

Adjunction in the absence of identity

Volodymyr Mazorchuk
(Uppsala University)

Topological Quantum Field Theory thematic sessions 2020

Joint with

Hankyung Ko (Uppsala University)

Xiaoting Zhang (Uppsala University/Capital Normal University, Beijing)

Classical adjoint functors

\mathcal{C}, \mathcal{D} — two categories

$F : \mathcal{C} \rightarrow \mathcal{D}$ — functor

$G : \mathcal{D} \rightarrow \mathcal{C}$ — functor

Definition. We say that (F, G) is an **adjoint pair of functors** provided that, for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there are isomorphisms $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$ natural in X and Y .

Claim. (F, G) is an adjoint pair of functors iff there exist **adjunction morphisms** $\varepsilon : FG \rightarrow \text{Id}_{\mathcal{D}}$ and $\eta : \text{Id}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G,$$

that is, the compositions

$$F \rightarrow FGF \rightarrow F \quad \text{and} \quad G \rightarrow GFG \rightarrow G$$

are the identities.

Classical adjoint functors

\mathcal{C}, \mathcal{D} — two categories

$F : \mathcal{C} \rightarrow \mathcal{D}$ — functor

$G : \mathcal{D} \rightarrow \mathcal{C}$ — functor

Definition. We say that (F, G) is an **adjoint pair of functors** provided that, for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there are isomorphisms $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$ natural in X and Y .

Claim. (F, G) is an adjoint pair of functors iff there exist **adjunction morphisms** $\varepsilon : FG \rightarrow \text{Id}_{\mathcal{D}}$ and $\eta : \text{Id}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G,$$

that is, the compositions

$$F \rightarrow FGF \rightarrow F \quad \text{and} \quad G \rightarrow GFG \rightarrow G$$

are the identities.

Classical adjoint functors

\mathcal{C}, \mathcal{D} — two categories

$F : \mathcal{C} \rightarrow \mathcal{D}$ — functor

$G : \mathcal{D} \rightarrow \mathcal{C}$ — functor

Definition. We say that (F, G) is an **adjoint pair of functors** provided that, for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there are isomorphisms $\mathcal{D}(F X, Y) \cong \mathcal{C}(X, G Y)$ natural in X and Y .

Claim. (F, G) is an adjoint pair of functors iff there exist **adjunction morphisms** $\varepsilon : FG \rightarrow \text{Id}_{\mathcal{D}}$ and $\eta : \text{Id}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G,$$

that is, the compositions

$$F \rightarrow FGF \rightarrow F \quad \text{and} \quad G \rightarrow GFG \rightarrow G$$

are the identities.

Classical adjoint functors

\mathcal{C}, \mathcal{D} — two categories

$F : \mathcal{C} \rightarrow \mathcal{D}$ — functor

$G : \mathcal{D} \rightarrow \mathcal{C}$ — functor

Definition. We say that (F, G) is an **adjoint pair of functors** provided that, for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there are isomorphisms $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$ natural in X and Y .

Claim. (F, G) is an adjoint pair of functors iff there exist **adjunction morphisms** $\varepsilon : FG \rightarrow \text{Id}_{\mathcal{D}}$ and $\eta : \text{Id}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G,$$

that is, the compositions

$$F \rightarrow FGF \rightarrow F \quad \text{and} \quad G \rightarrow GFG \rightarrow G$$

are the identities.

Classical adjoint functors

\mathcal{C}, \mathcal{D} — two categories

$F : \mathcal{C} \rightarrow \mathcal{D}$ — functor

$G : \mathcal{D} \rightarrow \mathcal{C}$ — functor

Definition. We say that (F, G) is an **adjoint pair of functors** provided that, for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there are isomorphisms $\mathcal{D}(F X, Y) \cong \mathcal{C}(X, G Y)$ natural in X and Y .

Claim. (F, G) is an adjoint pair of functors iff there exist **adjunction morphisms** $\varepsilon : FG \rightarrow \text{Id}_{\mathcal{D}}$ and $\eta : \text{Id}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G,$$

that is, the compositions

$$F \rightarrow FGF \rightarrow F \quad \text{and} \quad G \rightarrow GFG \rightarrow G$$

are the identities.

Classical adjoint functors

\mathcal{C}, \mathcal{D} — two categories

$F : \mathcal{C} \rightarrow \mathcal{D}$ — functor

$G : \mathcal{D} \rightarrow \mathcal{C}$ — functor

Definition. We say that (F, G) is an **adjoint pair of functors** provided that, for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there are isomorphisms $\mathcal{D}(F X, Y) \cong \mathcal{C}(X, G Y)$ natural in X and Y .

Claim. (F, G) is an adjoint pair of functors iff there exist **adjunction morphisms** $\varepsilon : FG \rightarrow \text{Id}_{\mathcal{D}}$ and $\eta : \text{Id}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G,$$

that is, the compositions

$$F \rightarrow FGF \rightarrow F \quad \text{and} \quad G \rightarrow GFG \rightarrow G$$

are the identities.

Adjoint objects of (strict) monoidal categories

\mathcal{C} — (strict) monoidal category

F, G — two objects in \mathcal{C}

Definition. We say that (F, G) is an **adjoint pair of objects** provided that there exist morphisms $\varepsilon : FG \rightarrow \mathbb{1}_{\mathcal{C}}$ and $\eta : \mathbb{1}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G.$$

Note. This extends to **2-categories** in the obvious way.

Question. Is it possible to get rid of $\mathbb{1}_{\mathcal{C}}$?

Adjoint objects of (strict) monoidal categories

\mathcal{C} — (strict) monoidal category

F, G — two objects in \mathcal{C}

Definition. We say that (F, G) is an **adjoint pair of objects** provided that there exist morphisms $\varepsilon : FG \rightarrow \mathbb{1}_{\mathcal{C}}$ and $\eta : \mathbb{1}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G.$$

Note. This extends to **2-categories** in the obvious way.

Question. Is it possible to get rid of $\mathbb{1}_{\mathcal{C}}$?

Adjoint objects of (strict) monoidal categories

\mathcal{C} — (strict) monoidal category

F, G — two objects in \mathcal{C}

Definition. We say that (F, G) is an **adjoint pair of objects** provided that there exist morphisms $\varepsilon : FG \rightarrow \mathbb{1}_{\mathcal{C}}$ and $\eta : \mathbb{1}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G.$$

Note. This extends to **2-categories** in the obvious way.

Question. Is it possible to get rid of $\mathbb{1}_{\mathcal{C}}$?

Adjoint objects of (strict) monoidal categories

\mathcal{C} — (strict) monoidal category

F, G — two objects in \mathcal{C}

Definition. We say that (F, G) is an **adjoint pair of objects** provided that there exist morphisms $\varepsilon : FG \rightarrow \mathbb{1}_{\mathcal{C}}$ and $\eta : \mathbb{1}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G.$$

Note. This extends to **2-categories** in the obvious way.

Question. Is it possible to get rid of $\mathbb{1}_{\mathcal{C}}$?

Adjoint objects of (strict) monoidal categories

\mathcal{C} — (strict) monoidal category

F, G — two objects in \mathcal{C}

Definition. We say that (F, G) is an **adjoint pair of objects** provided that there exist morphisms $\varepsilon : FG \rightarrow \mathbb{1}_{\mathcal{C}}$ and $\eta : \mathbb{1}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G.$$

Note. This extends to **2-categories** in the obvious way.

Question. Is it possible to get rid of $\mathbb{1}_{\mathcal{C}}$?

Adjoint objects of (strict) monoidal categories

\mathcal{C} — (strict) monoidal category

F, G — two objects in \mathcal{C}

Definition. We say that (F, G) is an **adjoint pair of objects** provided that there exist morphisms $\varepsilon : FG \rightarrow \mathbb{1}_{\mathcal{C}}$ and $\eta : \mathbb{1}_{\mathcal{C}} \rightarrow GF$ such that

$$(\varepsilon \circ_h \text{id}_F) \circ_v (\text{id}_F \circ_h \eta) = \text{id}_F \quad (\text{id}_G \circ_h \varepsilon) \circ_v (\eta \circ_h \text{id}_G) = \text{id}_G.$$

Note. This extends to **2-categories** in the obvious way.

Question. Is it possible to get rid of $\mathbb{1}_{\mathcal{C}}$?

Motivation: finitary 2-categories/bicategories

\mathcal{C} — 2-category

Definition. \mathcal{C} is called **finitary over some field \mathbb{k}** provided that

- ▶ it has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over a finite dimensional \mathbb{k} -algebra;
- ▶ compositions are biadditive and \mathbb{k} -bilinear.
- ▶ identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional Hopf algebra over \mathbb{k} of **finite representation type**.

Motivation: finitary 2-categories/bicategories

\mathcal{C} — 2-category

Definition. \mathcal{C} is called **finitary over some field \mathbb{k}** provided that

- ▶ it has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over a finite dimensional \mathbb{k} -algebra;
- ▶ compositions are biadditive and \mathbb{k} -bilinear.
- ▶ identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional Hopf algebra over \mathbb{k} of **finite representation type**.

Motivation: finitary 2-categories/bicategories

\mathcal{C} — 2-category

Definition. \mathcal{C} is called **finitary over some field \mathbb{k}** provided that

- ▶ it has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over a finite dimensional \mathbb{k} -algebra;
- ▶ compositions are biadditive and \mathbb{k} -bilinear.
- ▶ identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional Hopf algebra over \mathbb{k} of **finite representation type**.

Motivation: finitary 2-categories/bicategories

\mathcal{C} — 2-category

Definition. \mathcal{C} is called **finitary over some field \mathbb{k}** provided that

- ▶ it has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over a finite dimensional \mathbb{k} -algebra;
- ▶ compositions are biadditive and \mathbb{k} -bilinear.
- ▶ identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional Hopf algebra over \mathbb{k} of **finite representation type**.

Motivation: finitary 2-categories/bicategories

\mathcal{C} — 2-category

Definition. \mathcal{C} is called **finitary over some field \mathbb{k}** provided that

- ▶ it has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over a finite dimensional \mathbb{k} -algebra;
- ▶ compositions are biadditive and \mathbb{k} -bilinear.
- ▶ identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional Hopf algebra over \mathbb{k} of **finite representation type**.

Motivation: finitary 2-categories/bicategories

\mathcal{C} — 2-category

Definition. \mathcal{C} is called **finitary over some field \mathbb{k}** provided that

- ▶ it has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over a finite dimensional \mathbb{k} -algebra;
- ▶ compositions are biadditive and \mathbb{k} -bilinear.
- ▶ identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional Hopf algebra over \mathbb{k} of **finite representation type**.

Motivation: finitary 2-categories/bicategories

\mathcal{C} — 2-category

Definition. \mathcal{C} is called **finitary over some field \mathbb{k}** provided that

- ▶ it has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over a finite dimensional \mathbb{k} -algebra;
- ▶ compositions are biadditive and \mathbb{k} -bilinear.
- ▶ identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional **Hopf algebra** over \mathbb{k} of **finite representation type**.

Motivation: finitary 2-categories/bicategories

\mathcal{C} — 2-category

Definition. \mathcal{C} is called **finitary over some field \mathbb{k}** provided that

- ▶ it has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over a finite dimensional \mathbb{k} -algebra;
- ▶ compositions are biadditive and \mathbb{k} -bilinear.
- ▶ identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional **Hopf algebra** over \mathbb{k} of **finite representation type**.

Motivation: finitary 2-categories/bicategories

\mathcal{C} — 2-category

Definition. \mathcal{C} is called **finitary over some field \mathbb{k}** provided that

- ▶ it has finitely many objects;
- ▶ each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over a finite dimensional \mathbb{k} -algebra;
- ▶ compositions are biadditive and \mathbb{k} -bilinear.
- ▶ identity 1-morphisms are indecomposable.

Example. Finite dimensional modules over a finite dimensional **Hopf algebra** over \mathbb{k} of **finite representation type**.

Problematic example: projective bimodules

A — finite dimensional algebra (basic, connected, not semi-simple).

$A\text{-mod-}A$ — the category of A - A -bimodules (or, rather, its strictification)

Note. $A\text{-mod-}A$ is monoidal (= 2-category with one object)

Note. $A\text{-mod-}A$ is finitary iff $A \otimes_{\mathbb{k}} A^{\text{op}}$ has finite representation type.

Observation. $A\text{-proj-}A$ is closed under \otimes_A and is always “finitary”, but it is only a **sub-2-semicategory** as the identity ${}_A A_A$ is not projective.

Definition. The **2-category** \mathcal{C}_A of projective bimodules is defined as $\text{add}({}_A A_A \oplus A \otimes_{\mathbb{k}} A)$.

Problematic example: projective bimodules

A — finite dimensional algebra (basic, connected, not semi-simple).

$A\text{-mod-}A$ — the category of A - A -bimodules (or, rather, its strictification)

Note. $A\text{-mod-}A$ is monoidal (= 2-category with one object)

Note. $A\text{-mod-}A$ is finitary iff $A \otimes_{\mathbb{k}} A^{\text{op}}$ has finite representation type.

Observation. $A\text{-proj-}A$ is closed under \otimes_A and is always “finitary”, but it is only a **sub-2-semicategory** as the identity ${}_A A_A$ is not projective.

Definition. The **2-category** \mathcal{C}_A of projective bimodules is defined as $\text{add}({}_A A_A \oplus A \otimes_{\mathbb{k}} A)$.

Problematic example: projective bimodules

A — finite dimensional algebra (basic, connected, not semi-simple).

$A\text{-mod-}A$ — the category of A - A -bimodules (or, rather, its strictification)

Note. $A\text{-mod-}A$ is monoidal (= 2-category with one object)

Note. $A\text{-mod-}A$ is finitary iff $A \otimes_{\mathbb{k}} A^{\text{op}}$ has finite representation type.

Observation. $A\text{-proj-}A$ is closed under \otimes_A and is always “finitary”, but it is only a **sub-2-semicategory** as the identity ${}_A A_A$ is not projective.

Definition. The **2-category** \mathcal{C}_A of projective bimodules is defined as $\text{add}({}_A A_A \oplus A \otimes_{\mathbb{k}} A)$.

Problematic example: projective bimodules

A — finite dimensional algebra (basic, connected, not semi-simple).

$A\text{-mod-}A$ — the category of A - A -bimodules (or, rather, its strictification)

Note. $A\text{-mod-}A$ is monoidal (= 2-category with one object)

Note. $A\text{-mod-}A$ is finitary iff $A \otimes_{\mathbb{k}} A^{\text{op}}$ has finite representation type.

Observation. $A\text{-proj-}A$ is closed under \otimes_A and is always “finitary”, but it is only a **sub-2-semicategory** as the identity ${}_A A_A$ is not projective.

Definition. The **2-category** \mathcal{C}_A of projective bimodules is defined as $\text{add}({}_A A_A \oplus A \otimes_{\mathbb{k}} A)$.

Problematic example: projective bimodules

A — finite dimensional algebra (basic, connected, not semi-simple).

$A\text{-mod-}A$ — the category of A - A -bimodules (or, rather, its strictification)

Note. $A\text{-mod-}A$ is monoidal (= 2-category with one object)

Note. $A\text{-mod-}A$ is finitary iff $A \otimes_{\mathbb{k}} A^{\text{op}}$ has **finite representation type**.

Observation. $A\text{-proj-}A$ is closed under \otimes_A and is always “finitary”, but it is only a **sub-2-semicategory** as the identity ${}_A A_A$ is not projective.

Definition. The **2-category** \mathcal{C}_A of projective bimodules is defined as $\text{add}({}_A A_A \oplus A \otimes_{\mathbb{k}} A)$.

Problematic example: projective bimodules

A — finite dimensional algebra (basic, connected, not semi-simple).

$A\text{-mod-}A$ — the category of A - A -bimodules (or, rather, its strictification)

Note. $A\text{-mod-}A$ is monoidal (= 2-category with one object)

Note. $A\text{-mod-}A$ is finitary iff $A \otimes_{\mathbb{k}} A^{\text{op}}$ has **finite representation type**.

Observation. $A\text{-proj-}A$ is closed under \otimes_A and is always “finitary”, but it is only a **sub-2-semicategory** as the identity ${}_A A_A$ is not projective.

Definition. The 2-category \mathcal{C}_A of projective bimodules is defined as $\text{add}({}_A A_A \oplus A \otimes_{\mathbb{k}} A)$.

Problematic example: projective bimodules

A — finite dimensional algebra (basic, connected, not semi-simple).

$A\text{-mod-}A$ — the category of A - A -bimodules (or, rather, its strictification)

Note. $A\text{-mod-}A$ is monoidal (= 2-category with one object)

Note. $A\text{-mod-}A$ is finitary iff $A \otimes_{\mathbb{k}} A^{\text{op}}$ has **finite representation type**.

Observation. $A\text{-proj-}A$ is closed under \otimes_A and is always “finitary”, but it is only a **sub-2-semicategory** as the identity ${}_A A_A$ is not projective.

Definition. The **2-category** \mathcal{C}_A of projective bimodules is defined as $\text{add}({}_A A_A \oplus A \otimes_{\mathbb{k}} A)$.

Fiat/fiab

\mathcal{C} — finitary 2-category.

Definition. \mathcal{C} is fiat, (a.k.a. rigid or with duals) provided that

- ▶ \mathcal{C} has a weak involution \star ;
- ▶ \mathcal{C} has adjunction morphisms making each pair (F, F^*) into a pair of adjoint 1-morphisms.

Example. Modules over a finite dimensional Hopf algebra over k of finite representation type.

Example. \mathcal{C}_A if A is self-injective and the top of each projective is isomorphic to its socle (i.e. A is weakly symmetric).

Note. The identity ${}_A A_A$ is crucial for adjunction morphism.

Question. Can we still get rid of it, preserving the structure?

Fiat/fiab

\mathcal{C} — finitary 2-category.

Definition. \mathcal{C} is fiat, (a.k.a. rigid or with duals) provided that

- ▶ \mathcal{C} has a weak involution \star ;
- ▶ \mathcal{C} has adjunction morphisms making each pair (F, F^*) into a pair of adjoint 1-morphisms.

Example. Modules over a finite dimensional Hopf algebra over k of finite representation type.

Example. \mathcal{C}_A if A is self-injective and the top of each projective is isomorphic to its socle (i.e. A is weakly symmetric).

Note. The identity ${}_A A_A$ is crucial for adjunction morphism.

Question. Can we still get rid of it, preserving the structure?

Fiat/fiab

\mathcal{C} — finitary 2-category.

Definition. \mathcal{C} is **fiat**, (a.k.a. **rigid** or **with duals**) provided that

- ▶ \mathcal{C} has a weak involution \star ;
- ▶ \mathcal{C} has adjunction morphisms making each pair (F, F^*) into a pair of adjoint 1-morphisms.

Example. Modules over a finite dimensional Hopf algebra over k of finite representation type.

Example. \mathcal{C}_A if A is **self-injective** and the top of each projective is isomorphic to its socle (i.e. A is **weakly symmetric**).

Note. The identity ${}_A A_A$ is crucial for adjunction morphism.

Question. Can we still get rid of it, preserving the structure?

Fiat/fiab

\mathcal{C} — finitary 2-category.

Definition. \mathcal{C} is **fiat**, (a.k.a. **rigid** or **with duals**) provided that

- ▶ \mathcal{C} has a weak involution \star ;
- ▶ \mathcal{C} has adjunction morphisms making each pair (F, F^*) into a pair of adjoint 1-morphisms.

Example. Modules over a finite dimensional Hopf algebra over \mathbb{k} of finite representation type.

Example. \mathcal{C}_A if A is **self-injective** and the top of each projective is isomorphic to its socle (i.e. A is **weakly symmetric**).

Note. The identity ${}_A A_A$ is crucial for adjunction morphism.

Question. Can we still get rid of it, preserving the structure?

Fiat/fiab

\mathcal{C} — finitary 2-category.

Definition. \mathcal{C} is **fiat**, (a.k.a. **rigid** or **with duals**) provided that

- ▶ \mathcal{C} has a weak involution \star ;
- ▶ \mathcal{C} has adjunction morphisms making each pair (F, F^\star) into a pair of adjoint 1-morphisms.

Example. Modules over a finite dimensional Hopf algebra over \mathbb{k} of finite representation type.

Example. \mathcal{C}_A if A is **self-injective** and the top of each projective is isomorphic to its socle (i.e. A is **weakly symmetric**).

Note. The identity ${}_A A_A$ is crucial for adjunction morphism.

Question. Can we still get rid of it, preserving the structure?

Fiat/fiab

\mathcal{C} — finitary 2-category.

Definition. \mathcal{C} is **fiat**, (a.k.a. **rigid** or **with duals**) provided that

- ▶ \mathcal{C} has a weak involution \star ;
- ▶ \mathcal{C} has adjunction morphisms making each pair (F, F^\star) into a pair of adjoint 1-morphisms.

Example. Modules over a finite dimensional Hopf algebra over \mathbb{k} of finite representation type.

Example. \mathcal{C}_A if A is **self-injective** and the top of each projective is isomorphic to its socle (i.e. A is **weakly symmetric**).

Note. The identity ${}_A A_A$ is crucial for adjunction morphism.

Question. Can we still get rid of it, preserving the structure?

Fiat/fiab

\mathcal{C} — finitary 2-category.

Definition. \mathcal{C} is **fiat**, (a.k.a. **rigid** or **with duals**) provided that

- ▶ \mathcal{C} has a weak involution \star ;
- ▶ \mathcal{C} has adjunction morphisms making each pair (F, F^\star) into a pair of adjoint 1-morphisms.

Example. Modules over a finite dimensional Hopf algebra over \mathbb{k} of finite representation type.

Example. \mathcal{C}_A if A is **self-injective** and the top of each projective is isomorphic to its socle (i.e. A is **weakly symmetric**).

Note. The identity ${}_A A_A$ is crucial for adjunction morphism.

Question. Can we still get rid of it, preserving the structure?

Fiat/fiab

\mathcal{C} — finitary 2-category.

Definition. \mathcal{C} is **fiat**, (a.k.a. **rigid** or **with duals**) provided that

- ▶ \mathcal{C} has a weak involution \star ;
- ▶ \mathcal{C} has adjunction morphisms making each pair (F, F^\star) into a pair of adjoint 1-morphisms.

Example. Modules over a finite dimensional Hopf algebra over \mathbb{k} of finite representation type.

Example. \mathcal{C}_A if A is **self-injective** and the top of each projective is isomorphic to its socle (i.e. A is **weakly symmetric**).

Note. The identity ${}_A A_A$ is crucial for adjunction morphism.

Question. Can we still get rid of it, preserving the structure?

Fiat/fiab

\mathcal{C} — finitary 2-category.

Definition. \mathcal{C} is **fiat**, (a.k.a. **rigid** or **with duals**) provided that

- ▶ \mathcal{C} has a weak involution \star ;
- ▶ \mathcal{C} has adjunction morphisms making each pair (F, F^\star) into a pair of adjoint 1-morphisms.

Example. Modules over a finite dimensional Hopf algebra over \mathbb{k} of finite representation type.

Example. \mathcal{C}_A if A is **self-injective** and the top of each projective is isomorphic to its socle (i.e. A is **weakly symmetric**).

Note. The identity ${}_A A_A$ is crucial for adjunction morphism.

Question. Can we still get rid of it, preserving the structure?

The significance of the identity: semigroup detour

Definition. A semigroup/monoid S is called **simple** if it does not have any non-trivial quotients.

Problem. Classify all simple finite semigroups/monoids.

Answer for monoids. Simple finite monoids are exactly **simple finite groups** and the boolean monoid.

Observation related to semigroups. There are plenty of simple finite semigroups which are not monoids (they are classified).

Conclusion. Existence of the identity is a very serious restriction.

The significance of the identity: semigroup detour

Definition. A semigroup/monoid S is called **simple** if it does not have any non-trivial quotients.

Problem. Classify all simple finite semigroups/monoids.

Answer for monoids. Simple finite monoids are exactly **simple finite groups** and the boolean monoid.

Observation related to semigroups. There are plenty of simple finite semigroups which are not monoids (they are classified).

Conclusion. Existence of the identity is a very serious restriction.

The significance of the identity: semigroup detour

Definition. A semigroup/monoid S is called **simple** if it does not have any non-trivial quotients.

Problem. Classify all simple finite semigroups/monoids.

Answer for monoids. Simple finite monoids are exactly **simple finite groups** and the boolean monoid.

Observation related to semigroups. There are plenty of simple finite semigroups which are not monoids (they are classified).

Conclusion. Existence of the identity is a very serious restriction.

The significance of the identity: semigroup detour

Definition. A semigroup/monoid S is called **simple** if it does not have any non-trivial quotients.

Problem. Classify all simple finite semigroups/monoids.

Answer for monoids. Simple finite monoids are exactly **simple finite groups** and the boolean monoid.

Observation related to semigroups. There are plenty of simple finite semigroups which are not monoids (they are classified).

Conclusion. Existence of the identity is a very serious restriction.

The significance of the identity: semigroup detour

Definition. A semigroup/monoid S is called **simple** if it does not have any non-trivial quotients.

Problem. Classify all simple finite semigroups/monoids.

Answer for monoids. Simple finite monoids are exactly **simple finite groups** and the boolean monoid.

Observation related to semigroups. There are plenty of simple finite semigroups which are not monoids (they are classified).

Conclusion. Existence of the identity is a very serious restriction.

The significance of the identity: semigroup detour

Definition. A semigroup/monoid S is called **simple** if it does not have any non-trivial quotients.

Problem. Classify all simple finite semigroups/monoids.

Answer for monoids. Simple finite monoids are exactly **simple finite groups** and the boolean monoid.

Observation related to semigroups. There are plenty of simple finite semigroups which are not monoids (they are classified).

Conclusion. Existence of the identity is a very serious restriction.

Idea for the solution

Idea. Substitute the identity by a **lax identity** and (possibly different) **oplax identity**.

Credit for the idea: Marco Mackaay.

Cheating? Not really, in representations, (op)lax identities are usually not represented by the identity functors.

Lax identity. I together with $I \circ_h F \xrightarrow{l_F} F$ and $F \circ_h I \xrightarrow{r_F} F$, for each F , natural in F .



Idea for the solution

Idea. Substitute the identity by a **lax identity** and (possibly different) **oplax identity**.

Credit for the idea: Marco Mackaay.

Cheating? Not really, in representations, (op)lax identities are usually not represented by the identity functors.

Lax identity. I together with $I \circ_h F \xrightarrow{l_F} F$ and $F \circ_h I \xrightarrow{r_F} F$, for each F , natural in F .



Idea for the solution

Idea. Substitute the identity by a **lax identity** and (possibly different) **oplax identity**.

Credit for the idea: Marco Mackaay.

Cheating? Not really, in representations, (op)lax identities are usually not represented by the identity functors.

Lax identity. I together with $I \circ_h F \xrightarrow{l_F} F$ and $F \circ_h I \xrightarrow{r_F} F$, for each F , natural in F .



Idea for the solution

Idea. Substitute the identity by a **lax identity** and (possibly different) **oplax identity**.

Credit for the idea: Marco Mackaay.

Cheating? Not really, in representations, (op)lax identities are usually not represented by the identity functors.

Lax identity. I together with $I \circ_h F \xrightarrow{l_F} F$ and $F \circ_h I \xrightarrow{r_F} F$, for each F , natural in F .



Idea for the solution

Idea. Substitute the identity by a **lax identity** and (possibly different) **oplax identity**.

Credit for the idea: Marco Mackaay.

Cheating? Not really, in representations, (op)lax identities are usually not represented by the identity functors.

Lax identity. I together with $I \circ_h F \xrightarrow{l_F} F$ and $F \circ_h I \xrightarrow{r_F} F$, for each F , natural in F .



Idea for the solution

Idea. Substitute the identity by a **lax identity** and (possibly different) **oplax identity**.

Credit for the idea: Marco Mackaay.

Cheating? Not really, in representations, (op)lax identities are usually not represented by the identity functors.

Lax identity. I together with $I \circ_h F \xrightarrow{l_F} F$ and $F \circ_h I \xrightarrow{r_F} F$, for each F , natural in F .



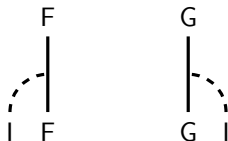
Idea for the solution

Idea. Substitute the identity by a **lax identity** and (possibly different) **oplax identity**.

Credit for the idea: Marco Mackaay.

Cheating? Not really, in representations, (op)lax identities are usually not represented by the identity functors.

Lax identity. I together with $I \circ_h F \xrightarrow{l_F} F$ and $F \circ_h I \xrightarrow{r_F} F$, for each F , natural in F .



Axioms

$$\begin{array}{c} G & F \\ | & | \\ G & I & F \end{array} = \begin{array}{c} G & F \\ | & | \\ G & I & F \end{array}$$

$$\begin{array}{c} F & G \\ | & | \\ I & F & G \end{array} = \begin{array}{c} FG \\ | \\ I & FG \end{array} \quad \text{and} \quad \begin{array}{c} F & G \\ | & | \\ F & G & I \end{array} = \begin{array}{c} FG \\ | \\ FG & I \end{array}$$

Dual for the oplax identity I' .

Axioms

$$\begin{array}{c} G \\ | \\ G \end{array} \begin{array}{c} F \\ \curvearrowright \\ I \end{array} \begin{array}{c} F \\ | \\ F \end{array} = \begin{array}{c} G \\ | \\ G \end{array} \begin{array}{c} F \\ \curvearrowright \\ I \end{array} \begin{array}{c} F \\ | \\ F \end{array}$$

$$\begin{array}{c} F \\ \curvearrowright \\ I \end{array} \begin{array}{c} F \\ | \\ F \end{array} \begin{array}{c} G \\ | \\ G \end{array} = \begin{array}{c} FG \\ \curvearrowright \\ I \end{array} \begin{array}{c} FG \\ | \\ FG \end{array} \quad \text{and} \quad \begin{array}{c} F \\ | \\ F \end{array} \begin{array}{c} G \\ | \\ G \end{array} \begin{array}{c} I \\ \curvearrowright \\ I \end{array} = \begin{array}{c} FG \\ \curvearrowright \\ FG \end{array} \begin{array}{c} FG \\ | \\ FG \end{array}$$

Dual for the oplax identity I' .

Axioms

A diagrammatic equation showing the identity axiom for the adjunction. On the left, a blue vertical line labeled 'G' at both top and bottom is connected to a black vertical line labeled 'F' at both top and bottom by a dashed arc on the right side. This is equal to the right-hand side, where the blue 'G' line is connected to the black 'F' line by a dashed arc on the left side.

Two diagrammatic equations for the Frobenius axiom. The first equation shows a dashed arc on the left connecting a black 'F' line and a blue 'G' line, which is equal to a pink 'FG' line with a dashed arc on the left. The second equation shows a dashed arc on the right connecting a black 'F' line and a blue 'G' line, which is equal to a pink 'FG' line with a dashed arc on the right. The word 'and' is placed between the two equations.

Dual for the oplax identity I' .

Axioms

A diagrammatic equation showing the interchange of objects G and F. On the left, a blue vertical line labeled G at both ends is on the left, and a black vertical line labeled F at both ends is on the right. A dashed arc connects the top of the G line to the top of the F line. On the right, the same diagram is shown but with the lines swapped: the blue G line is on the right and the black F line is on the left. The two diagrams are separated by an equals sign.

Two diagrammatic equations. The first equation shows a diagram with a dashed arc on the left, a black vertical line labeled F at both ends in the middle, and a blue vertical line labeled G at both ends on the right. This is equal to a diagram with a dashed arc on the left and a pink vertical line labeled FG at both ends in the middle. The second equation shows a diagram with a black vertical line labeled F at both ends on the left, a blue vertical line labeled G at both ends in the middle, and a dashed arc on the right. This is equal to a diagram with a pink vertical line labeled FG at both ends in the middle and a dashed arc on the right. The two equations are separated by the word "and".

Dual for the oplax identity I' .

New setup: definitions

Definition. A **bilax unital 2-category** is a 2-semicategory with a choice of a lax unit l_i and an oplax unit l'_i , for each object.

\mathcal{C} — bilax unital 2-category.

$F \in \mathcal{C}(i, j)$ and $G \in \mathcal{C}(j, i)$

Definition. (F, G) is a pair of **adjoint 1-morphisms** in \mathcal{C} provided that there exist $\varepsilon : FG \rightarrow l_j$ and $\eta : l'_i \rightarrow GF$ such that the compositions

$$F \rightarrow Fl'_i \rightarrow FGF \rightarrow l_j F \rightarrow F$$

and

$$G \rightarrow l'_i G \rightarrow GFG \rightarrow Gl_j \rightarrow G$$

are the identities.

New setup: definitions

Definition. A **bilax unital 2-category** is a 2-semicategory with a choice of a lax unit I_i and an oplax unit I'_i , for each object.

\mathcal{C} — bilax unital 2-category.

$F \in \mathcal{C}(i, j)$ and $G \in \mathcal{C}(j, i)$

Definition. (F, G) is a pair of **adjoint 1-morphisms** in \mathcal{C} provided that there exist $\varepsilon : FG \rightarrow I_j$ and $\eta : I'_i \rightarrow GF$ such that the compositions

$$F \rightarrow FI'_i \rightarrow FGF \rightarrow I_j F \rightarrow F$$

and

$$G \rightarrow I'_i G \rightarrow GFG \rightarrow GI_j \rightarrow G$$

are the identities.

New setup: definitions

Definition. A **bilax unital 2-category** is a 2-semicategory with a choice of a lax unit I_i and an oplax unit I'_i , for each object.

\mathcal{C} — bilax unital 2-category.

$F \in \mathcal{C}(i, j)$ and $G \in \mathcal{C}(j, i)$

Definition. (F, G) is a pair of **adjoint 1-morphisms** in \mathcal{C} provided that there exist $\varepsilon : FG \rightarrow I_j$ and $\eta : I'_i \rightarrow GF$ such that the compositions

$$F \rightarrow FI'_i \rightarrow FGF \rightarrow I_j F \rightarrow F$$

and

$$G \rightarrow I'_i G \rightarrow GFG \rightarrow GI_j \rightarrow G$$

are the identities.

New setup: definitions

Definition. A **bilax unital 2-category** is a 2-semicategory with a choice of a lax unit I_i and an oplax unit I'_i , for each object.

\mathcal{C} — bilax unital 2-category.

$F \in \mathcal{C}(i, j)$ and $G \in \mathcal{C}(j, i)$

Definition. (F, G) is a pair of **adjoint 1-morphisms** in \mathcal{C} provided that there exist $\varepsilon : FG \rightarrow I_j$ and $\eta : I'_i \rightarrow GF$ such that the compositions

$$F \rightarrow FI'_i \rightarrow FGF \rightarrow I_j F \rightarrow F$$

and

$$G \rightarrow I'_i G \rightarrow GFG \rightarrow GI_j \rightarrow G$$

are the identities.

New setup: definitions

Definition. A **bilax unital 2-category** is a 2-semicategory with a choice of a lax unit I_i and an oplax unit I'_i , for each object.

\mathcal{C} — bilax unital 2-category.

$F \in \mathcal{C}(i, j)$ and $G \in \mathcal{C}(j, i)$

Definition. (F, G) is a pair of **adjoint 1-morphisms** in \mathcal{C} provided that there exist $\varepsilon : FG \rightarrow I_j$ and $\eta : I'_i \rightarrow GF$ such that the compositions

$$F \rightarrow FI'_i \rightarrow FGF \rightarrow I_j F \rightarrow F$$

and

$$G \rightarrow I'_i G \rightarrow GFG \rightarrow GI_j \rightarrow G$$

are the identities.

Diagrammatically

The diagram shows two equations. The first equation shows a complex diagram with two vertical lines labeled 'F' at the top and bottom. A blue dashed line labeled 'I' connects the top 'F' to the top of a loop. A red dashed line labeled 'I'' connects the bottom of the loop to the bottom 'F'. The loop itself is labeled 'G'. This is shown to be equal to a single vertical line labeled 'F'. The second equation shows a similar diagram with two vertical lines labeled 'G' at the top and bottom. A blue dashed line labeled 'I' connects the top 'G' to the top of a loop. A red dashed line labeled 'I'' connects the bottom of the loop to the bottom 'G'. The loop is labeled 'F'. This is shown to be equal to a single vertical line labeled 'G'. The entire set of equations is labeled (1).

$$\begin{array}{c} \text{F} \\ | \\ \text{---} \text{I} \text{---} \\ | \\ \text{G} \\ | \\ \text{---} \text{I}' \text{---} \\ | \\ \text{F} \end{array} = \text{F} \quad \text{and} \quad \begin{array}{c} \text{G} \\ | \\ \text{---} \text{I} \text{---} \\ | \\ \text{F} \\ | \\ \text{---} \text{I}' \text{---} \\ | \\ \text{G} \end{array} = \text{G} \quad (1)$$

Diagrammatically

The diagram shows two equations. The first equation shows a complex diagram with two vertical lines labeled 'F' at the top and bottom. A blue dashed arc labeled 'I' connects the top of the left 'F' line to the top of the right 'F' line. A red dashed arc labeled 'I'' connects the bottom of the left 'F' line to the bottom of the right 'F' line. A teal loop labeled 'G' is formed by a solid teal line that starts at the top of the right 'F' line, goes up, then down, then up, then down, ending at the bottom of the right 'F' line. This diagram is equal to a single vertical line labeled 'F'. The second equation shows a similar diagram but with the vertical lines labeled 'G' at the top and bottom. A blue dashed arc labeled 'I' connects the top of the left 'G' line to the top of the right 'G' line. A red dashed arc labeled 'I'' connects the bottom of the left 'G' line to the bottom of the right 'G' line. A teal loop labeled 'F' is formed by a solid teal line that starts at the top of the right 'G' line, goes up, then down, then up, then down, ending at the bottom of the right 'G' line. This diagram is equal to a single vertical line labeled 'G'. The entire set of equations is labeled (1).

$$\begin{array}{c} \text{F} \\ | \\ \text{F} \end{array} \begin{array}{c} \text{I} \\ \text{---} \\ \text{G} \\ \text{---} \\ \text{I}' \end{array} \begin{array}{c} \text{F} \\ | \\ \text{F} \end{array} = \begin{array}{c} \text{F} \\ | \\ \text{F} \end{array} \text{ and } \begin{array}{c} \text{G} \\ | \\ \text{G} \end{array} \begin{array}{c} \text{I} \\ \text{---} \\ \text{F} \\ \text{---} \\ \text{I}' \end{array} \begin{array}{c} \text{G} \\ | \\ \text{G} \end{array} = \begin{array}{c} \text{G} \\ | \\ \text{G} \end{array} \quad (1)$$

New setup: remarks

Note. The adjunction defined in the way is **equivalent** to the classical adjunction.

Reason for that: Despite of the fact that the (op)lax units are not usually represented by the identity functors, they are represented by functors which **have non-trivial natural transformations to (from) the identity functors**. These allow us to compare our adjunction to the classical adjunction.

New setup: remarks

Note. The adjunction defined in the way is **equivalent** to the classical adjunction.

Reason for that: Despite of the fact that the (op)lax units are not usually represented by the identity functors, they are represented by functors which **have non-trivial natural transformations to (from) the identity functors**. These allow us to compare our adjunction to the classical adjunction.

New setup: remarks

Note. The adjunction defined in the way is **equivalent** to the classical adjunction.

Reason for that: Despite of the fact that the (op)lax units are not usually represented by the identity functors, they are represented by functors which **have non-trivial natural transformations to (from) the identity functors**. These allow us to compare our adjunction to the classical adjunction.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \dots + e_n$ — primitive decomposition of $1 \in A$

Definition. The bilax unital 2-category \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $\mathbb{k} \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \dots + e_n$ — primitive decomposition of $1 \in A$

Definition. The bilax unitary 2-category \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $k \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \cdots + e_n$ — primitive decomposition of $1 \in A$

Definition. The bilax unital 2-category \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $k \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \cdots + e_n$ — primitive decomposition of $1 \in A$

Definition. The **bilax unitax 2-category** \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $k \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \cdots + e_n$ — primitive decomposition of $1 \in A$

Definition. The **bilax unitary 2-category** \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $\mathbb{k} \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \dots + e_n$ — primitive decomposition of $1 \in A$

Definition. The **bilax unitary 2-category** \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $\mathbb{k} \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \cdots + e_n$ — primitive decomposition of $1 \in A$

Definition. The **bilax unitary 2-category** \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $\mathbb{k} \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \cdots + e_n$ — primitive decomposition of $1 \in A$

Definition. The **bilax unital 2-category** \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $\mathbb{k} \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \dots + e_n$ — primitive decomposition of $1 \in A$

Definition. The **bilax unital 2-category** \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $\mathbb{k} \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Example

A — finite dimensional \mathbb{k} -algebra, basic, connected, self-injective, weakly symmetric

$1 = e_1 + e_2 + \cdots + e_n$ — primitive decomposition of $1 \in A$

Definition. The **bilax unital 2-category** \mathcal{D}_A is defined to have:

- ▶ objects: $1, \dots, n$, where $\mathbb{k} \leftrightarrow e_k A e_k\text{-mod}$;
- ▶ 1-morphisms: functors isomorphic to tensoring with $X \in \text{add}(Ae_i \otimes_{\mathbb{k}} e_j A)$;
- ▶ 2-morphisms: natural transformations of functors.

Observe: No genuine identities!!!!

Note: Each $Ae_i \otimes_{\mathbb{k}} e_j A$ is a lax identity via the multiplication map $ae_i \otimes e_j b \mapsto ae_i b$ (a morphism from $Ae_i \otimes_{\mathbb{k}} e_j A$ to A).

Note: Using the weak involution on \mathcal{C}_A , each $Ae_i \otimes_{\mathbb{k}} e_j A$ is also an oplax identity.

Generalization: bilax 2-categories of \mathcal{J} -cell, preliminaries

\mathcal{C} — fiat

$F \sim_L G$ if and only if $\text{add}(\mathcal{C} \circ F) = \text{add}(\mathcal{C} \circ G)$

\sim_R and \sim_J are defined similarly

\mathcal{J} — an equivalence class for \sim_J (two-sided cell)

$\mathcal{L} \subset \mathcal{J}$ — an equivalence class for \sim_L (left cell)

Observation. [Mazorchuk-Miemietz] \mathcal{L} contains a unique 1-morphism F (called **Duflo element**) for which there is a homomorphism $\xi : F \rightarrow \mathbb{1}_i$ such that $G(\xi)$ is **right split**, for every $G \in \mathcal{L}$.

Generalization: bilax 2-categories of \mathcal{J} -cell, preliminaries

\mathcal{C} — fiat

$F \sim_L G$ if and only if $\text{add}(\mathcal{C} \circ F) = \text{add}(\mathcal{C} \circ G)$

\sim_R and \sim_J are defined similarly

\mathcal{J} — an equivalence class for \sim_J (two-sided cell)

$\mathcal{L} \subset \mathcal{J}$ — an equivalence class for \sim_L (left cell)

Observation. [Mazorchuk-Miemietz] \mathcal{L} contains a unique 1-morphism F (called **Duflo element**) for which there is a homomorphism $\xi : F \rightarrow \mathbb{1}_i$ such that $G(\xi)$ is **right split**, for every $G \in \mathcal{L}$.

Generalization: bilax 2-categories of \mathcal{J} -cell, preliminaries

\mathcal{C} — fiat

$F \sim_L G$ if and only if $\text{add}(\mathcal{C} \circ F) = \text{add}(\mathcal{C} \circ G)$

\sim_R and \sim_J are defined similarly

\mathcal{J} — an equivalence class for \sim_J (two-sided cell)

$\mathcal{L} \subset \mathcal{J}$ — an equivalence class for \sim_L (left cell)

Observation. [Mazorchuk-Miemietz] \mathcal{L} contains a unique 1-morphism F (called **Duflo element**) for which there is a homomorphism $\xi : F \rightarrow \mathbb{1}_i$ such that $G(\xi)$ is **right split**, for every $G \in \mathcal{L}$.

Generalization: bilax 2-categories of \mathcal{J} -cell, preliminaries

\mathcal{C} — fiat

$F \sim_L G$ if and only if $\text{add}(\mathcal{C} \circ F) = \text{add}(\mathcal{C} \circ G)$

\sim_R and \sim_J are defined similarly

\mathcal{J} — an equivalence class for \sim_J (two-sided cell)

$\mathcal{L} \subset \mathcal{J}$ — an equivalence class for \sim_L (left cell)

Observation. [Mazorchuk-Miemietz] \mathcal{L} contains a unique 1-morphism F (called Duflo element) for which there is a homomorphism $\xi : F \rightarrow \mathbb{1}_i$ such that $G(\xi)$ is right split, for every $G \in \mathcal{L}$.

Generalization: bilax 2-categories of \mathcal{J} -cell, preliminaries

\mathcal{C} — fiat

$F \sim_L G$ if and only if $\text{add}(\mathcal{C} \circ F) = \text{add}(\mathcal{C} \circ G)$

\sim_R and \sim_J are defined similarly

\mathcal{J} — an equivalence class for \sim_J (**two-sided cell**)

$\mathcal{L} \subset \mathcal{J}$ — an equivalence class for \sim_L (**left cell**)

Observation. [Mazorchuk-Miemietz] \mathcal{L} contains a unique 1-morphism F (called **Duflo element**) for which there is a homomorphism $\xi : F \rightarrow \mathbb{1}_i$ such that $G(\xi)$ is **right split**, for every $G \in \mathcal{L}$.

Generalization: bilax 2-categories of \mathcal{J} -cell, preliminaries

\mathcal{C} — fiat

$F \sim_L G$ if and only if $\text{add}(\mathcal{C} \circ F) = \text{add}(\mathcal{C} \circ G)$

\sim_R and \sim_J are defined similarly

\mathcal{J} — an equivalence class for \sim_J (**two-sided cell**)

$\mathcal{L} \subset \mathcal{J}$ — an equivalence class for \sim_L (**left cell**)

Observation. [Mazorchuk-Miemietz] \mathcal{L} contains a unique 1-morphism F (called **Duflo element**) for which there is a homomorphism $\xi : F \rightarrow \mathbb{1}_i$ such that $G(\xi)$ is **right split**, for every $G \in \mathcal{L}$.

Generalization: bilax 2-categories of \mathcal{J} -cell, preliminaries

\mathcal{C} — fiat

$F \sim_L G$ if and only if $\text{add}(\mathcal{C} \circ F) = \text{add}(\mathcal{C} \circ G)$

\sim_R and \sim_J are defined similarly

\mathcal{J} — an equivalence class for \sim_J (**two-sided cell**)

$\mathcal{L} \subset \mathcal{J}$ — an equivalence class for \sim_L (**left cell**)

Observation. [Mazorchuk-Miemietz] \mathcal{L} contains a unique 1-morphism F (called **Duflo element**) for which there is a homomorphism $\xi : F \rightarrow \mathbb{1}$; such that $G(\xi)$ is **right split**, for every $G \in \mathcal{L}$.

Generalization: bilax 2-categories of \mathcal{J} -cell, definition

Definition. The bilax unital 2-category $\mathcal{D}_{\mathcal{J}}$ is defined to have:

- ▶ objects are in bijection with Duflo elements in \mathcal{J} ;
- ▶ 1-morphisms from Duflo F to Duflo G : the additive closure of the intersection of the left cell of F and the right cell of G ;
- ▶ 2-morphisms: induced from \mathcal{C} modulo those which factor through "higher" \mathcal{J} -cells.
- ▶ composition is induced from \mathcal{C} modulo "higher" \mathcal{J} -cells.
- ▶ lax units: Duflo 1-morphisms.
- ▶ oplax units: coDuflo 1-morphisms (i.e. F^* , for F Duflo).

Generalization: bilax 2-categories of \mathcal{J} -cell, definition

Definition. The bilax unital 2-category $\mathcal{D}_{\mathcal{J}}$ is defined to have:

- ▶ objects are in bijection with Duflo elements in \mathcal{J} ;
- ▶ 1-morphisms from Duflo F to Duflo G : the additive closure of the intersection of the left cell of F and the right cell of G ;
- ▶ 2-morphisms: induced from \mathcal{C} modulo those which factor through "higher" \mathcal{J} -cells.
- ▶ composition is induced from \mathcal{C} modulo "higher" \mathcal{J} -cells.
- ▶ lax units: Duflo 1-morphisms.
- ▶ oplax units: coDuflo 1-morphisms (i.e. F^* , for F Duflo).

Generalization: bilax 2-categories of \mathcal{J} -cell, definition

Definition. The bilax unital 2-category $\mathcal{D}_{\mathcal{J}}$ is defined to have:

- ▶ objects are in bijection with Duflo elements in \mathcal{J} ;
- ▶ 1-morphisms from Duflo F to Duflo G : the additive closure of the intersection of the left cell of F and the right cell of G ;
- ▶ 2-morphisms: induced from \mathcal{C} modulo those which factor through "higher" \mathcal{J} -cells.
- ▶ composition is induced from \mathcal{C} modulo "higher" \mathcal{J} -cells.
- ▶ lax units: Duflo 1-morphisms.
- ▶ oplax units: coDuflo 1-morphisms (i.e. F^* , for F Duflo).

Generalization: bilax 2-categories of \mathcal{J} -cell, definition

Definition. The bilax unital 2-category $\mathcal{D}_{\mathcal{J}}$ is defined to have:

- ▶ objects are in bijection with Duflo elements in \mathcal{J} ;
- ▶ 1-morphisms from Duflo F to Duflo G : the additive closure of the intersection of the left cell of F and the right cell of G ;
- ▶ 2-morphisms: induced from \mathcal{C} modulo those which factor through “higher” \mathcal{J} -cells.
- ▶ composition is induced from \mathcal{C} modulo “higher” \mathcal{J} -cells.
- ▶ lax units: Duflo 1-morphisms.
- ▶ oplax units: coDuflo 1-morphisms (i.e. F^* , for F Duflo).

Generalization: bilax 2-categories of \mathcal{J} -cell, definition

Definition. The bilax unital 2-category $\mathcal{D}_{\mathcal{J}}$ is defined to have:

- ▶ objects are in bijection with Duflo elements in \mathcal{J} ;
- ▶ 1-morphisms from Duflo F to Duflo G : the additive closure of the intersection of the left cell of F and the right cell of G ;
- ▶ 2-morphisms: induced from \mathcal{C} modulo those which factor through “higher” \mathcal{J} -cells.
- ▶ composition is induced from \mathcal{C} modulo “higher” \mathcal{J} -cells.
- ▶ lax units: Duflo 1-morphisms.
- ▶ oplax units: coDuflo 1-morphisms (i.e. F^* , for F Duflo).

Generalization: bilax 2-categories of \mathcal{J} -cell, definition

Definition. The bilax unital 2-category $\mathcal{D}_{\mathcal{J}}$ is defined to have:

- ▶ objects are in bijection with Duflo elements in \mathcal{J} ;
- ▶ 1-morphisms from Duflo F to Duflo G : the additive closure of the intersection of the left cell of F and the right cell of G ;
- ▶ 2-morphisms: induced from \mathcal{C} modulo those which factor through “higher” \mathcal{J} -cells.
- ▶ composition is induced from \mathcal{C} modulo “higher” \mathcal{J} -cells.
- ▶ lax units: Duflo 1-morphisms.
- ▶ oplax units: coDuflo 1-morphisms (i.e. F^* , for F Duflo).

Generalization: bilax 2-categories of \mathcal{J} -cell, definition

Definition. The bilax unital 2-category $\mathcal{D}_{\mathcal{J}}$ is defined to have:

- ▶ objects are in bijection with Duflo elements in \mathcal{J} ;
- ▶ 1-morphisms from Duflo F to Duflo G : the additive closure of the intersection of the left cell of F and the right cell of G ;
- ▶ 2-morphisms: induced from \mathcal{C} modulo those which factor through “higher” \mathcal{J} -cells.
- ▶ composition is induced from \mathcal{C} modulo “higher” \mathcal{J} -cells.
- ▶ lax units: Duflo 1-morphisms.
- ▶ oplax units: coDuflo 1-morphisms (i.e. F^* , for F Duflo).

Generalization: bilax 2-categories of \mathcal{J} -cell, definition

Definition. The bilax unital 2-category $\mathcal{D}_{\mathcal{J}}$ is defined to have:

- ▶ objects are in bijection with Duflo elements in \mathcal{J} ;
- ▶ 1-morphisms from Duflo F to Duflo G : the additive closure of the intersection of the left cell of F and the right cell of G ;
- ▶ 2-morphisms: induced from \mathcal{C} modulo those which factor through “higher” \mathcal{J} -cells.
- ▶ composition is induced from \mathcal{C} modulo “higher” \mathcal{J} -cells.
- ▶ lax units: Duflo 1-morphisms.
- ▶ oplax units: coDuflo 1-morphisms (i.e. F^* , for F Duflo).

Discussion

This allows us to define a setup in which we can talk about adjoint 1-morphisms without having genuine identities.

A lot of choice involved: (op)lax identities are in no way unique (in particular, they are closed under composition).

The genuine identity is the **coequalizer** of $\Pi \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{\eta} \end{array} \mathbb{I}$.

Good case: Unitors split (true in our main examples).

Some results from 2-representation theory of finitary 2-categories generalize (sometimes in a “cleaner” form).

Very technical.

Many open questions.

Discussion

This allows us to define a setup in which we can talk about adjoint 1-morphisms without having genuine identities.

A lot of choice involved: (op)lax identities are in no way unique (in particular, they are closed under composition).

The genuine identity is the **coequalizer** of $\Pi \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{\eta} \end{array} I$.

Good case: Unitors split (true in our main examples).

Some results from 2-representation theory of finitary 2-categories generalize (sometimes in a “cleaner” form).

Very technical.

Many open questions.

Discussion

This allows us to define a setup in which we can talk about adjoint 1-morphisms without having genuine identities.

A lot of choice involved: (op)lax identities are in no way unique (in particular, they are closed under composition).

The genuine identity is the **coequalizer** of $\Pi \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{\eta} \end{array} I$.

Good case: Unitors split (true in our main examples).

Some results from 2-representation theory of finitary 2-categories generalize (sometimes in a “cleaner” form).

Very technical.

Many open questions.

Discussion

This allows us to define a setup in which we can talk about adjoint 1-morphisms without having genuine identities.

A lot of choice involved: (op)lax identities are in no way unique (in particular, they are closed under composition).

The genuine identity is the **coequalizer** of $\text{II} \begin{array}{c} \xrightarrow{l_I} \\ \xrightarrow{r_I} \end{array} \text{I}$.

Good case: Unitors split (true in our main examples).

Some results from 2-representation theory of finitary 2-categories generalize (sometimes in a “cleaner” form).

Very technical.

Many open questions.

Discussion

This allows us to define a setup in which we can talk about adjoint 1-morphisms without having genuine identities.

A lot of choice involved: (op)lax identities are in no way unique (in particular, they are closed under composition).

The genuine identity is the **coequalizer** of $\text{II} \begin{array}{c} \xrightarrow{l_I} \\ \xrightarrow{r_I} \end{array} \text{I}$.

Good case: Unitors split (true in our main examples).

Some results from 2-representation theory of finitary 2-categories generalize (sometimes in a “cleaner” form).

Very technical.

Many open questions.

Discussion

This allows us to define a setup in which we can talk about adjoint 1-morphisms without having genuine identities.

A lot of choice involved: (op)lax identities are in no way unique (in particular, they are closed under composition).

The genuine identity is the **coequalizer** of $\text{II} \begin{array}{c} \xrightarrow{l_I} \\ \xrightarrow{r_I} \end{array} \text{I}$.

Good case: Unitors split (true in our main examples).

Some results from 2-representation theory of finitary 2-categories generalize (sometimes in a “cleaner” form).

Very technical.

Many open questions.

Discussion

This allows us to define a setup in which we can talk about adjoint 1-morphisms without having genuine identities.

A lot of choice involved: (op)lax identities are in no way unique (in particular, they are closed under composition).

The genuine identity is the **coequalizer** of $\text{II} \begin{array}{c} \xrightarrow{l_I} \\ \xrightarrow{r_I} \end{array} \text{I}$.

Good case: Unitors split (true in our main examples).

Some results from 2-representation theory of finitary 2-categories generalize (sometimes in a “cleaner” form).

Very technical.

Many open questions.

Discussion

This allows us to define a setup in which we can talk about adjoint 1-morphisms without having genuine identities.

A lot of choice involved: (op)lax identities are in no way unique (in particular, they are closed under composition).

The genuine identity is the **coequalizer** of $\text{II} \begin{array}{c} \xrightarrow{l_I} \\ \xrightarrow{r_I} \end{array} \text{I}$.

Good case: Unitors split (true in our main examples).

Some results from 2-representation theory of finitary 2-categories generalize (sometimes in a “cleaner” form).

Very technical.

Many open questions.

THANK YOU!!!