Mirror symmetry for Painlevé surfaces

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Painlevé equations

The Painlevé equations are second-order differential equations whose only movable singularities are poles

$$\begin{tikzcd}
 & & P_{III}^{(D_6)} \arrow[r, dash] \arrow[dl, dash] \arrow[d, equal] \arrow[dr, dash] & (P_{III}^{(D_7)} \arrow[r, dash] \arrow[dl, dash] \arrow[dr, dash] & P_{III}^{(D_8)}) \\
P_{VI} \arrow[r, dash] & P_V \arrow[r, dash] \arrow[dr, dash] & P_V^{deg} \arrow[dr, dash] & P_{II} \arrow[r, dash] \arrow[dl, dash] \arrow[d, equal] & P_I \arrow[dl, dash] \\
 & & P_{IV} \arrow[r, dash] & P_{II}^{FN} & \\
4 & 3 & 2 & 1 & 0
\end{tikzcd}$$

Their solutions provide examples of special functions beyond abelian integrals.

Hamiltonian description

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$$q' = -\frac{\partial H}{\partial p} = 2p \quad p' = \frac{\partial H}{\partial q} = 3q^2 + t$$

After a rational (P_I, P_{II}, P_{IV}), trigonometric (P_{III}, P_V) or elliptic (P_{VI}) transformation of the time variable, all have the form of a particle moving in a time-dependent potential.

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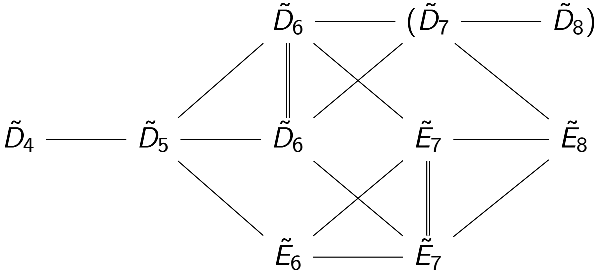
We need to compactify $\mathbb{C}^2$ to account for solutions with poles.

Spaces of initial conditions

The required compactification is the complement of an anti-canonical divisor of the projective plane blown-up in nine (infinitely near) points.
The canonical holomorphic symplectic form extends to have poles (with multiplicities) on the anti-canonical divisor



The Hamiltonian dynamics of the Painlevé equations can be recovered from (the) locally trivial deformation of the pair.

Limit to rational elliptic surfaces

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In the limit $\lambda \to 0$, we find a fast dynamics which limits to a flow around tori.

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Geometrically, the space of initial conditions limits to a rational elliptic surface by moving the ninth blow-up point so that all nine points lie on a pencil of cubic curves

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For $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, Painlevé VI is equivalent to the Schlesinger equations

$$\frac{\partial A_0}{\partial t} = \frac{[A_0, A_t]}{t} \quad \frac{\partial A_1}{\partial t} = \frac{[A_1, A_t]}{t-1} \quad \frac{\partial A_t}{\partial t} = -\left(\frac{[A_0, A_t]}{t} + \frac{[A_1, A_t]}{t-1}\right)$$

for $A_i = A_i(p, q, t)$ 2-by-2 traceless matrices.

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for $A_i = A_i(p, q, t)$ 2-by-2 traceless matrices.

Solutions of the Schlesinger equations define an isomonodromic family of flat connections on the trivial rank 2 bundle on $\mathbb{P}^1$

$$d + \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}$$

Isomonodromy for the other Painlevé equations

The remaining Painlevé equations have a similar interpretation as isomonodromic deformations of connections with higher order poles on the projective line.
The monodromy data must be taken to include Stokes data at the higher-order poles, and the complex structure to include local data at these points.

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Solutions to Painlevé I describe a family of connections on the projective line whose five Stokes matrices at the unique singular point at ∞ are constant

$$d + \begin{pmatrix} p & z^2 + qz + q + t \\ z - q & -p \end{pmatrix}$$

Character varieties

A moduli space of local systems with simple poles on the four-punctured sphere is a classical character variety

$$\mathrm{Hom}(\pi_1(\mathbb{P}^1 \backslash \{0, 1, t, \infty\}), \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2 \cong \mathrm{SL}_2^3//\mathrm{SL}_2$$

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Similar "wild" character varieties parameterise the monodromy data of connections with higher order poles, e.g. for Painlevé I

$$((\mathbb{P}^1)^5\backslash\Delta)//\mathrm{SL}(2,\mathbb{C})$$

Cubic surfaces

The character varieties are all families of affine cubic surfaces

$$XYZ = f_2(X, Y, Z; a, b, c, \dots)$$

For example, Painlevé VI gives the Fricke-Klein family of cubic surfaces

$$\begin{aligned} XYZ =& X^2 + Y^2 + Z^2 + (ab + cd)X + (ac + bd)Y + (ad + bc)Z \\ &+ (a^2 + b^2 + c^2 + d^2 + abcd - 4) \end{aligned}$$

The three coordinate functions X, Y and Z correspond to traces of loops around the three pants curves.
The coefficients a, b, c and d are given by traces around the four simple loops.

Compactification of Painlevé VI

The affine cubic surfaces admit a compactification by a triangle of lines, over which the holomorphic symplectic form extends with simple poles.
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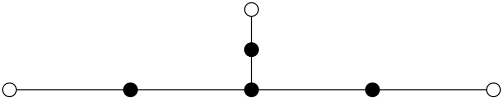
The orthogonal complement to the triangle of lines defines a D_4 sublattice of the $\tilde{E}_6$ Picard lattice.



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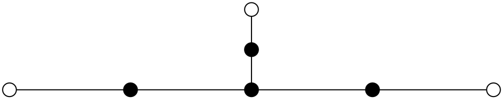

The $24 = 27 - 3$ lines in the interior represent partially reducible local systems, which correspond to special "truncated" solutions of P_{VI}.

Compactification of the other Painlevé varieties

In the remaining cases, some of the intersections of the triangle of lines at infinity meet at singular points of the cubic surface. These surfaces can be realised by blowing up the projective plane in six points in special position and blowing down effective (-2)-curves.

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These surfaces can be realised by blowing up the projective plane in six points in special position and blowing down effective (-2)-curves.
The orthogonal complement to these (-2)-classes defines a sublattice of the D_4 lattice.

$$
\begin{tikzcd}
 & & 2A_1 \arrow[dr, dash] & & \\
D_4 \arrow[r, dash] & A_3 \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & 2A_1 \arrow[dr, dash] & A_1 \arrow[r, dash] \arrow[dl, dash] & A_0 \\
 & & A_2 \arrow[r, dash] & A_1 \arrow[ur, dash] &
\end{tikzcd}
$$

Non-abelian Hodge and SYZ

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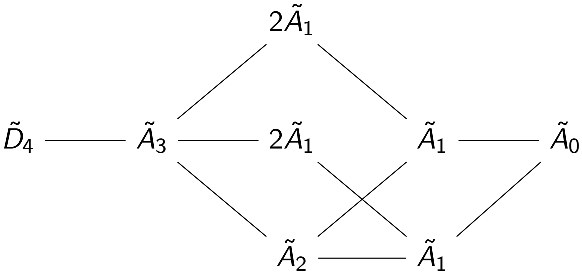
Via hyperkahler rotation, we may view the elliptic fibration as fibration by special Lagrangian tori, the input data for the SYZ picture of mirror symmetry. We can reconstruct the cubic surface from a scattering diagram drawn in the base of the fibration, together with its coordinate functions as theta functions.

Cohomological consequences

The middle cohomology of the complement of the singular fibre in the rational elliptic surfaces is represented by an affine Dynkin diagram, obtained as the orthogonal complement of the components of the singular fibre in the $\tilde{E}_8$ lattice



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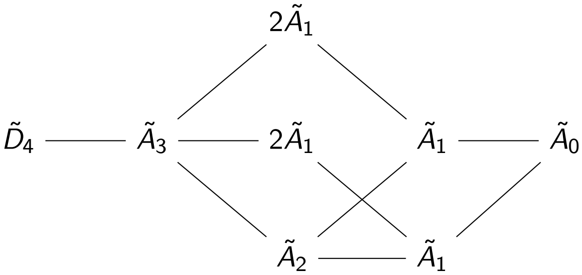

The complement to the class of a section is a sublattice isomorphic to the lattice corresponding to the respective finite Dynkin diagram.

Quivers

We can associate a mutation-equivalence class of quivers to each of the Painlevé equations.

$$
\begin{tikzcd}
 & & \tilde{A}_3 \arrow[r, dash] \arrow[dr, dash] & \tilde{A}_2 \arrow[r, dash] \arrow[dl, dash] \arrow[dr, dash] & \tilde{A}_1 \\
\tilde{\tilde{D}}_4 \arrow[r, dash] & \tilde{D}_4 \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \tilde{D}_3 \arrow[dr, dash] & A_3 \arrow[r, dash] \arrow[dl, dash] & A_2 \arrow[dl, dash] \\
 & & D_4 \arrow[r, dash] & D_3 &
\end{tikzcd}
$$

The mutation equivalence classes each contain a quiver of Dynkin, affine Dynkin or elliptic Dynkin type, in correspondence with the rational, trigonometric and elliptic types of the Painlevé equations.

Stability conditions

The bases of the elliptic fibrations have interpretations as a slice of the space of stability conditions of a Calabi-Yau-3 category associated to the quiver.

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The central charge is computed by integrating a meromorphic 1-form along loops in the fibres, whose exterior derivative is the holomorphic symplectic form.

$$Z(S) = \int_\alpha \sqrt{z^3 + cz + H}dz$$

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The scattering diagram from this perspective is understood through counts of stable objects of a a given phase, and the theta functions can be computed via counts of stable objects of a framed quiver.

Cluster varieties

The total spaces of the families of affine cubic surfaces are isomorphic in codimension two to the cluster $\mathcal{X}$-variety of the corresponding quiver.
The scattering diagram can be produced by an iterative process from the data of the quiver and an atlas of toric charts with certain birational maps as transition functions.

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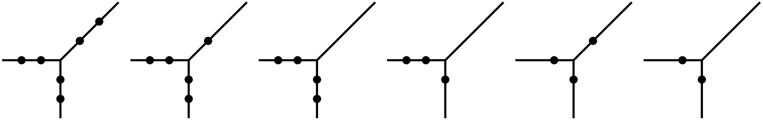
The natural functions on the character varieties can be written in any chart as Laurent polynomials in Fock-Goncharov coordinates, which are holonomies of $\mathbb{C}^*$-local systems on the fibres of the elliptic fibration.
It is expected that this abelianisation procedure provides an expression for the theta functions.

Scattering diagrams

Scattering diagrams for the Painlevé surfaces have (at most) three incoming rays, corresponding to the three lines at infinity.



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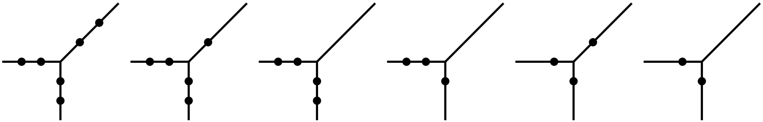
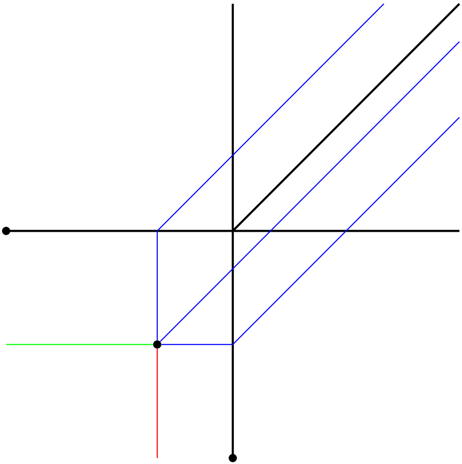


Can express the three theta functions as Laurent polynomials via counting broken lines (cf X, Y, Z in Fock-Goncharov coordinates)

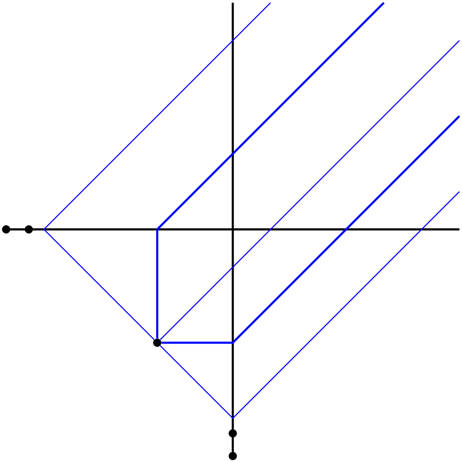
Can compute products of theta functions via counts of tropical curves (cf cubic equation satisfied by X, Y, Z)

Painlevé I



$$XYZ = X + Y + 1 \qquad X = x \quad Y = y \quad Z = \frac{1 + x + y}{xy}$$

Painlevé VI

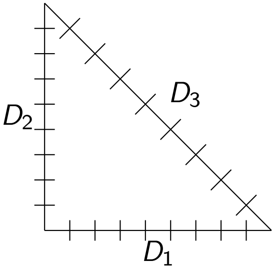


$$Z = \frac{y}{x} + \frac{2}{x} + \frac{1}{xy} + \frac{2}{y} + \frac{x}{y}$$

Theta functions for Painlevé VI

In "The mirror of the cubic surface" we find the equation

$$\vartheta_X \vartheta_Y \vartheta_Z = \vartheta_X^2 + \vartheta_Y^2 + \vartheta_Z^2 + \Big(\sum_{L \cap D_1} z^L\Big)\vartheta_X + \Big(\sum_{L \cap D_2} z^L\Big)\vartheta_Y + \Big(\sum_{L \cap D_3} z^L\Big)\vartheta_Z + \Big(\sum_{\alpha \in D_4} z^\alpha - 4\Big)$$



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The sums are over lines meeting a given component D_i of the boundary respectively roots of the D_4 lattice, which correspond bijectively with tropical curves.
After appropriate identifications, this recovers the Fricke-Klein family.
An analogous result holds for the remaining Painlevé surfaces.

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Thanks!