Adjunction in the absence of identity

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Classical adjoint functors

$\mathcal{C}$, $\mathcal{D}$ — two categories

$F : \mathcal{C} \to \mathcal{D}$ — functor

$G : \mathcal{D} \to \mathcal{C}$ — functor

Definition. We say that (F, G) is an adjoint pair of functors provided that, for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there are isomorphisms $\mathcal{D}(F\,X, Y) \cong \mathcal{C}(X, G\,Y)$ natural in X and Y.

Claim. (F, G) is an adjoint pair of functors iff there exist adjunction morphisms $\varepsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ and $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ such that

$$(\varepsilon \circ_h \mathrm{id}_F) \circ_v (\mathrm{id}_F \circ_h \eta) = \mathrm{id}_F \qquad (\mathrm{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \mathrm{id}_G) = \mathrm{id}_G,$$

that is, the compositions

$$F \to FGF \to F \quad \text{and} \quad G \to GFG \to G$$

are the identities.

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F, G — two objects in $\mathscr{C}$

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Note. This extends to 2-categories in the obvious way.

Question. Is it possible to get rid of $\mathbb{1}_{\mathscr{C}}$?

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Motivation: finitary 2-categories/bicategories

$\mathscr{C}$ — 2-category

Definition. $\mathscr{C}$ is called finitary over some field $\Bbbk$ provided that

- it has finitely many objects;
- each $\mathscr{C}(\mathtt{i},\mathtt{j})$ is equivalent to the category of projective modules over a finite dimensional $\Bbbk$-algebra;
- compositions are biadditive and $\Bbbk$-bilinear.
- identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional Hopf algebra over $\Bbbk$ of finite representation type.

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Problematic example: projective bimodules

A — finite dimensional algebra (basic, connected, not semi-simple).

A-mod-A — the category of A-A-bimodules (or, rather, its strictification)

Note. A-mod-A is monoidal (= 2-category with one object)

Note. A-mod-A is finitary iff $A \otimes_{\Bbbk} A^{\mathrm{op}}$ has finite representation type.

Observation. A-proj-A is closed under $\otimes_A$ and is always "finitary", but it is only a sub-2-semicategory as the identity ${}_AA_A$ is not projective.

Definition. The 2-category $\mathscr{C}_A$ of projective bimodules is defined as $\mathrm{add}({}_AA_A \oplus A \otimes_{\Bbbk} A)$.

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- $\mathscr{C}$ has a weak involution $\star$;
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Example. Modules over a finite dimensional Hopf algebra over $\Bbbk$ of finite representation type.

Example. $\mathscr{C}_A$ if A is self-injective and the top of each projective is isomorphic to its socle (i.e. A is weakly symmetric).

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Definition. A semigroup/monoid S is called simple if it does not have any non-trivial quotients.

Problem. Classify all simple finite semigroups/monoids.

Answer for monoids. Simple finite monoids are exactly simple finite groups and the boolean monoid.

Observation related to semigroups. There are plenty of simple finite semigroups which are not monoids (they are classified).

Conclusion. Existence of the identity is a very serious restriction.

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Problem. Classify all simple finite semigroups/monoids.

Answer for monoids. Simple finite monoids are exactly simple finite groups and the boolean monoid.

Observation related to semigroups. There are plenty of simple finite semigroups which are not monoids (they are classified).

Conclusion. Existence of the identity is a very serious restriction.

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Idea. Substitute the identity by a lax identity and (possibly different) oplax identity.

Credit for the idea: Marco Mackaay.

Cheating? Not really, in representations, (op)lax identities are usually not represented by the identity functors.

Lax identity. I together with $\mathrm{I} \circ_h \mathrm{F} \xrightarrow{l_{\mathrm{F}}} \mathrm{F}$ and $\mathrm{F} \circ_h \mathrm{I} \xrightarrow{r_{\mathrm{F}}} \mathrm{F}$, for each F, natural in F.

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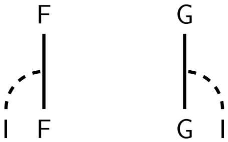

Axioms

Dual for the oplax identity I'.

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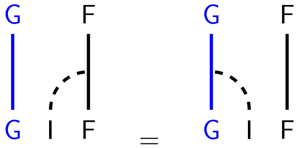

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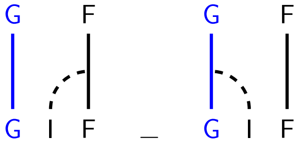

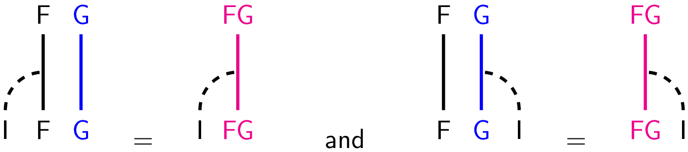

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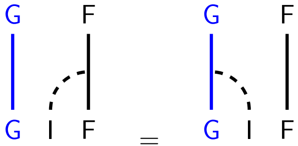

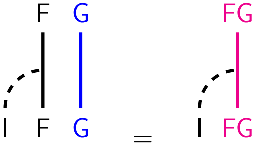

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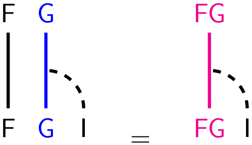

Dual for the oplax identity I'.

New setup: definitions

Definition. A bilax unital 2-category is a 2-semicategory with a choice of a lax unit $I_{\mathtt{i}}$ and an oplax unit $I'_{\mathtt{i}}$, for each object.

$\mathscr{C}$ — bilax unital 2-category.

$F \in \mathscr{C}(\mathtt{i},\mathtt{j})$ and $G \in \mathscr{C}(\mathtt{j},\mathtt{i})$

Definition. (F, G) is a pair of adjoint 1-morphisms in $\mathscr{C}$ provided that there exist $\varepsilon : FG \to I_{\mathtt{j}}$ and $\eta : I'_{\mathtt{i}} \to GF$ such that the compositions

$$F \to FI'_{\mathtt{i}} \to FGF \to I_{\mathtt{j}}F \to F$$

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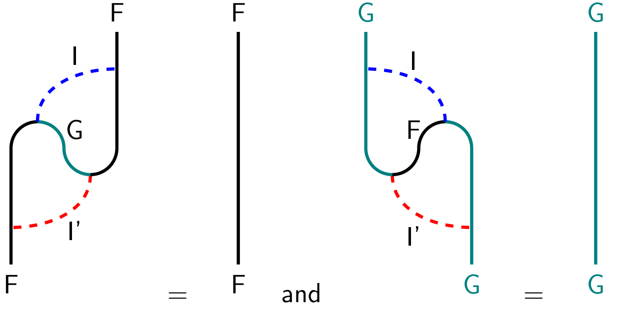

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New setup: remarks

Note. The adjunction defined in the way is equivalent to the classical adjunction.

Reason for that: Despite of the fact that the (op)lax units are not usually represented by the identity functors, they are represented by functors which have non-trivial natural transformations to (from) the identity functors. These allow us to compare our adjunction to the classical adjunction.

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Example

A — finite dimensional $\Bbbk$-algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \cdots + e_n$ — primitive decomposition of $1 \in A$

Definition. The bilax unital 2-category $\mathscr{D}_A$ is defined to have:

- objects: $\mathtt{1}, \dots, \mathtt{n}$, where $\mathtt{k} \leftrightarrow e_k A e_k$-mod;
- 1-morphisms: functors isomorphic to tensoring with $X \in \mathrm{add}(Ae_i \otimes_\Bbbk e_j A)$;
- 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_\Bbbk e_i A$ is a lax identity via the multiplication map $ae_i \otimes e_i b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_\Bbbk e_i A$ to A).

Note: Using the weak involution on $\mathscr{C}_A$, each $Ae_i \otimes_\Bbbk e_i A$ is also an oplax identity.

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Generalization: bilax 2-categories of $\mathcal{J}$-cell, preliminaries

$\mathscr{C}$ — fiat

$F \sim_L G$ if and only if $\mathrm{add}(\mathscr{C} \circ F) = \mathrm{add}(\mathscr{C} \circ G)$

$\sim_R$ and $\sim_J$ are defined similarly

$\mathcal{J}$ — an equivalence class for $\sim_J$ (two-sided cell)

$\mathcal{L} \subset \mathcal{J}$ — an equivalence class for $\sim_L$ (left cell)

Observation. [Mazorchuk-Miemietz] $\mathcal{L}$ contains a unique 1-morphism F (called Duflo element) for which there is a homomorphism $\xi : F \to \mathbb{1}_i$ such that $G(\xi)$ is right split, for every $G \in \mathcal{L}$.

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Generalization: bilax 2-categories of $\mathcal{J}$-cell, definition

Definition. The bilax unital 2-category $\mathscr{D}_J$ is defined to have:

- objects are in bijection with Duflo elements in $\mathcal{J}$;
- 1-morphisms from Duflo F to Duflo G: the additive closure of the intersection of the left cell of F and the right cell of G;
- 2-morphisms: induced from $\mathscr{C}$ modulo those which factor through "higher" $\mathcal{J}$-cells.
- composition is induced from $\mathscr{C}$ modulo "higher" $\mathcal{J}$-cells.
- lax units: Duflo 1-morphisms.
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Discussion

This allows us to define a setup in which we can talk about adjoint 1-morphisms without having genuine identities.

A lot of choice involved: (op)lax identities are in no way unique (in particular, they are closed under composition).

The genuine identity is the coequalizer of $\mathrm{II} \underset{r_{\mathrm{I}}}{\overset{l_{\mathrm{I}}}{\rightrightarrows}} \mathrm{I}$.

Good case: Unitors split (true in our main examples).

Some results from 2-representation theory of finitary 2-categories generalize (sometimes in a "cleaner" form).

Very technical.

Many open questions.

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