# Four-dimensional semimetals with tensor monopoles: from surface states to topological responses

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Y.-Q. Zhu, N. Goldman and G.P., arXiv:2007.00549
G. P. and N. Goldman, Phys. Rev. B 99, 045154 (2019)
G. P. and N. Goldman, Phys. Rev. Lett. 121, 170401 (2018)

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# Outline

- Gauge theories and Abelian monopoles
- Dirac monopoles in 3D Weyl semimetals
- Tensor Berry connections
- Tensor monopoles in 4D topological semimetals
- 3D boundary states
- Parity magnetic effect

# Topological phases of matter

- States of matter beyond Landau's theory
- Topological quantum numbers: $Z$, $Z_2$, etc.
- Bulk-edge correspondence
- Effective Dirac Hamiltonians
- Berry connections and curvatures

# Vector gauge fields vs tensor gauge fields

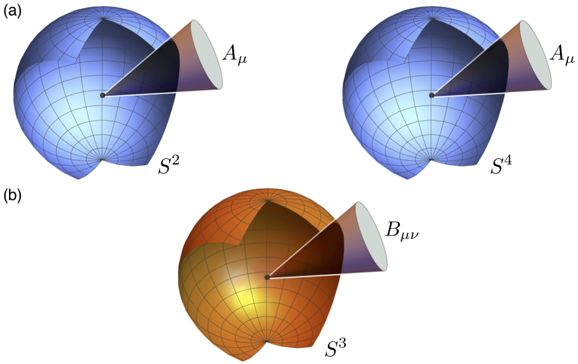
**U(1) gauge theories**

- $L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu$
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- $A_\mu \to A_\mu - \partial_\mu \lambda$
- $E_i = F_{i0}, \quad \tilde{B}_i = \epsilon_{ijk} F^{jk}$
- $\partial^i \tilde{B}_i = 0$
- Dirac monopoles violate the magnetic Gauss law

- $L = \frac{1}{4} \mathcal{H}_{\mu\nu\lambda} \mathcal{H}^{\mu\nu\lambda} + B_{\mu\nu} J^{\mu\nu}$
- $\mathcal{H}_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$
- $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$
- $\tilde{E}_{ij} = \mathcal{H}_{ij0}, \quad \tilde{B}_i = \epsilon_{ijkl} \mathcal{H}^{jkl}$
- $\partial^i \tilde{B}_i = 0$
- Tensor monopoles violate the magnetic Gauss law

# Monopoles and topological invariants

Monopoles allow us to prove the quantization of the electric charge.




The first and second **Chern numbers** characterize monopoles in 3D (Dirac monopole) and 5D (Yang monopole), respectively.

Tensor monopoles in 4D are characterized by the **Dixmier-Douady** ($\mathcal{DD}$) invariant (Murray, 1996).

# 3D Dirac monopoles

U(1) vector gauge field: $A_\mu \to A_\mu - \partial_\mu \lambda$.

Curvature tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$$\tilde{B}^j = \frac{r^j}{|r|^3},$$
$$r^j = \{x, y, z\}, \quad r = \sqrt{x^2 + y^2 + z^2}.$$

First Chern number:

$$Q = \frac{1}{2\pi} \int_{S^2} dx^k \wedge dx^l \, F_{kl} = \pm \frac{1}{2\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta = \pm 1.$$

# 4D tensor monopoles

U(1) tensor gauge field: $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$.

Curvature tensor: $\mathcal{H}_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$.

$$\tilde{B}^j = \frac{r^j}{|r|^4},$$
$$r^j = \{x, y, z, w\}, \quad r = \sqrt{x^2 + y^2 + z^2 + w^2}.$$

First DD number:

$$Q_T = \frac{1}{2\pi^2} \int_{S^3} dx^k \wedge dx^l \wedge dx^m \, \mathcal{H}_{klm} =$$
$$\pm \frac{1}{2\pi^2} \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^{2\pi} d\varphi \sin^2\theta_1 \sin\theta_2 = \pm 1.$$

# Berry connection

Bloch wavefunctions: $|u(\mathbf{q})\rangle = (u^1(\mathbf{q}), u^2(\mathbf{q}), ..., u^{\aleph}(\mathbf{q}))^\top$.

The gauge redundancy in a non-degenerate Bloch state is encoded in the arbitrary phase in $|u\rangle$

$$|u\rangle \to e^{i\alpha(\mathbf{q})}|u\rangle,$$

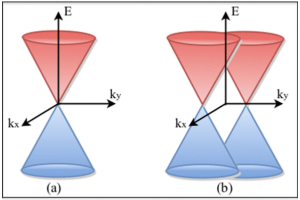
where $\alpha(\mathbf{q})$ is a momentum-dependent function.
We can build an Abelian gauge connection in momentum space:

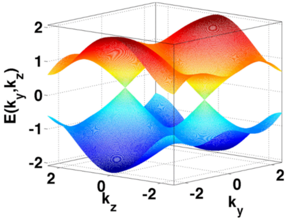
$$A_j = i\langle u|\partial_j|u\rangle, \quad A_j \to A_j - \partial_j\alpha$$

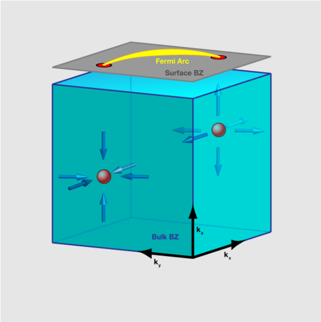
with $\partial_j \equiv \partial_{q_j}$, while the gauge-invariant Berry curvature is given by

$$\Omega_{jk} = \partial_j A_k - \partial_k A_j.$$

# 3D Weyl semimetals



# Momentum-space Dirac monopole

$$H_{3\mathrm{D}} = q_x\sigma^x + q_y\sigma^y + q_z\sigma^z,$$

where $\boldsymbol{q} = (q_x, q_y, q_z)$ are the momenta and $\sigma^{x,y,z}$ are the Pauli matrices.

In 3D Weyl semimetals, the Berry curvature of the Dirac monopole is given by

$$\Omega_{jk} = \epsilon_{jkl} \frac{q_l}{2(q_x^2 + q_y^2 + q_z^2)^{3/2}},$$

$$Q = \frac{1}{2\pi} \int_{S^2} dq^j \wedge dq^k \Omega_{jk} = \pm 1.$$

# Tensor Berry connection

$$B_{jk} = \frac{i}{3}\,\epsilon^{abc}\phi_a\partial_j\phi_b\partial_k\phi_c,$$

where $a, b, c = \{1, 2, 3\}$.

Scalars can be complex or (pseudo-)real depending on their gauge transformations

$$\phi \to e^{i\alpha(\mathbf{q})}\phi,$$
$$\varphi \to \varphi + \alpha(\mathbf{q}).$$

In our case, we consider the following scalars

$$\phi_1 \equiv \varphi \to \varphi + \alpha(\mathbf{q}), \quad \phi_2 \to e^{-i\alpha(\mathbf{q})}\phi_2, \quad \phi_3 \to e^{i\alpha(\mathbf{q})}\phi_3,$$

such that

$$B_{jk} \to B_{jk} + \Lambda_{jk},$$
$$\mathcal{H}_{jkl} = \partial_j B_{kl} + \partial_k B_{lj} + \partial_l B_{jk}.$$

# 4D topological semimetals

$$H(\boldsymbol{k}) = d_x \tilde{\Gamma}_x + d_y \tilde{\Gamma}_y + d_z \tilde{\Gamma}_z + d_w \tilde{\Gamma}_w, \tag{1}$$

with the four-component Bloch vector defined as

$$\begin{aligned} d_x &= 2J \sin k_x, \;\; d_y = 2J \sin k_y, \;\; d_z = 2J \sin k_z, \\ d_w &= 2J(M - \cos k_x - \cos k_y - \cos k_z - \cos k_w). \end{aligned} \tag{2}$$

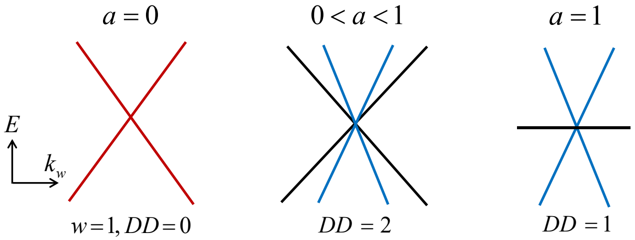
with

$$\begin{aligned} \tilde{\Gamma}_x &= \sigma_0 \otimes \sigma_1 + a\,\sigma_1 \otimes \sigma_0, \quad \tilde{\Gamma}_y = \sigma_2 \otimes \sigma_3 + a\,\sigma_3 \otimes \sigma_2, \\ \tilde{\Gamma}_z &= \sigma_0 \otimes \sigma_2 + a\,\sigma_2 \otimes \sigma_0, \quad \tilde{\Gamma}_w = \sigma_1 \otimes \sigma_3 + a\,\sigma_3 \otimes \sigma_1. \end{aligned}$$

These matrices only satisfy the Clifford algebra for $a = 0$ ("Dirac regime"). When $a \neq 0$, the Hamiltonian (1) supports spin-3/2-like quasiparticles (birefringent fermions).

# Monopole-to-monopole phase transition

For $a = 0$ we have a $Z_2$ monopole protected by CP symmetry.




$$E(\boldsymbol{k}) = \pm(1 \pm a)\sqrt{d_x^2 + d_y^2 + d_z^2 + d_w^2}. \quad (3)$$

For $2 < M < 4$ and $a \neq 0$, there exists a single pair of Dirac-like cones in the first Brillouin zone (BZ) separated along the $k_w$ axis and located at $\boldsymbol{K}_\pm = (0, 0, 0, \pm \arccos k_m)$ with $k_m = M - 3$.

# DD invariant

$$\mathcal{DD} = \frac{1}{2\pi^2} \int_{\mathbb{S}^3} dq^j \wedge dq^k \wedge dq^l \sum_{n=1,2} \mathcal{H}^n_{jkl}, \tag{4}$$

where

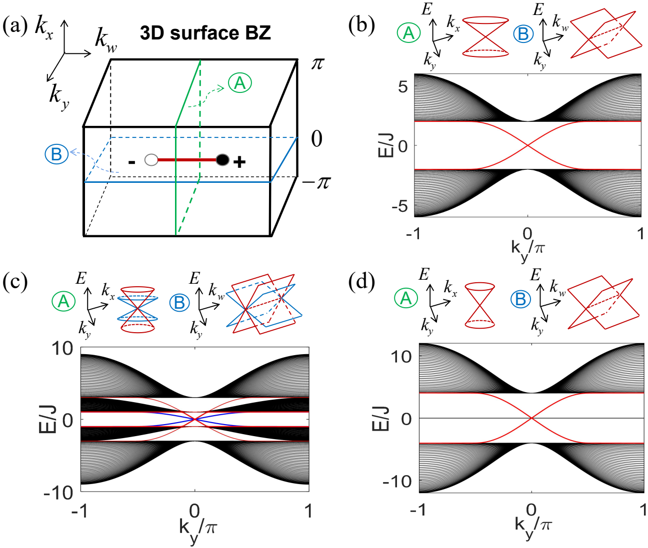
$$\mathcal{H}^n_{jkl} = \partial_j B^n_{kl} + \partial_k B^n_{lj} + \partial_l B^n_{jk}, \tag{5}$$

denotes the 3-form Berry curvature associated with the $n$-th eigenstate $|u_n(\boldsymbol{q})\rangle$. Only the two lowest bands $(n=1,2)$ contribute to the $\mathcal{DD}$ invariant, as required by the half-filling condition.

$$B^n_{jk} = \phi_n \mathcal{F}^n_{jk}, \;\; \phi_n = -\frac{i}{2} \log \prod_{\aleph=1}^{4} u^{\aleph}_{-,n}, \tag{6}$$

where $u^{\aleph}_n$ denotes the components of $|u_n\rangle$.

# 3D boundary states



we introduce the dipolar momentum $b_\mu$, which denotes the separation of the two monopoles in momentum space, with vector $\boldsymbol{b} = (\boldsymbol{K}_+ - \boldsymbol{K}_-)/2$, and in energy with offset $2b_t$.

$$H_{\text{eff}} = k_i \tilde{G}^i - b_\mu \tilde{G}_b^\mu, \tag{7}$$

where $i = x, y, z, w$ and $\mu = t, x, y, z, w$.
By implementing a Legendre transformation on Eq. (7), the action can be written in terms of a first-order Lagrangian,

$$S[\bar{\psi}, \psi, b] = \int d^5x \, \bar{\psi}(i\tilde{\gamma}^\mu \partial_\mu - \tilde{\gamma}_b^\mu b_\mu)\psi. \tag{8}$$

By integrating out the fermion field, we obtain the effective action

$$S_{\text{eff}} = -i \ln \det(i\tilde{\gamma}^\mu D_\mu - \tilde{\gamma}_b^\mu b_\mu), \tag{9}$$

where $D_\mu = \partial_\mu - iA_\mu$ is the gauge covariant derivative.

# Parity anomaly

This action can be regularized by employing the standard Pauli-Villars method which consists in introducing a mass term $\tilde{m}\bar{\psi}\psi$ with $\tilde{m} = m - \alpha k^2$.
We determine the effective Chern-Simons action, by calculating a one-loop triangle diagram

$$S_{\rm top} = \frac{C_2}{4\pi^2} \int d^5x\, \epsilon^{\mu\nu\lambda\rho\sigma} b_\mu \partial_\nu A_\lambda \partial_\rho A_\sigma, \tag{10}$$

where $C_2 = -\left[\mathrm{sgn}(m) + \mathrm{sgn}(\alpha)\right]/2$ is the second Chern number.

Our massless Hamiltonian in the Dirac case, supports a reflection symmetry along the $w$ direction, namely

$$U_r^{-1} H(k_x, k_y, k_z, k_w) U_r = H(k_x, k_y, k_z, -k_w), \tag{11}$$

with $U_r = \sigma_2 \otimes \sigma_0$, which is broken at quantum level by the Pauli-Villars mass regulator. This is the essence of the parity anomaly for $a = 0$ (similar situation with the spin-3/2-like fermions and the sublattice chiral symmetry).

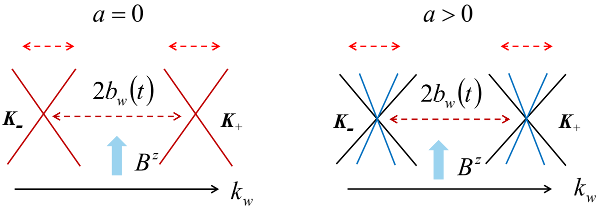
# Topological current

$$J^{\mu} = \frac{\delta S_{\text{top}}}{\delta A_{\mu}} = \frac{C_2}{2\pi^2}\epsilon^{\mu\nu\lambda\rho\sigma}\partial_{\nu} b_{\lambda}\partial_{\rho} A_{\sigma}. \qquad (12)$$

For simplicity, we consider the response of our system to a static and uniform magnetic field (i.e. $A_y = xB^z$ and $A_{x,z,w,t} = 0$), and to a simultaneous time-dependent modulation of the cones separation, $b_w = \arccos[M(t) - 3]$:

$$J^z = \frac{C_2}{2\pi^2}(\partial_t b_w) B^z. \qquad (13)$$

This is the parity magnetic effect, where $b_w$ plays the role of an effective axial gauge field.



# Conclusions

- 4D topological semimetals protected by chiral symmetry.
- Monopole-to-monopole phase transition ($Z_2 \to Z$).
- Robust 3D edge states.
- 4D parity magnetic effect.

# Experimental Observation of Tensor Monopoles

The three-level model proposed in "G.P. and N. Goldman, PRL (2018)" has been recently implemented in superconducting qudits:

**Experimental Observation of Tensor Monopoles with a Superconducting Qudit**

Xinsheng Tan,[1,*] Dan-Wei Zhang,[2,3,†] Danyu Li,[1] Xiaopei Yang,[1] Shuqing Song,[1] Zhikun Han,[1] Yuqian Dong,[1] Dong Lan,[1] Hui Yan,[2,3] Shi-Liang Zhu,[2,3,‡] and Yang Yu[1,§]

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ArXiv:2006.11770.

They have measured the DD invariant from the quantum metric following "G.P. and N. Goldman, PRL (2018)".