Motivation

* Application of higher category theory to physics
  - Classification of phases of matter beyond Landau-Ginzburg symmetry breaking
  - phenomenology of such systems
  - Applications to quantum information/computation

Topological phase of matter

* Equivalence class of gapped, local quantum many-body systems.

* Equivalence relation => Two systems in same topological phase if they share a common TQFT description of ter infra-red limit.

Physically:

* Example
  - Fractional quantum hall effect, Chern-Simons effective field theory

* Topological excitations
  - Anyons
  - Candidate physics for engineering fault-tolerant quantum computer. Topological protection from errors.

TQFT in a nutshell

Atiyah, TQFT is a symmetric monoidal functor

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Essentially a set of rules

$$Z : M^n \mapsto \mathcal{H}[M^n] \quad \triangleq \text{ state-space}$$

$$Z : C : M^n \to M'^n \mapsto Z(C) : \mathcal{H}[M^n] \to \mathcal{H}[M'^n]$$

$\triangleq$ depends only on diffeomorphism class.

TQFT in a nutshell

Pros: nice concise defn ✓

Cons: lots of data required ✗

lack of phenomenology ✗

- only tells us $\mathcal{H}[M^n]$ as rep of $MCG(M^n)$
- nothing about excitations
- real materials have boundaries, only tells us about closed spatial materials

No 1-1 between TQFT and physically realisable TPM ✗

- not necessarily local.

TQFT in a nutshell

* Extended TQFT - addresses some of problems

$Z(M^{n+1}) \in \mathbb{C}$

$Z(M^{n}) \in \mathrm{Vect}$

$Z(M^{n-1}) \in 2\mathrm{Vect}$

$\vdots$

$Z(*) \in n\mathrm{Vect}$

↑ provides notion of locality

Idea: swap topological complexity for algebraic data

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However: What do we assign to lower dim manifolds

cobordism hypothesis

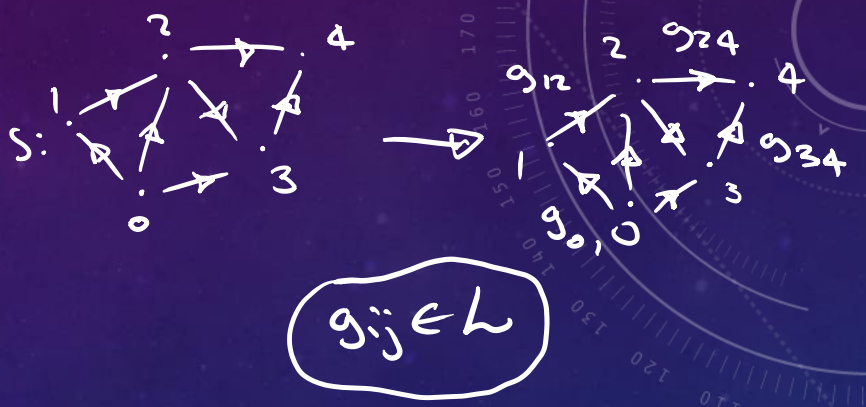
$\cong$ can reconstruct rest of theory from point

HAMILTONIAN MODELS OF TPM

Local, lattice Hamiltonian scheme

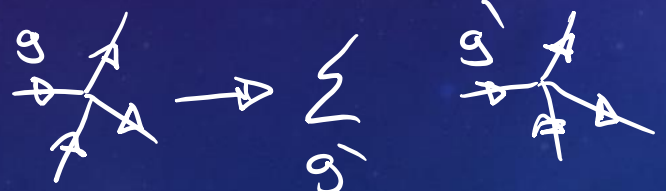
defn: pick spatial dimension n, then for all triangulated n-manifolds M^n_Δ

1) $\{ s : M^n_\Delta \to h \}$ ← set of classical field configurations



2) $H = \sum_{\Delta^0 \subseteq Int(M^n_\Delta)} H_{\Delta^0}$ ← Hamiltonian

where $H : Span_{\mathbb{C}} \{ s : M^n_\Delta \to h \} \longrightarrow Span_{\mathbb{C}} \{ s : M^n_\Delta \to h \} \equiv V[M^n_\Delta]$

$H_{\Delta_i} :$  $\phi(g', \ldots)$ ← phase factor!

– changes field configuration in local neighbourhood of a vertex

Exactly solvable LLHS

$$V[M^{\wedge}_{\Delta}] := \mathrm{Span}_{\mathbb{C}}\{ s : M^{n}_{\Delta} \to \mathbb{L} \} \quad , \quad H = \sum_{\Delta^{\circ} \subseteq \mathrm{Int}(M^{n}_{\Delta})} H_{\Delta^{\circ}}$$

- Exactly solvable $\Rightarrow$ $H_{\Delta^{\circ}} \cdot H_{\Delta^{\circ}} = H_{\Delta^{\circ}}$ ← projector

$$[H_{\Delta^{\circ}}, H_{\Delta'^{\circ}}] = 0 \quad \forall \Delta^{\circ} \subseteq M^{\wedge}_{\Delta}$$

← mutually commuting projectors!

- ground state subspace $\Rightarrow \mathcal{H}[M^{n}_{\Delta}] \equiv \mathrm{Im}\left[\prod_{\Delta^{\circ}} H_{\Delta^{\circ}}\right]$

Models for TPM

* Given exactly solvable LLHS

if $\forall M^n$ w. triangulations M^n_Δ, M'^n_Δ s.t. $\partial M^n_\Delta = \partial M'^n_\Delta$

$\exists$ unitary isomorphism s.t. $\prod_{\Delta^0 \subseteq Int(M^n_\Delta)} H_{\Delta^0} \circ U = U \circ \prod_{\Delta^0 \subseteq Int(M'^n_\Delta)} H_{\Delta^0}$ and

$$\mathcal{H}[M^n_\Delta] \xrightarrow{U} \mathcal{H}[M'^n_\Delta] \xrightarrow{U'} \mathcal{H}[M''^n_\Delta] = \mathcal{H}[M^n_\Delta] \xrightarrow{U' \circ U} \mathcal{H}[M''^n_\Delta]$$

- we say we have a topological lattice model

* such models expected to capture infra-red limit effective field theory of condensed matter lattice models eg. TPM.

State-Sum TQFT

Given data of topological lattice model we can construct state-sum TQFT

Roughly: ssTQFT computes TQFT on simplicial model of space-time

To find ssTQFT we use unitary isomorphisms U to define partition on local balls of spacetime and glue to evaluate a full partition function.

Continuum theory

- Using colimit over all triangulations we can define $\mathcal{H}[\hat{M}]$ for closed M^n via a colimit construction
- Such construction defines continuum theory which can be lifted to Atiyah TQFT.

Folklore: ssTQFT in 1-1-correspondence w. fully extended TQFT

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Assuming true $\Rightarrow$

Question : What do our TLM / ssTQFT assign to lower dimensional manifolds

: How can we compute properties of lattice model from this Vantage point ?

TUBE ALGEBRAS

Tube algebras:

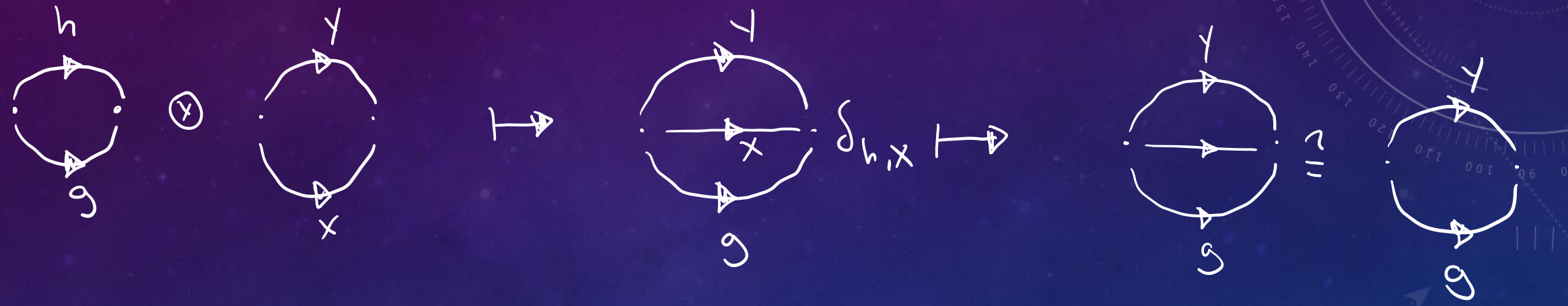
So far given a TLM we showed how to define a Hilbert space $\mathcal{H}[M_\Delta]$ to a triangulated n-manifold

- the idea of the tube algebra is to associate a "2-Hilbert space" to a triangulated (n-1)-manifold.

* in the following a (finite dim) 2-Hilbert space := semisimple $\mathbb{C}$-linear Abelian category.
  (I won't discuss categorified inner product structure but can be added!)

Tube algebra

$$H[\bigcirc] \otimes H[\bigcirc] \hookrightarrow V[\ominus] \hookrightarrow H[\ominus] \cong H[\bigcirc]$$



(* - structure)