A CATEGORIFICATION OF THE TUBE ALGEBRA

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Based around work in Arxiv: 2006.06536 w. Delcamp
Motivation

- Application of higher category theory to physics
- Classification of phases of matter beyond Landau-Ginzburg symmetry breaking
- Phenomenology of such systems
- Applications to quantum information/computation
Topological phase of matter

* Equivalence class of gapped, local quantum many-body systems.

* Equivalence relation $\Rightarrow$ Two systems in same topological phase if they share a common TQFT description of far infrared limit.
Physically:

* Example
  - Fractional quantum hall effect, Chern-Simons effective field theory

* Topological excitations
  - Anyons
  - Candidate physics for engineering fault-tolerant quantum computer. Topological protection from errors.
TQFT in a nutshell

Atiyah, TQFT is a symmetric monoidal functor

\[ Z : (n+1) \text{Cob} \to \text{Vect} \]
TQFT in a nutshell

Atiyah, TQFT is a symmetric monoidal functor

\[ Z : (n+1)\text{Cob} \rightarrow \text{Vect} \]

Essentially a set of rules

\[ Z : M^n \rightarrow H(M^n) \quad \text{state-space} \]

\[ Z : C : M^n \rightarrow M^n \rightarrow Z(C) : H(M^n) \rightarrow H(M^n) \]

\[ \text{depends only on diffeomorphism class} \].
TQFT in a nutshell

Pros: nice concise theory ✓
Cons: lots of data required ×
  lack of phenomenology ×

- only tells us $H[M^3]$ as rep of $\text{MCG}(M^3)$
- nothing about excitations
- real materials have boundaries, only tells us about closed spatial materials
- not physically realisable TQFT ×
  and physically realisable TPM ×
- not necessarily local .
TQFT in a nutshell

# Extended TQFT - addresses some of problems

\[ Z(M^{n+1}) \in \mathbb{C} \]
\[ Z(M^n) \in \text{Vect} \]
\[ Z(M^{n-1}) \in 2\text{Vect} \]
\[ \vdots \]
\[ Z(*) \in n\text{Vect} \]

*provides notion of locality
TQFT in a nutshell

*Extended TQFT* - addresses some of problems

\[ Z(M^{n+1}) \in \mathbb{C} \]
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\[ Z(M^{n-1}) \in 2\text{Vect} \]
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- cobordism hypothesis
  - can reconstruct rest of $g$ theory from point
TQFT in a nutshell

*Extended TQFT* - addresses some of problems

\[ Z(M^{n+1}) \in \mathbb{C} \]
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\[ Z(*) \in n\text{Vect} \]

However: What do we assign to lower dim manifolds?

Cobordism hypothesis

Can reconstruct rest of theory from point
HAMILTONIAN MODELS OF TPM
local, lattice Hamiltonian schema
clearly pick spatial dimension $n$, then
for all triangulated $n$-manifolds $M^\Delta_n$

1) $i_s: M^\Delta_n \rightarrow \mathbb{H}^3 \quad \text{set of classical field configurations}$

2) $H = \sum_{\Delta \in \text{Tri}(M^\Delta_n)} H_{\Delta}$ \quad $\text{Hamiltonian}$

where $H: \text{Span}_\mathbb{C} \times s: M^\Delta_n \rightarrow \mathbb{H}^3 \rightarrow \text{Span}_\mathbb{C} \times s: M^\Delta_n \rightarrow \mathbb{H}^3 \equiv V[M^\Delta_n]$

$H_{\Delta}: \mathbb{H}^3 \rightarrow \mathbb{C} \sum_{g} \phi(s, \ldots) \quad \phi$ \quad phase factor

- changes field configuration in local neighbourhood of $g$ a vertex
Models for TPM

*Given exactly solvable LLHS

if \( A M^n \) w. triangulations \( M^\wedge_A, M^\sim_A \) s.t. \( \partial M^\wedge_A = \partial M^\sim_A \)

\[ \exists \text{ unitary isomorphism s.t. } \Pi_{\Delta_0} U = U \circ \Pi_{\Delta_0} \text{ and } \Delta_0 \in \text{Int}(M^\wedge_A) \]

\[ H[M^\wedge_A] \stackrel{U}{\longrightarrow} H[M^\sim_A] \stackrel{U}{\longrightarrow} H[M^\wedge_A] = H[M^\wedge_A] \stackrel{U \circ U}{\longrightarrow} H[M^\wedge_A] \]

- we say we have a topological lattice model

* such models expected to capture infrared limit effective field theory of condensed matter lattice models eg. TPM.
State-Sum TQFT

Given data of topological lattice model we can construct state-sum TQFT

Roughly: ssTQFT computes TQFT on simplicial model of space-time

To find ssTQFT we use unitary isomorphisms $U$ to define partition on local balls of spacetime and glue to evaluate a full partition function.
Continuum theory

- Using colimit over all triangulations we can define $H(M^4)$ for closed $M^3$ via a colimit construction.
- Such construction defines continuum theory which can be lifted to Atiyah TQFT.
Folk lore: ssTQFT in 1-1-correspondence w. fully extended TQFT
Folk lore: ssTQFT in 1-1-correspondence w. fully extended TQFT

Assuming true =>

Question: What do our TLM/ssTQFT assign to lower dimensional manifolds

- How can we compute properties of lattice model from this vantage point?
TUBE ALGEBRAS
Tube algebras

So far given a TLM we showed how to define a Hilbert space $H[C_{M_0}]$ to a triangulated n-manifold.

- The idea of the tube algebra is to associate a "2-Hilbert space" to a triangulated $(n-1)$-manifold.

* In the following a (finite din) 2-Hilbert space := semisimple C-linear Abelian category.

(I won’t discuss category-od inner product structure but can be added!)
Tube algebra

\[ H[\mathcal{O}] \times H[\mathcal{O}] \to H[\mathcal{O}] \to H[\mathcal{O}] \]

\[ \ast \left( \begin{array}{c} h \\ \circ \end{array} \right) \to \begin{array}{c} g \\ \uparrow \end{array} = \begin{array}{c} g \\ \downarrow \end{array} \quad (\ast \text{- structure}) \]
Tube algebras

Let $N_\Delta$ be a triangulated $(n,1)$-manifold and $\Delta N_\Delta$ a triangulation of $N \times I$.

$N \times I = \underbrace{N \times I}_{\sim}$

$(\eta,i) \sim (\eta,j)$ \quad $\forall (\eta,i),(\eta,j) \in \partial N \times I$

**Example:** $I \times I = \begin{pmatrix} (0,0) & (1,0) \\ (0,1) & (1,1) \end{pmatrix} = \bigcirc$

**Example:** $S^1 \times I = S^1 \times I = \text{cylinder}$
Tube algebras

ΔNΔ U ΔNΔ \rightarrow ΔNΔ U NΔ ΔNΔ \cong ΔNΔ

NΔ = I

NΔ = S
Tube algebra

Given the tube algebra on $\mathcal{H}[\Lambda_N^d]$ we define semisimple Abelian category $\text{Mod}(N_\Lambda)$ as category of $\mathcal{H}[\Lambda_N^d]$ modules.

$\text{Mod}(N_\Lambda)$ = 2-Hilbert space we associate to triangulated (n-1)-manifold $N_\Lambda$.
Given the tube algebra on $\mathcal{H}[\Delta N_a]$ we define the semisimple Abelian category $\text{Mod}(N_a)$ as the category of $\mathcal{H}[\Delta N_a]$ modules.

$\text{Mod}(N_a) = \mathcal{H}$-Hilbert space we associate to triangulated $(n, 1)$-manifold $N_a$.

*Importantly* $\text{Mod}(N_a) \cong \text{Mod}(N_{a'})$ (equivalence of ss. Abelian categories) for all $N_a, N_{a'}$ st. $\partial N_a = \partial N_{a'}$.

- this follows from Morita equivalence of $\mathcal{H}[\Delta N_a]$ and $\mathcal{H}[\Delta N_{a'}]$. 

Tube algebra
Morita equivalence:

two equivalent definitions: two algebras $A, B$ are Morita equivalent if

1) there exists an $A$-$B$-bimodule $AQB$ and a $B$-$A$-bimodule $BPA$

$s.t.\ AQB \otimes_B BPA \cong A$

$BPA \otimes_A AQB \cong B$ isomorphic as $A$-$A$-bimodules
$B$-$B$-bimodules

2) $\text{Mod}(A)$ is equivalent to $\text{Mod}(B)$
Morita equivalence:

two equivalent definitions: two algebras $A, B$ are Morita equivalent if

1) there exists an $A$-$B$-bimodule $AQB$ and a $B$-$A$-bimodule $BPA$
\
s.t. $AQB \otimes_B BPA \cong A$
\
$BPA \otimes_A AQB \cong B$ — isomorphic as $A$-$A$-bimodules
\
$B$-$B$-bimodules

2) $\text{Mod}(A)$ is equivalent to $\text{Mod}(B)$

To see $H[\triangleangledown N_{\lambda}]$ is Morita to $H[\triangleangledown N_{\lambda}]$ we make following observations:

* $H[\bigtriangleup]\bigtriangleup[\bigtriangleup]$ defines a right $H[\bigtriangleup]\bigtriangleup[\bigtriangleup]$ and left $H[\bigtriangleup]\bigtriangleup[\bigtriangleup]$ module!

* $H[\bigtriangleup]\bigtriangleup[\bigtriangleup]$
\
$\otimes H[\bigtriangleup]\bigtriangleup[\bigtriangleup] \cong H[\bigtriangleup]\bigtriangleup[\bigtriangleup]$ as $H[\bigtriangleup]\bigtriangleup[\bigtriangleup]$ bimodules and similarly in other direction!
Crossing with circle

defn: the dimension of a category $\equiv \text{Nat}(\text{id}, \text{id})$

facts: * for an algebra $A$ $\dim[\text{Mod}(A)] \cong Z(A)$ as commutative algebras
  * if $A$ is Morita $B$ $Z(A) \cong Z(B)$

$$\dim \text{Mod}(N_A) \equiv \dim \text{Mod}(N_{A'}) \cong \mathcal{H}(N_A \times S')$$

& isomorphism of Hilbert spaces

for all closed $(n.1)$-manifolds $N$

$$\mathcal{H}[\begin{array}{c} \infty \\ \infty \end{array}] \subset \mathcal{H}[\begin{array}{c} \infty \\ \infty \end{array}]$$

identify boundary + $H$ operators on "gluing seam"
CATEGORIZED TUBE ALGEBRAS
(and some physics... )
Given a monoidal category $C$ an algebra $(A,p)$ is an object $A \in C^0$ and morphism $p : A \otimes A \rightarrow A$ s.t. following commutes

$$(A \otimes A) \otimes A \xrightarrow{p \otimes A} A \otimes A \xrightarrow{p} A$$

$A \otimes (A \otimes A) \xrightarrow{A \otimes p} A \otimes A \xrightarrow{p} A$$
2-Algebra

Given a monoidal bicategory $B$ a 2-algebra $(A, p, Q)$ is an object $A \in B$, a morphism $p : A \otimes A \to A$ and 2-morphism

$$(A \otimes A) \otimes A \xrightarrow{p \otimes A} A \otimes A \xrightarrow{p} A$$

$satisfying some coherence data...$
2-Algebra

Given a monoidal bicategory $B$ a 2-algebra $(A, p, Q)$ is an object $A \in B$, a morphism $p : A \otimes A \rightarrow A$ and 2-morphism

\[
\begin{array}{ccc}
(A \otimes A) & \xrightarrow{p \otimes A} & A \otimes A \\
\downarrow & & \downarrow p \\
A \otimes (A \otimes A) & \xrightarrow{Q} & A
\end{array}
\]

satisfying some coherence data...

* example in $2Vect$ (Bicategory of Vect-Module categories) are tensor categories

* semisimple 2-algebras in $2Vect$ are multifusion categories

see e.g. EGNO, Douglas+Reutter 1812.11933
Categorified tube algebras

* semisimple 2-algebra in 2Vect, multifusion categories.

* Let $O_\Delta$ be a closed triangulated $(n-2)$-monoidal eg $n = 2$

* Can define $\text{Mod}(O_\Delta \times I)$ eg $\text{Mod}(\cdot \otimes \cdot)$ $n=2$

* want to define linear monoidal structure

\[ \boxtimes : \text{Mod}(O_\Delta \times I) \boxtimes \text{Mod}(O_\Delta \times I) \to \text{Mod}(O_\Delta \times I) \]

\[ \Rightarrow \text{Categorified tube algebra for } O_\Delta \]
Categorified tube algebra for $x$ in 2+1D

$\text{Mod}(\rightarrow) \boxtimes \text{Mod}(\rightarrow) \rightarrow \text{Mod}(\rightarrow)$

$\otimes_{s,s'} \tilde{S}_{s,s}' \otimes_{M_1 M_2} \mathcal{H}^* [\langle \rangle] \rightarrow \circ_{\mathcal{H}}$

* action on morphisms similarly defined
Categorified tube algebra for $*$ in $2+1D$

$\text{Mod}(\rightarrow) \boxtimes \text{Mod}(\rightarrow) \rightarrow \text{Mod}(\rightarrow)$

$\otimes_s x \otimes_t \rightarrow \bar{S}_{st}$

$\text{H}^*[\langle \rangle]$?

Induced from Mod $\text{Cat}^N$.

$\Rightarrow \text{Mod}(\rightarrow) \rightarrow \otimes_s \otimes_t$.

- Action on morphisms similarly defined
- Now we have a natural choice for associator

triangulation change defines module intertwiner. Triangulation of partition function guarantees solution to pentagon equation.

*unit + dualisability are consequences of existence of 2-inner product
Examples

$\Omega_\Delta = *$ in 2+1D THT GT with $G = (\mathbb{C} : E \to G, \triangleright)$

let $\tilde{G}$ denote corresponding monoidal groupoid

$[\tilde{G}, \text{Vect}]^\otimes = \text{multifusion cat of monoidal functors + monoidal nat trans}$

$\Omega_\Delta = *$ in 2+1D TGT theory $\cong \text{Vect}_G^\otimes$, multifusion cat of $G$-graded vector spaces

Now we define

$\text{MOD}(\Omega_\Delta) = \text{bicategory of } \text{Mod}^\otimes(\Delta_0 \Delta)$ - module categories, nat-trans, modifications

$\cong$ 3-Hilbert space assigned to $\mathcal{N}_\Delta$

$\Rightarrow \text{MOD}(\star)$ in 2+1D THT GT $\cong 2\text{Rep}(G)$
and some physics...

- How can we interpret $\text{MOD}(\#)$?

$\text{ob}$ morphism

2-morphisms define fusion
Defn: Dimension of bicategory \(\equiv\) braided monoidal category of pseudo-natural transformations of identity bimodule

\[
\dim \text{MOD}(O_A) \equiv \text{Mod}^{\otimes, \otimes}(O_A \times S') \cong \mathbb{Z}[\text{Mod}^{\otimes} (O_A \times I)]
\]

\[
\dim \text{MOD}(\times) \equiv \mathbb{Z}[\text{Mod}(\otimes) \otimes] \leftarrow \text{Drinfeld center}
\]

\(\Rightarrow\) algebraic data of anyons w. fusion + braiding
and some more physics...

In 3+1D $\text{MOD}(S')$

[for DW theory see 2006.06536 w. Donnelly]
and some more physics... 

In 3+1D $\text{MOD}(S')$

$\dim[\text{MOD}(S')] \cong \text{Mod}(S' \times S')$ and describes closed loop excitations with braiding and fusion!

(coming soon w. Delcamp)
Thanks for listening!
Algebras and chunks of space

\[ \mathcal{B} \simeq \mathcal{B} \]

\[ \mathcal{P}_{\mathcal{B}} \subseteq \mathcal{H} \left[ \begin{array}{c} \mathcal{B} \end{array} \right] \]
Modules and chunks of space

$\text{(M} \otimes \text{B}) \otimes \text{B} \xrightarrow{\alpha_{M,B,B}} \text{M} \otimes (\text{B} \otimes \text{B})$
Bimodules and chunks of space

\[ A \otimes \left[ A \otimes (M \otimes B) \right] \otimes B \]

\[ \downarrow \quad A \otimes \phi_A, M \otimes B, B \]

\[ A \otimes \left( A \otimes \left[ (M \otimes B) \otimes B \right] \right) \]

\[ \downarrow \quad A \otimes (A \otimes \phi_M, B, B) \]

\[ A \otimes \left( A \otimes \left[ M \otimes (B \otimes B) \right] \right) \]

\[ \downarrow \quad A \otimes (A \otimes (M \otimes P_B)) \]

\[ A \otimes (A \otimes (M \otimes B)) \]

\[ \phi_A, A, M \otimes B \]

\[ (A \otimes A) \otimes (M \otimes B) \]

\[ m_A \otimes (M \otimes B) \]

\[ A \otimes (M \otimes B) \]
Bimodules and chunks of space

\[(A \otimes M) \otimes B \rightarrow M \otimes B\]

\[\alpha_{A,M,B}\]

\[A \otimes (M \otimes B) \rightarrow A \otimes M\]

\[A \otimes M_B\]
- boundary tube algebra

\[ A \rightarrow M \rightarrow B \rightarrow A \]

\[ P_A \rightarrow M \rightarrow P_B \]

- described renormalization properties for boundary excitations. Rep \(\rightarrow\) excite