(X)$ (Wagner-Preston's theorem).

$I(X)$ - injective partial transformations on X.

Definition (Word problem for an inverse semigroup)

An inverse semigroup S generated by $X = \{x_1, \ldots, x_n, \ldots\}$, has a *solvable word problem with respect to* X, if the set

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The word problem for the free group

The free group FG_X on a set $X \neq \emptyset$, as a solvable word problem.

Sketch of a decision process:

- Input w ;
- Compute $\overline{w}$;
- If $\overline{w} = \varepsilon$, write $w = 1$ in FG_X and stop
  else write $w \neq 1$ in FG_X and stop .

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The word problem for the free inverse semigroup

The free inverse semigroup FIS_X on a set $X \neq \emptyset$, as a solvable word problem.

But, things get more complicated…

reduced words on $X \rightsquigarrow$ Munn trees on X (MT_X)

Munn tree of both $w = aaa^{-1}cc^{-1}b$ and $z = abb^{-1}cc^{-1}aa^{-1}b$.

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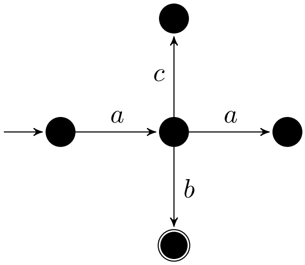

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Theorem (Munn, 1974)

There exists an isomorphism between FIS_X and MT_X.

Sketch of a decision process:

- Input (w, z) ;
- Compute Munn trees of w and z ;
- Compare the respective Munn trees ;
  If they are equal write $w = z$ in FIS_X and stop
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Anisimov's theorem original statement

Up to now, we have been working on the decidability of the word problem in certain algebraic structures by Turing Machines.

From now on, we are going to think about the question: how does the word problem decidability by a "weaker" Turing Machine (automaton) interacts with the finiteness of the algebraic structure?

We will present some concepts to state Anisimov's theorem, which relates both.

Anisimov's theorem original statement

Definition (Finite State Automaton (FSA))

A *Finite State Automaton* $\mathcal{A}$ is a tuple $\mathcal{A} = \langle Q, X, q_0, F, \delta \rangle$, with:

- Q a finite set of *states*;
- X an *alphabet*;
- $q_0 \in Q$ an *initial state*;
- $F \subseteq Q$ a set of *final states*;
- $\delta \subseteq Q \times (X \cup \{\varepsilon\}) \times Q$ a *transition relation*.

Examples

Munn trees are automata;

The automaton given by: $Q = \{q_0, q_1\}$, $X = \{x, y\}$, $F = \{q_1\}$ and $\delta = \{(q_0, x, q_1), (q_0, y, q_1), (q_1, y, q_1)\}$.

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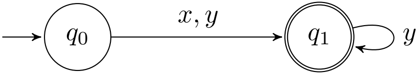

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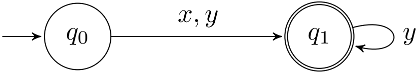


Definitions

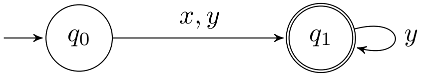
- A *computation* on $\mathcal{A}$ from q_1 to q_{n+1} is a finite sequence of transitions:

$$q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} q_3 \ldots \xrightarrow{x_{n-1}} q_n \xrightarrow{x_n} q_{n+1}$$

- A word $w = x_1x_2\ldots x_{n-1}x_n$ is *accepted* by $\mathcal{A}$ if in the above computation $q_1 = q_0$ and $q_{n+1} \in F$.

For example, $w = xy$ is accepted by the above automaton.

Anisimov's theorem original statement



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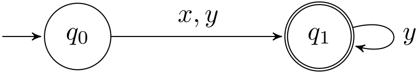

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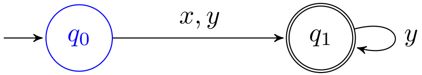

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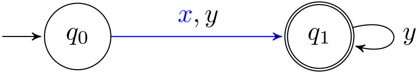

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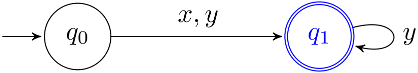

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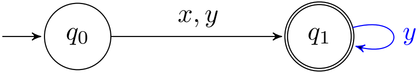

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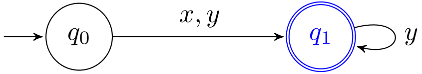

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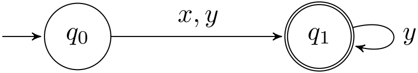

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A language L is a subset of X^*.

If G is a group generated by X, then its word problem, $\{w \in (X \cup X^{-1})^+ : w = 1 \text{ in } G\}$ is a language on X.

Definition (Regular Language)

A language is regular if there is an FSA accepting precisely its words.

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Generalizations to inverse semigroups

Is Anisimov's theorem true for inverse semigroups?

The word problem for groups can be stated in two equivalent ways:

$$w = z \tag{1}$$

$$u = 1 \tag{2}$$

For inverse semigroups (1) was mentioned before and (2) is generalized by the idempotent problem

$$w^2 = w$$

observing that idempotents in inverse semigroups are closely related with the identity on groups.

Analogous of Anisimov's theorem will be considered for these two different problems.

First generalization to inverse semigroups

N. D. Gilbert, R. N. Heale & M. Kambites' view ($w \equiv e$)

When does a word represent an idempotent?

Definition (Idempotent problem)

For an inverse semigroup S generated by X, the *idempotent problem of S with respect to X is regular*, if the language

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Proposition (Gilbert & Heale, 2013)

If S is a finite inverse semigroup generated by X, then its idempotent problem is regular.

The proof relies on the construction of the Cayley graph $Cay(S, X)$, which is then converted into an automaton.

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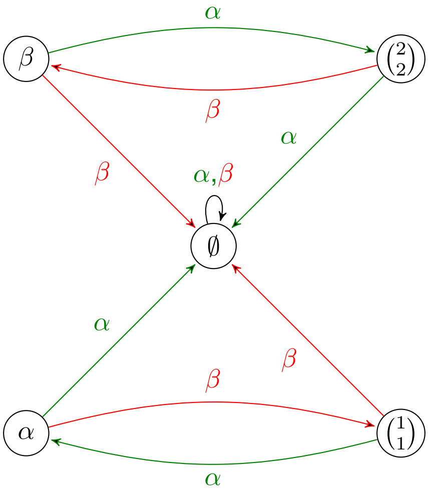

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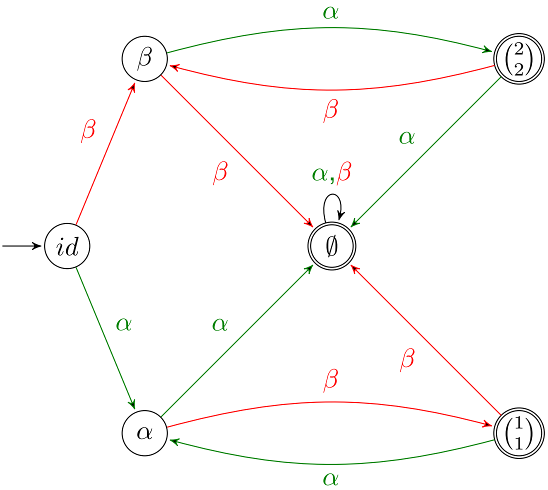

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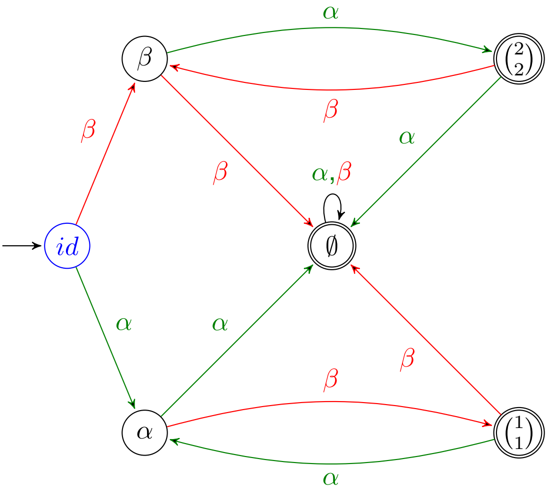

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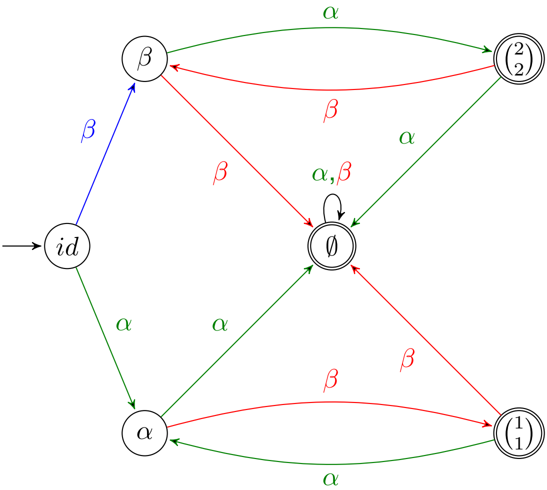

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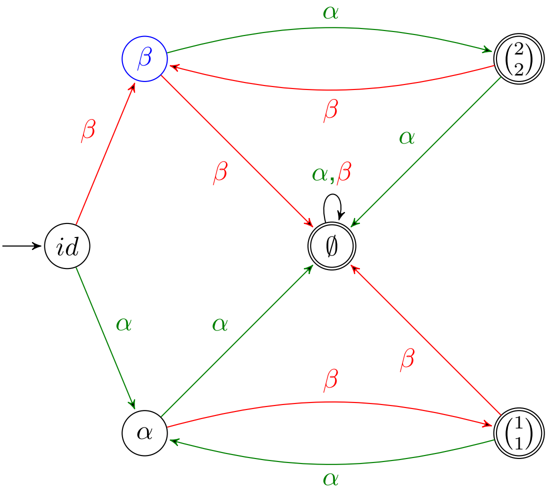

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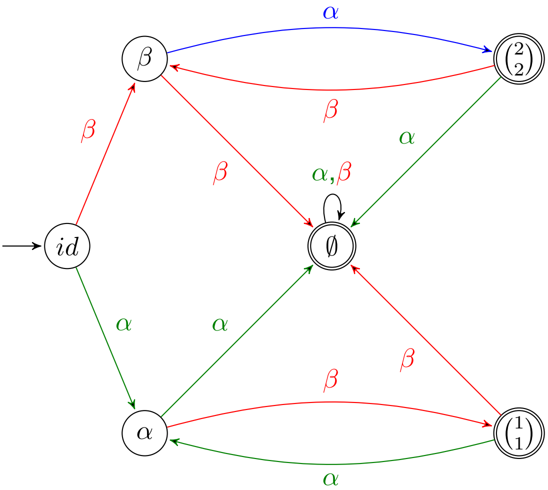

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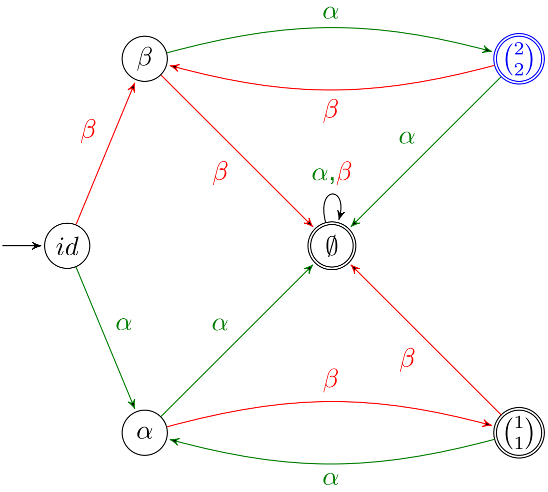

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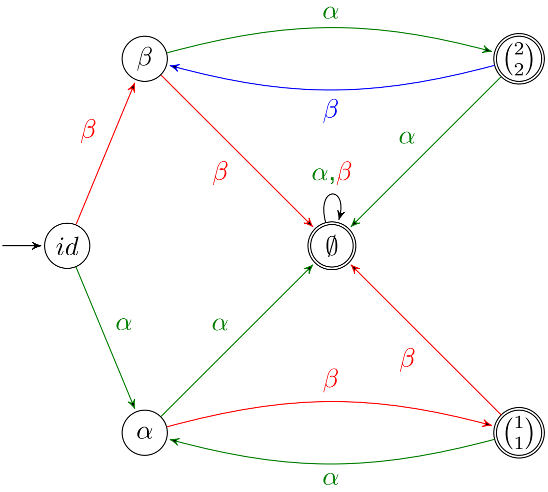

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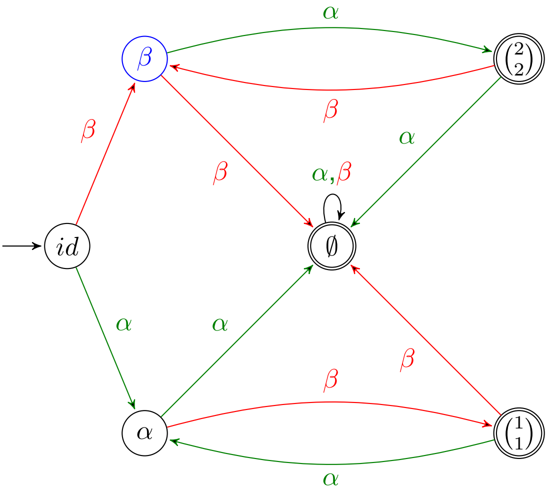

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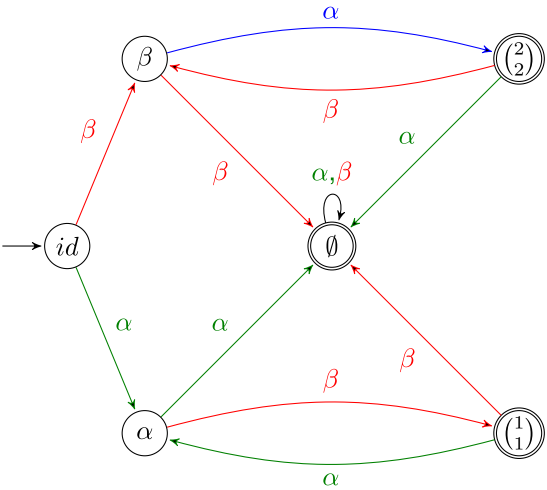

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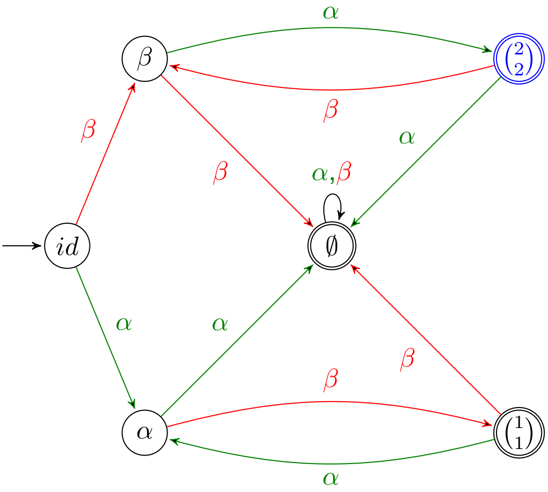

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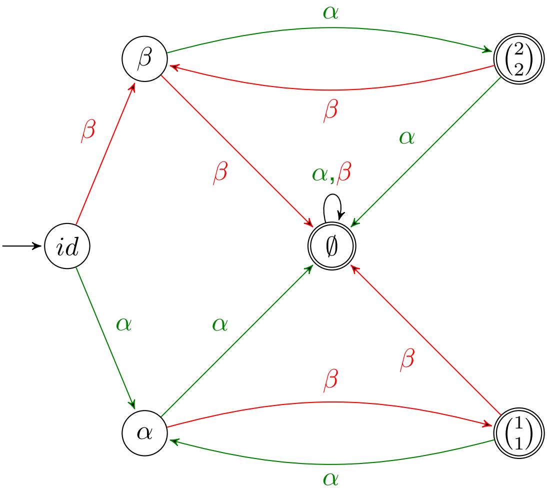

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Question (Gilbert & Heale)

Is the converse of the above proposition true? Yes!

Theorem (Kambites, 2013)

If S is a finitely generated inverse semigroup with regular idempotent problem, then S is finite.

This is an heavy proof, that is basically the paper of Kambites. It relies on a theorem of Billhardt, which allows to embed an inverse semigroup S into a λ-semidirect product of a quotient of S by a certain semilattice related to S and on the characterization of the syntactic monoid of the idempotent problem of S with respect to X.

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When do w and z are equal in the inverse semigroup?

To take an FSA is no longer enough, one must require two tapes.

Definition (Asynchronous FSA (AFSA))

An *AFSA* $\mathcal{A}$ is a tuple $\mathcal{A} = \langle Q, X, Y, q_0, F, \delta \rangle$, with:

- Q a finite set of *states*;
- X and Y *alphabets*;
- $q_0 \in Q$ an *initial state*;
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- $\delta \subseteq Q \times (X \cup \{\varepsilon\}) \times (Y \cup \{\varepsilon\}) \times Q$ a *transition relation*.

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Proposition (Neunhöffer, Pfeiffer & Ruškuc, 2013)

Let $T = \langle x, y : x^2 = x, yx = y \rangle$ *and* $S = \langle x, y : (xy^n x = xyx)_{n \geq 2} \rangle$. *These semigroups are infinite and the respective word problems are rational.*

The following automaton decides the word problem of T with respect to $\{x, y\}$:

For example, the pairs (x^2, x) and (xyx, xy) are accepted.

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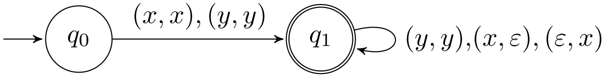

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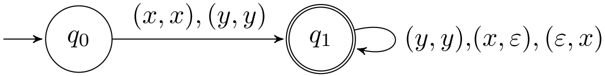

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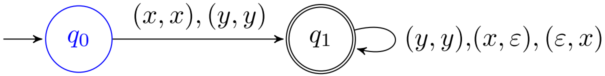

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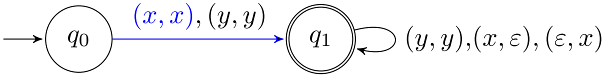

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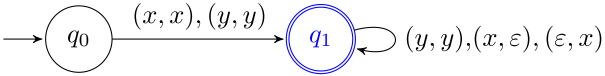

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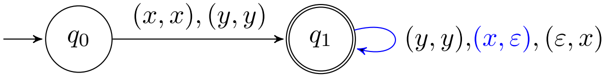

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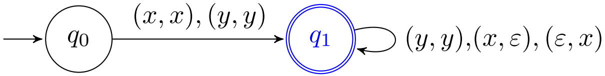

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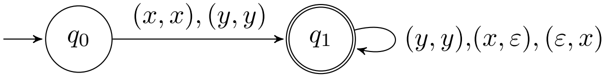

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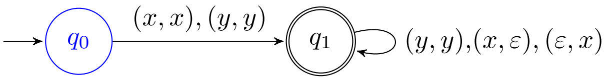

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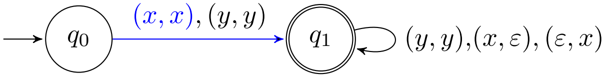

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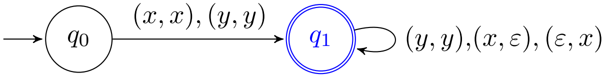

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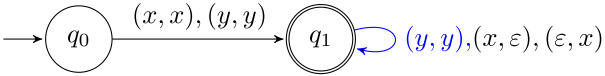

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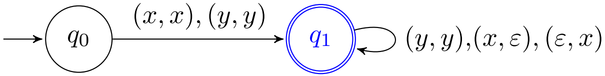

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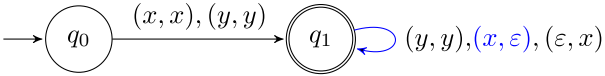

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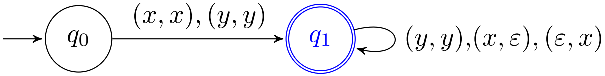

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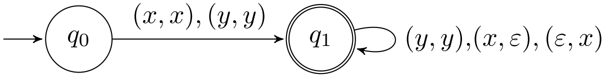

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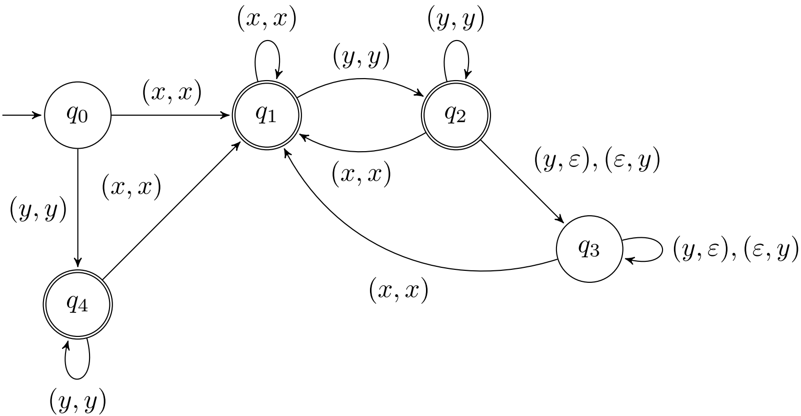

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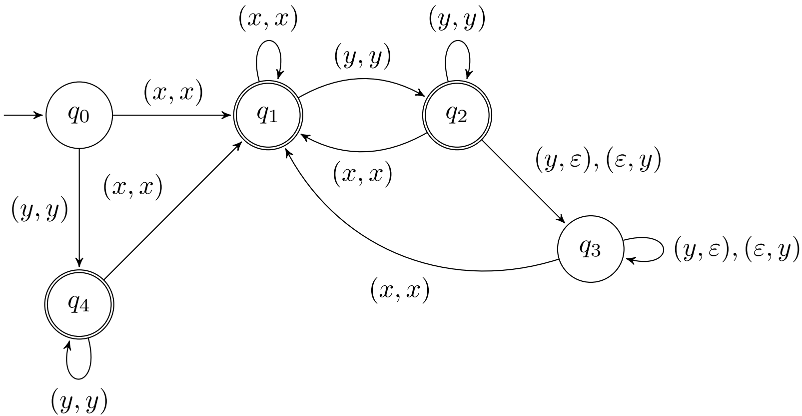

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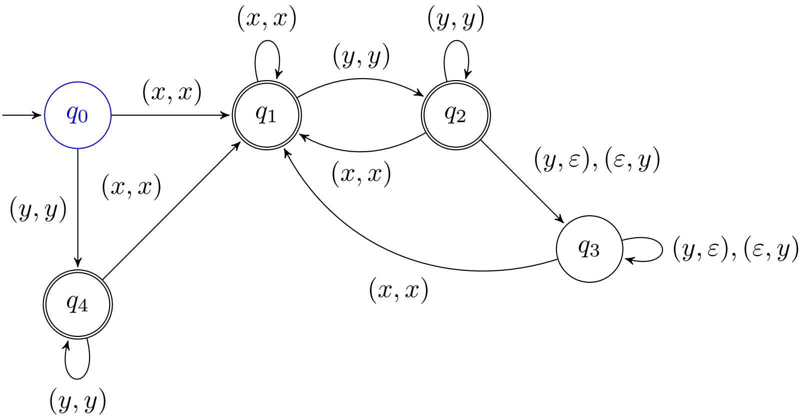

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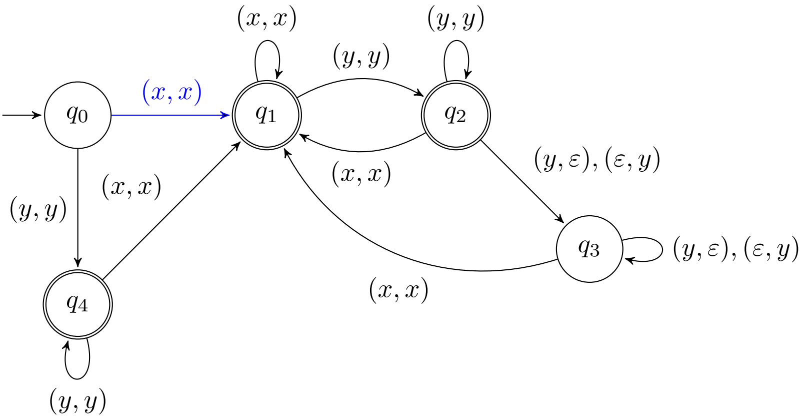

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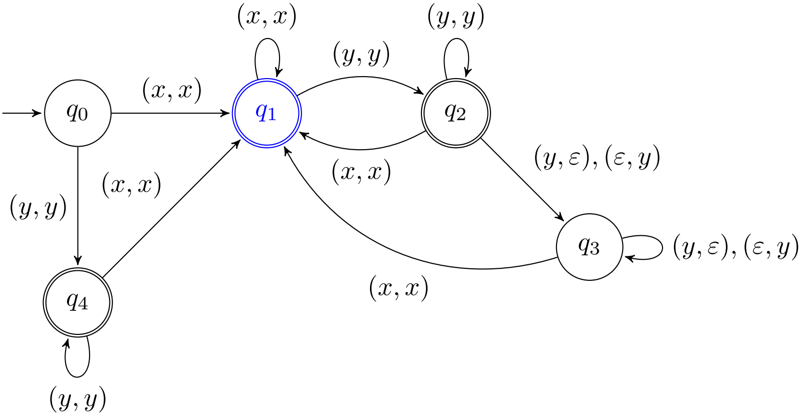

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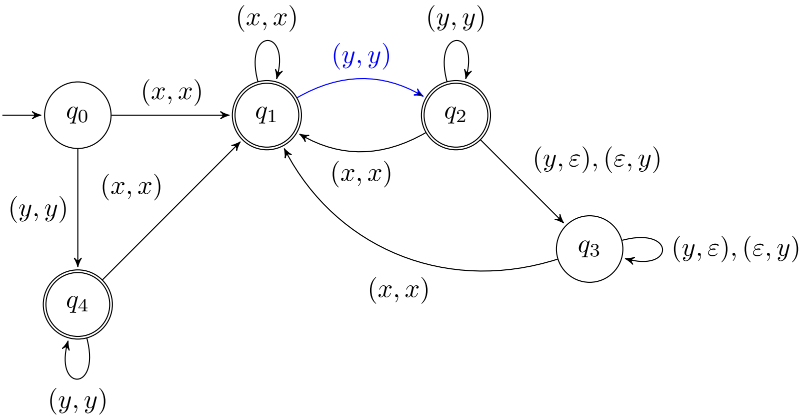

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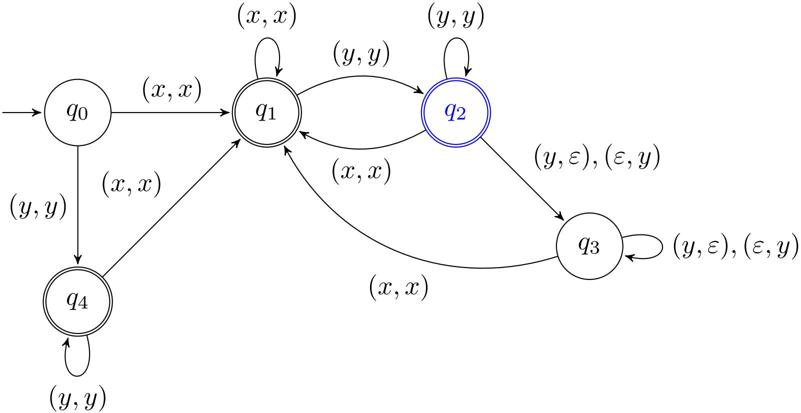

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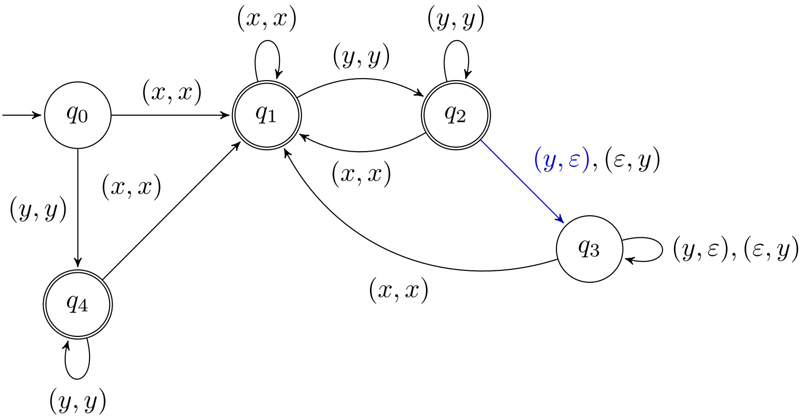

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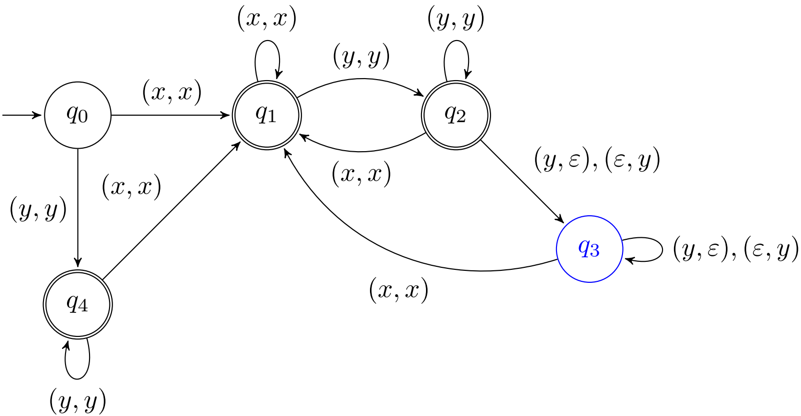

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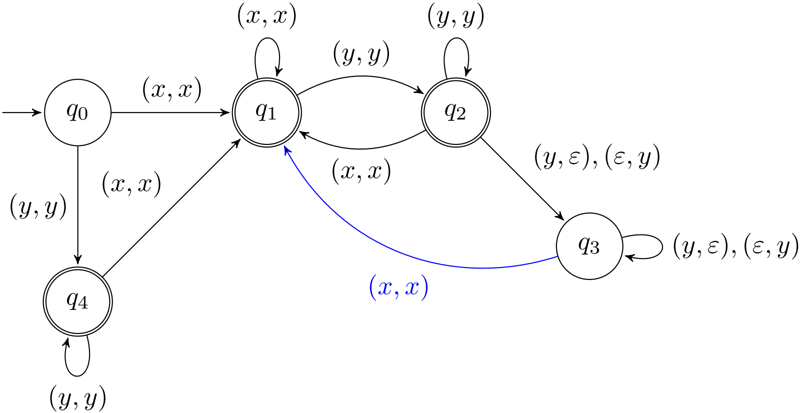

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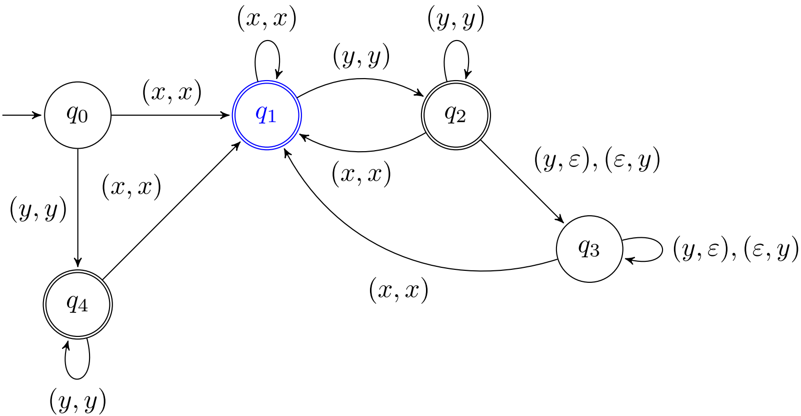

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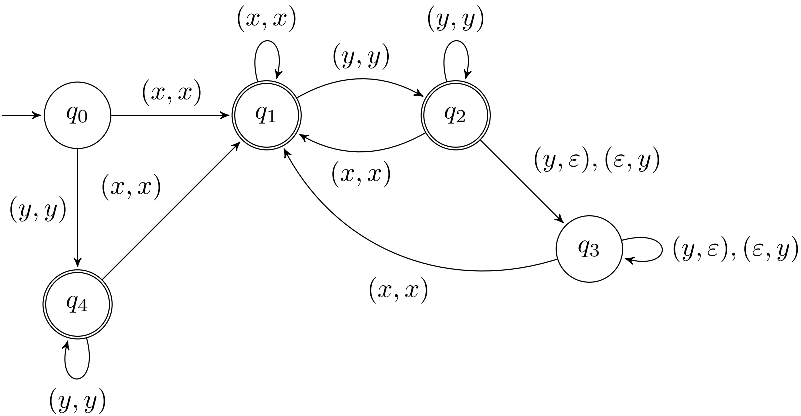

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When is the converse of the above proposition true?

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