Sharp lower bounds for Neumann eigenvalues

Barbara Brandolini

Dipartimento di Matematica “R. Caccioppoli”
Università degli Studi di Napoli Federico II (Italy)

Lisbon WADE June 30, 2020

Neumann eigenvalue problem

$\Omega \subset \mathbb{R}^n$ bounded, Lipschitz domain, $p > 1$

$$\begin{cases} -\Delta_p u = \mu |u|^{p-2} u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

It is well-known that the first nontrivial eigenvalue can be variationally characterized as

$$\mu_{1,p}(\Omega) = \min \left\{ \frac{\displaystyle\int_\Omega |D\varphi|^p}{\displaystyle\int_\Omega |\varphi|^p} : \varphi \in W^{1,p}(\Omega) \setminus \{0\}, \int_\Omega |\varphi|^{p-2}\varphi = 0 \right\}$$

and $\mu_{1,p}(\Omega)^{1/p}$ coincides with the best constant in the Poincaré-Wirtinger inequality

$$C_{\Omega,p} \inf_{t \in \mathbb{R}} ||\varphi - t||_p \leq ||D\varphi||_p, \qquad \varphi \in W^{1,p}(\Omega).$$

The Szegő-Weinberger inequality (1954, 1956)

Since exact values of $\mu_{1,p}(\Omega)$ are known only for specific values of p and special domains Ω, it is natural to look for (sharp) estimates for $\mu_{1,p}(\Omega)$ in terms of (simple) geometric quantities such as measure, perimeter, diameter and so on.

Unfortunately, as is well-known, many difficulties arise in estimating $\mu_{1,p}(\Omega)$. One reason for this is the lack of monotonicity of eigenvalues with respect to set inclusion. Another is the fact that eigenfunctions corresponding to $\mu_{1,p}(\Omega)$ must change sign, and localizing the nodal line seems to be a hard problem.

The Szegő-Weinberger inequality (1954, 1956)

Since exact values of $\mu_{1,p}(\Omega)$ are known only for specific values of p and special domains Ω, it is natural to look for (sharp) estimates for $\mu_{1,p}(\Omega)$ in terms of (simple) geometric quantities such as measure, perimeter, diameter and so on.

Unfortunately, as is well-known, many difficulties arise in estimating $\mu_{1,p}(\Omega)$. One reason for this is the lack of monotonicity of eigenvalues with respect to set inclusion. Another is the fact that eigenfunctions corresponding to $\mu_{1,p}(\Omega)$ must change sign, and localizing the nodal line seems to be a hard problem.

The most celebrated example is the Szegő-Weinberger inequality ($p = 2$):

(SW) $$\mu_{1,2}(\Omega)|\Omega|^{2/n} \leq \mu_{1,2}(B)|B|^{2/n},$$

equality holding if and only if Ω is a n-dimensional ball B.

We recall that the proof of (SW) crucially exploits some peculiarities of the Laplacian, like linearity and the knowledge of the explicit form of eigenfunctions on balls. Its validity is still an open problem for $p \neq 2$.

A remark on the Szegő-Weinberger inequality

We recall that when $n = 2$, (SW) inequality can be sharpened. Namely, Weinberger noticed that Szegő's proof gives

$$(1) \qquad \frac{1}{|\Omega|}\left(\frac{1}{\mu_{1,2}(\Omega)} + \frac{1}{\mu_{2,2}(\Omega)}\right) \geq \frac{1}{|B|}\left(\frac{1}{\mu_{1,2}(B)} + \frac{1}{\mu_{2,2}(B)}\right)$$

for every simply connected domain, where B is any open disc and $\mu_{2,2}$ means the second nontrivial Neumann eigenvalue.
By recalling that for a disc $\mu_{1,2} = \mu_{2,2}$, (1) immediately implies (SW) for simply connected sets in $\mathbb{R}^2$.

A remark on the Szegő-Weinberger inequality

We recall that when $n = 2$, (SW) inequality can be sharpened. Namely, Weinberger noticed that Szegő's proof gives

$$\frac{1}{|\Omega|}\left(\frac{1}{\mu_{1,2}(\Omega)}+\frac{1}{\mu_{2,2}(\Omega)}\right) \geq \frac{1}{|B|}\left(\frac{1}{\mu_{1,2}(B)}+\frac{1}{\mu_{2,2}(B)}\right) \tag{1}$$

for every simply connected domain, where B is any open disc and $\mu_{2,2}$ means the second nontrivial Neumann eigenvalue.
By recalling that for a disc $\mu_{1,2} = \mu_{2,2}$, (1) immediately implies (SW) for simply connected sets in $\mathbb{R}^2$.

Remark

A quantitative improvement of (1) was made by Nadirashvilli in 1997 (see also [Brasco - de Philippis, 2016]). He proved that there exists a constant $C > 0$ such that for every smooth simply connected open set $\Omega \subset \mathbb{R}^2$ we have

$$\frac{1}{|\Omega|}\left(\frac{1}{\mu_{1,2}(\Omega)}+\frac{1}{\mu_{2,2}(\Omega)}\right) - \frac{1}{|B|}\left(\frac{1}{\mu_{1,2}(B)}+\frac{1}{\mu_{2,2}(B)}\right) \geq C\mathcal{A}(\Omega)$$

where $\mathcal{A}(\Omega)$ is the Fraenkel asymmetry of Ω.

A remark on the Szegő-Weinberger inequality

We recall that when $n = 2$, (SW) inequality can be sharpened. Namely, Weinberger noticed that Szegő's proof gives

$$(1) \qquad \frac{1}{|\Omega|}\left(\frac{1}{\mu_{1,2}(\Omega)} + \frac{1}{\mu_{2,2}(\Omega)}\right) \geq \frac{1}{|B|}\left(\frac{1}{\mu_{1,2}(B)} + \frac{1}{\mu_{2,2}(B)}\right)$$

for every simply connected domain, where B is any open disc and $\mu_{2,2}$ means the second nontrivial Neumann eigenvalue.

By recalling that for a disc $\mu_{1,2} = \mu_{2,2}$, (1) immediately implies (SW) for simply connected sets in $\mathbb{R}^2$.

Remark

Inequality (1) in turn can be sharpened. Indeed, Hersch and Monkewitz in 1971 have shown that there exists a constant $c > 0$ such that for every simply connected open set $\Omega \subset \mathbb{R}^2$ we have

$$\frac{1}{|\Omega|}\left(\frac{1}{\mu_{1,2}(\Omega)} + \frac{1}{\mu_{2,2}(\Omega)} + \frac{c}{\lambda_{1,2}(\Omega)}\right) \geq \frac{1}{|B|}\left(\frac{1}{\mu_{1,2}(B)} + \frac{1}{\mu_{2,2}(B)} + \frac{c}{\lambda_{1,2}(B)}\right).$$

A remark on the Szegő-Weinberger inequality

We recall that when $n = 2$, (SW) inequality can be sharpened. Namely, Weinberger noticed that Szegő's proof gives

$$(1) \qquad \frac{1}{|\Omega|}\left(\frac{1}{\mu_{1,2}(\Omega)} + \frac{1}{\mu_{2,2}(\Omega)}\right) \geq \frac{1}{|B|}\left(\frac{1}{\mu_{1,2}(B)} + \frac{1}{\mu_{2,2}(B)}\right)$$

for every simply connected domain, where B is any open disc and $\mu_{2,2}$ means the second nontrivial Neumann eigenvalue.
By recalling that for a disc $\mu_{1,2} = \mu_{2,2}$, (1) immediately implies (SW) for simply connected sets in $\mathbb{R}^2$.

Remark

For a general sharp quantitative version of (SW) we have to wait until 2012, when Brasco and Pratelli proved that, if $\Omega \subset \mathbb{R}^n$ is a connected, open set with Lipschitz boundary, then

$$\frac{\mu_{1,2}(B)|B|^{2/n} - \mu_{1,2}(\Omega)|\Omega|^{2/n}}{\mu_{1,2}(B)|B|^{2/n}} \geq c_n \mathcal{A}(\Omega)^2.$$

A remark on the Szegő-Weinberger inequality

We recall that when $n = 2$, (SW) inequality can be sharpened. Namely, Weinberger noticed that Szegő's proof gives

$$\frac{1}{|\Omega|}\left(\frac{1}{\mu_{1,2}(\Omega)}+\frac{1}{\mu_{2,2}(\Omega)}\right) \geq \frac{1}{|B|}\left(\frac{1}{\mu_{1,2}(B)}+\frac{1}{\mu_{2,2}(B)}\right) \tag{1}$$

for every simply connected domain, where B is any open disc and $\mu_{2,2}$ means the second nontrivial Neumann eigenvalue.

By recalling that for a disc $\mu_{1,2} = \mu_{2,2}$, (1) immediately implies (SW) for simply connected sets in $\mathbb{R}^2$.

Remark

The higher dimensional analogue of (1), conjectured by Ashbaugh-Benguria in 1993, would be

$$\frac{1}{|\Omega|^{2/n}} \sum_{k=1}^{n} \frac{1}{\mu_{k,2}(\Omega)} \geq \frac{1}{|B|^{2/n}} \sum_{k=1}^{n} \frac{1}{\mu_{k,2}(B)},$$

but its validity is still an open problem. Recently, in 2018, Wang and Xia proved

$$\frac{1}{|\Omega|^{2/n}} \sum_{k=1}^{n-1} \frac{1}{\mu_{k,2}(\Omega)} \geq \frac{1}{|B|^{2/n}} \sum_{k=1}^{n-1} \frac{1}{\mu_{k,2}(B)}.$$

The Payne-Weinberger inequality (1960)

$\Omega \subset \mathbb{R}^n$ convex, bounded domain, $p = 2$

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Downarrow$$

(PW) $$\mu_{1,2}(\Omega) \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2}$$

Remarks.

1. (PW) is sharp since $\mu_{1,2}(\Omega)\mathrm{diam}(\Omega)^2$ goes to π^2 for a parallelepiped all but one of whose dimensions shrink to zero.
2. The convexity assumption cannot be relaxed. It is enough to consider the classical example of a planar domain made by two equal squares connected by a thin corridor.

The Payne-Weinberger inequality (1960)

$\Omega \subset \mathbb{R}^n$ convex, bounded domain, $p = 2$

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Downarrow$$

(PW)
$$\mu_{1,2}(\Omega) \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2}$$

Remarks.

3. Generalizations of (PW) can be found, for example, in [Acosta - Duran 2003], [V. Ferone - Nitsch - Trombetti 2012], [Esposito - Nitsch - Trombetti 2013], [Valtorta 2012], [Esposito - Kawohl - Nitsch - Trombetti 2015], [Rossi - Saintier 2016], [Della Pietra - Gavitone - Piscitelli 2017].

In literature:

[Bandle, 1980]
[Chavel, 1980]
[Dacorogna - Gangbo - Subia, 1992]
[Kröger, 1992]
[Ashbaugh - Benguria, 1993]
[Avinyo - Mora 1997]
[Bakry - Qian, 2000]
[Croce - Dacorogna, 2003]
[Ricciardi, 2005]
[Chua - Wheeden, 2006]
[Laugesen - Siudeja, 2009-2010]
[Chiacchio - di Blasio, 2012]
[Enache - Philippin, 2013]
[B. - Chiacchio - Henrot - C. Trombetti, 2013]
[Brasco - Nitsch - C. Trombetti, 2016]
[Brock - Chiacchio - di Blasio, 2016]
[Burenkov - Gol'dshtein - Ukhlov, 2016]
[B. - Chiacchio - Krejcirik - C. Trombetti, 2016]
[Harrell-Stubbe, 2016]
[Henrot (Ed.) *Shape Optimization and Spectral Theory*, 2017]
[Brasco - de Philippis, 2017]
[B. - Chiacchio - Langford, 2017]
[Koerber, 2018]
[Gol'dshtein - Ukhlov, 2016, 2017, 2019]
[Benguria - B. - Chiacchio, 2020]
[Gol'dshtein - Pchelintsev - Ukhlov, 2020]
[Gold'shtein - Hurri Syrjänen - Pchelintsev - Ukhlov, 2020]

....

[B. - Chiacchio - C. Trombetti, 2009] [B. - Chiacchio - C. Trombetti, 2015] [B. - Chiacchio - Dryden - Langford, 2017] [B. - Chiacchio - Langford, in preparation]

A lower bound for $\mu_{1,p}$

We allow the set to be non-convex and in place of the diameter our estimate will involve $K_n(\Omega)$, the best isoperimetric constant relative to Ω, that is

$$K_n(\Omega) = \inf_{E \subset \Omega} \frac{P_\Omega(E)}{(\min\{|E|, |\Omega \setminus E|\})^{1-1/n}}.$$

Theorem

Let Ω be a bounded, Lipschitz domain of $\mathbb{R}^n$. Then

(2) $$\mu_{1,p}(\Omega) \geq 2^{p/n} \left(\frac{K_n(\Omega)}{K_n(\mathbb{R}^n)} \right)^p \lambda_{1,p}(\Omega^\sharp),$$

where $\lambda_{1,p}(\Omega^\sharp)$ is the first Dirichlet eigenvalue of $-\Delta_p$ in the ball $\Omega^\sharp$ having the same measure as Ω.
Furthermore (2) is sharp at least in the case $n = p = 2$.

$K_n(\mathbb{R}^n) = n\omega_n^{1/n}$ classical isoperimetric constant

$\omega_n = |B_1|$

A remark

Let $\Psi_p(r) > 0$ be a solution to the following Sturm-Liouville problem

$$\begin{cases} -(p-1)\left|\Psi_p'\right|^{p-2}\Psi_p'' - \frac{n-1}{r}\left|\Psi_p'\right|^{p-1} = \Psi_p^{p-1} & \text{in } (0,\psi_p) \\ \Psi_p'(0) = \Psi_p(\psi_p) = 0. \end{cases}$$

A remark

Let $\Psi_p(r) > 0$ be a solution to the following Sturm-Liouville problem

$$\begin{cases} -(p-1)\left|\Psi_p'\right|^{p-2}\Psi_p'' - \frac{n-1}{r}\left|\Psi_p'\right|^{p-1} = \Psi_p^{p-1} \quad \text{in } (0,\psi_p) \\ \Psi_p'(0) = \Psi_p(\psi_p) = 0. \end{cases}$$

Clearly, when $p = 2$, $\Psi_p(r)$ coincides with $r^{1-n/2}J_{n/2-1,1}$ and ψ_p is the first positive zero $j_{n/2-1,1}$ of the Bessel function of the first kind $J_{n/2-1,1}$.

A remark

Let $\Psi_p(r) > 0$ be a solution to the following Sturm-Liouville problem

$$\begin{cases} -(p-1)\left|\Psi_p'\right|^{p-2}\Psi_p'' - \frac{n-1}{r}\left|\Psi_p'\right|^{p-1} = \Psi_p^{p-1} \quad \text{in } (0,\psi_p) \\ \Psi_p'(0) = \Psi_p(\psi_p) = 0. \end{cases}$$

Estimate (2) can be rewritten as

$$\mu_{1,p}(\Omega) \geq 2^{p/n}\frac{K_n(\Omega)^p}{|\Omega|^{p/n}}\left(\frac{\psi_p}{n}\right)^p.$$

When $p = n = 2$:

$$\mu_{1,2}(\Omega) \geq 2\,\frac{K_2(\Omega)^2}{|\Omega|}\left(\frac{j_{0,1}}{2}\right)^2$$

$j_{0,1}$ first zero of the Bessel function of the first kind J_0

A special class of planar domains

If $\Omega \subset \mathbb{R}^2$ is convex and it is symmetric about a point, then (see [Cianchi '89])

$$K_2(\Omega)^2 = \frac{2w(\Omega)^2}{|\Omega|},$$

where $w(\Omega)$ stands for the width of Ω. In this case

$$\mu_{1,2}(\Omega) \geq j_{0,1}^2 \frac{w(\Omega)^2}{|\Omega|^2}$$

A special class of planar domains

If $\Omega \subset \mathbb{R}^2$ is convex and it is symmetric about a point, then (see [Cianchi '89])

$$K_2(\Omega)^2 = \frac{2w(\Omega)^2}{|\Omega|},$$

where $w(\Omega)$ stands for the width of Ω. In this case

$$\mu_{1,2}(\Omega) \geq j_{0,1}^2 \frac{w(\Omega)^2}{|\Omega|^2}$$

If Ω satisfies

$$|\Omega| < \frac{j_{0,1}}{\pi} w(\Omega) \mathrm{diam}(\Omega) \qquad \left(\frac{j_{0,1}}{\pi} \approx 0.7655\right),$$

we get

$$\mu_{1,2}(\Omega) \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2}\left(1 + \delta(\Omega)\right),$$

with $\delta(\Omega) = \left(\frac{j_{0,1} w(\Omega) \mathrm{diam}(\Omega)}{\pi |\Omega|}\right)^2 - 1 > 0$.

Example

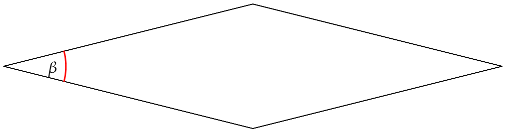
In the class of rhombi with side 1 and acute angle β, condition

$$|\Omega| < \frac{j_{0,1}}{\pi} w(\Omega)\mathrm{diam}(\Omega) \qquad \left(\frac{j_{0,1}}{\pi} \approx 0.7655\right),$$

is satisfied if

$$\cos\frac{\beta}{2} > \frac{\pi}{2j_{0,1}} \approx 0.6532.$$

This last condition is always fulfilled since $\frac{\beta}{2} < \frac{\pi}{4}$.



$\mu_{1,p}(\Omega) \geq 2^{p/n} \frac{K_n(\Omega)^p}{|\Omega|^{p/n}} \left(\frac{\psi_p}{n}\right)^p$: main steps of the proof

Let u_1 be an eigenfunction corresponding to $\mu_{1,p}(\Omega)$ such that

$$|\Omega^+| = \left|\text{supp}\left(u_1^+\right)\right| \leq \frac{|\Omega|}{2}.$$

1. We first prove a reverse Hölder inequality for u_1^+. Namely, we show that

   $$||u_1^+||_{L^q(\Omega)} \leq C||u_1^+||_{L^r(\Omega)}, \qquad 0 < r < q < +\infty,$$

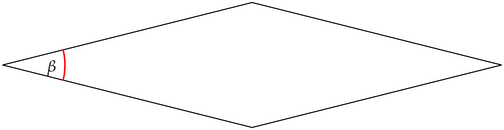
   where $C = C\left(n, p, q, r, K_n(\Omega), \mu_{1,p}(\Omega)\right) > 0$ is explicitly given.

2. From the above inequality we deduce the lower bound for $\mu_{1,p}$.

3. Finally we prove that such a bound is sharp, for $n = p = 2$, by considering a sequence of rhombi R_{β_m} having side 1 and acute angle $\beta_m = \frac{2\pi}{m}$ $(m > 4)$.

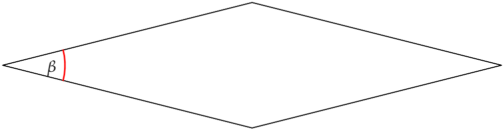
Step 3: optimality for n=p=2

Consider a rhombus R_β having side 1 and acute angle β



Step 3: optimality for n=p=2

Consider a rhombus R_β having side 1 and acute angle β

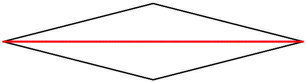


$\mu_{1,2}(R_\beta)$ is simple [Banuelos - Burdzy '99, Jerison-Nadirashvili '00, Atar - Burdzy '04]

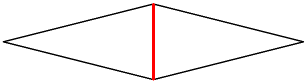
$+$

The nodal line cannot enclose a subdomain

$\Longrightarrow$ The nodal line of u_1 has just two possibilities



Case 1



Case 2

The nodal line of u_1

Proposition

Case 2 occurs: the nodal line of u_1 is the shortest diagonal for each $\beta \in (0, \tilde{\beta})$, where $\tilde{\beta}$ is the unique zero in $(0, \pi/2)$ of the function $g(\beta) = \frac{\sin\beta}{\cos^2(\beta/2)} + \beta - \pi$ $\left(\tilde{\beta} \simeq 1.4209\right)$.

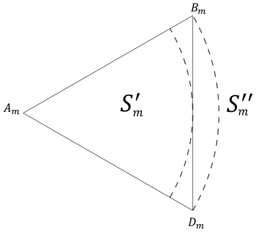
The nodal line of u_1

Proposition

Case 2 occurs: the nodal line of u_1 is the shortest diagonal for each $\beta \in (0, \tilde{\beta})$, where $\tilde{\beta}$ is the unique zero in $(0, \pi/2)$ of the function $g(\beta) = \frac{\sin\beta}{\cos^2(\beta/2)} + \beta - \pi$ $\left(\tilde{\beta} \simeq 1.4209\right)$.

We consider a sequence of rhombi with acute angles $\beta_m = \frac{2\pi}{m}$ $(m > 4)$ and side 1. Then

$$\mu_{1,2}(R_{\beta_m}) = \lambda_{1,2}^{DN}(T_{\beta_m})$$



Step 3: optimality for n=p=2

$$j_{0,1}^2 = \lambda_{1,2}^{DN}(S_m^{''}) \leq \lambda_{1,2}^{DN}(T_{\beta_m}) \leq \lambda_{1,2}^{DN}(S_m^{'}) = \frac{j_{0,1}^2}{\cos^2\left(\frac{\beta_m}{2}\right)}$$

Step 3: optimality for n=p=2

$$j_{0,1}^2 = \lambda_{1,2}^{DN}(S_m^{''}) \leq \lambda_{1,2}^{DN}(T_{\beta_m}) \leq \lambda_{1,2}^{DN}(S_m^{'}) = \frac{j_{0,1}^2}{\cos^2\left(\frac{\beta_m}{2}\right)}$$

Hence

$$\lim_{m\to+\infty} \mu_{1,2}(R_{\beta_m}) = \lim_{m\to+\infty} \lambda_{1,2}^{DN}(T_{\beta_m}) = j_{0,1}^2.$$

On the other side our estimate reads as

$$\mu_{1,2}(R_{\beta_m}) \geq 2\left(\frac{K_2(R_{\beta_m})}{2\sqrt{\pi}}\right)^2 \lambda_{1,2}(R_{\beta_m}^\sharp) = 2\left(\frac{\sqrt{2\sin\beta_m}}{2\sqrt{\pi}}\right)^2 \frac{\pi j_{0,1}^2}{\sin\beta_m} = j_{0,1}^2.$$

Step 1: a reverse Hölder inequality

Proposition

Let u_1 be an eigenfunction corresponding to $\mu_{1,p}(\Omega)$ and $0 < r < q$. There exists a positive constant $C = C(n, p, q, r, \mu_{1,p}(\Omega), K_n(\Omega))$ such that

$$||u_1^+||_{L^q(\Omega)} \leq C||u_1^+||_{L^r(\Omega)}.$$

Actually

$$C = \frac{||v_1||_{L^q(B_R)}}{||v_1||_{L^r(B_R)}},$$

where v_1 is any eigenfunction of the following Dirichlet eigenvalue problem in B_R corresponding to $\lambda_{1,p}(B_R) = \left(\frac{n\omega_n^{1/n}}{K_n(\Omega)}\right)^p \mu_{1,p}(\Omega)$, i.e.

$$\begin{cases} -\Delta_p v_1 = \lambda_{1,p}(B_R) v_1 & \text{in } B_R \\ v_1 = 0 & \text{on } \partial B_R. \end{cases}$$

Reverse Hölder Inequalities imply estimates for Dirichlet eigenvalues

$$\begin{cases} -\Delta w_1 = \lambda_{1,2}(\Omega)\, w_1 & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega \end{cases}$$

When $q = 2$ and $r = 1$, [Payne-Rayner '73] $n = 2$

$$||w_1||_{L^2(\Omega)} \leq \sqrt{\frac{\lambda_{1,2}(\Omega)}{4\pi}}||w_1||_{L^1(\Omega)}$$

Reverse Hölder Inequalities imply estimates for Dirichlet eigenvalues

$$\begin{cases} -\Delta w_1 = \lambda_{1,2}(\Omega)\, w_1 & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega \end{cases}$$

When $q = 2$ and $r = 1$, [Payne-Rayner '73] $n = 2$

$$\frac{||w_1||_{L^1(\Omega)}}{|\Omega|^{1/2}} \leq ||w_1||_{L^2(\Omega)} \leq \sqrt{\frac{\lambda_{1,2}(\Omega)}{4\pi}}||w_1||_{L^1(\Omega)} \Rightarrow \lambda_{1,2}(\Omega) \geq \frac{4\pi}{|\Omega|}$$

Reverse Hölder Inequalities imply estimates for Dirichlet eigenvalues

$$\begin{cases} -\Delta w_1 = \lambda_{1,2}(\Omega)\, w_1 & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega \end{cases}$$

When $q = 2$ and $r = 1$, [Payne-Rayner '73] $n = 2$, [Kohler-Jobin '77] $n \geq 2$:

$$||w_1||_{L^2(\Omega)} \leq \frac{\lambda_{1,2}(\Omega)^{n/4}}{\sqrt{2n\omega_n}\, j_{n/2-1,1}^{n/2-1}} ||w_1||_{L^1(\Omega)} \Rightarrow \lambda_{1,2}(\Omega) \geq \left(\frac{2n\omega_n\, j_{n/2-1,1}^{n-2}}{|\Omega|} \right)^{2/n}$$

$j_{n/2-1,1}$ is the first positive zero of the Bessel function $J_{n/2-1,1}$

$\omega_n = |B_1|$

Reverse Hölder Inequalities imply estimates for Dirichlet eigenvalues

$$\begin{cases} -\Delta w_1 = \lambda_{1,2}(\Omega)\, w_1 & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta z_1 = \lambda_{1,2}(\Omega^\sharp)\, z_1 & \text{in } B_R \\ z_1 = 0 & \text{on } \partial B_R \end{cases}$$

When $q = 2$ and $r = 1$, [Payne-Rayner '73] $n = 2$, [Kohler-Jobin '77] $n \geq 2$:

$$||w_1||_{L^2(\Omega)} \leq ||w_1||_{L^1(\Omega)} \Rightarrow \lambda_{1,2}(\Omega) \geq \left(\frac{2n\omega_n j_{n/2-1,1}^{n-2}}{|\Omega|} \right)^{2/n}$$

When $n \geq 2$, if $q \to \infty$ and $r \to 0^+$ in the Chiti's inequality (1982)

$$||w_1||_{L^q(\Omega)} \leq \frac{||z_1||_{L^q(B_R)}}{||z_1||_{L^r(B_R)}} ||w_1||_{L^r(\Omega)}$$

we get

$$\lambda_{1,2}(\Omega) \geq \frac{\omega_n^{2/n} j_{n/2-1,1}^2}{|\Omega|^{2/n}} = \lambda_{1,2}(\Omega^\sharp)$$

$j_{n/2-1,1}$ is the first positive zero of the Bessel function $J_{n/2-1,1}$
B_R is the ball with the same first Dirichlet eigenvalue as Ω

$\omega_n = |B_1|$

Step 2: eigenvalue estimate

$$||u_1^+||_{L^q(\Omega)} \leq \frac{||v_1||_{L^q(B_R)}}{||v_1||_{L^r(B_R)}}||u_1^+||_{L^r(\Omega)} \quad \overset{\text{Höld. Ineq.}}{\Rightarrow} \quad \left|\Omega^+\right|^{\frac{1}{q}-\frac{1}{r}} \leq \frac{||v_1||_{L^q(B_R)}}{||v_1||_{L^r(B_R)}}$$

Step 2: eigenvalue estimate

$$||u_1^+||_{L^q(\Omega)} \leq \frac{||v_1||_{L^q(B_R)}}{||v_1||_{L^r(B_R)}} ||u_1^+||_{L^r(\Omega)} \quad \overset{\text{Höld. Ineq.}}{\Rightarrow} \quad \left|\Omega^+\right|^{\frac{1}{q}-\frac{1}{r}} \leq \frac{||v_1||_{L^q(B_R)}}{||v_1||_{L^r(B_R)}}$$

We choose

$$v_1(x) = \Psi_p \left(\left(\frac{\mu_{1,p}(\Omega)}{\alpha} \right)^{1/p} |x| \right),$$

where $\alpha = \left(\frac{K_n(\Omega)}{n\omega_n^{1/n}} \right)^p$ and Ψ_p is the solution to

$$\begin{cases} -(p-1)|\Psi_p'|^{p-2}\Psi_p'' - \frac{n-1}{r}|\Psi_p'|^{p-1} = \Psi_p^{p-1} & \text{in } (0, \psi_p) \\ \Psi_p'(0) = \Psi_p(\psi_p) = 0, \end{cases}$$

normalized in such a way that $\Psi_p(0) = 1$.

Step 2: eigenvalue estimate

$$\frac{||v_1||_{L^q(B_R)}}{||v_1||_{L^r(B_R)}} = \frac{\left(n\omega_n \int_0^R t^{n-1}\Psi_p\left(\left(\frac{\mu_{1,p}(\Omega)}{\alpha}\right)^{1/p} t\right)^q dt\right)^{1/q}}{\left(n\omega_n \int_0^R t^{n-1}\Psi_p\left(\left(\frac{\mu_{1,p}(\Omega)}{\alpha}\right)^{1/p} t\right)^r dt\right)^{1/r}}$$

$$= (n\omega_n)^{1/q-1/r}\left(\frac{\alpha}{\mu_{1,p}(\Omega)}\right)^{n/(pq)-n/(pr)} \frac{\left(\int_0^{\psi_p} t^{n-1}\Psi_p(t)^q dt\right)^{1/q}}{\left(\int_0^{\psi_p} t^{n-1}\Psi_p(t)^r dt\right)^{1/r}}$$

Step 2: eigenvalue estimate

$$\frac{||v_1||_{L^q(B_R)}}{||v_1||_{L^r(B_R)}} = \frac{\left(n\omega_n \int_0^R t^{n-1}\Psi_p\left(\left(\frac{\mu_{1,p}(\Omega)}{\alpha}\right)^{1/p} t\right)^q dt\right)^{1/q}}{\left(n\omega_n \int_0^R t^{n-1}\Psi_p\left(\left(\frac{\mu_{1,p}(\Omega)}{\alpha}\right)^{1/p} t\right)^r dt\right)^{1/r}}$$

$$= (n\omega_n)^{1/q-1/r}\left(\frac{\alpha}{\mu_{1,p}(\Omega)}\right)^{n/(pq)-n/(pr)} \frac{\left(\int_0^{\psi_p} t^{n-1}\Psi_p(t)^q dt\right)^{1/q}}{\left(\int_0^{\psi_p} t^{n-1}\Psi_p(t)^r dt\right)^{1/r}}$$

$+$

$$|\Omega^+|^{\frac{1}{q}-\frac{1}{r}} \leq \frac{||v_1||_{L^q(B_R)}}{||v_1||_{L^r(B_R)}}$$

Step 2: eigenvalue estimate

$$\Downarrow$$

$$\mu_{1,p}(\Omega) \geq \alpha \left(\frac{n\omega_n}{|\Omega^+|} \right)^{p/n} \frac{\left(\int_0^{\psi_p} t^{n-1} \Psi_p(t)^r dt \right)^{pq/n(q-r)}}{\left(\int_0^{\psi_p} t^{n-1} \Psi_p(t)^q dt \right)^{pr/n(q-r)}}$$

Step 2: eigenvalue estimate

$$\Downarrow$$

$$\mu_{1,p}(\Omega) \geq \alpha \left(\frac{n\omega_n}{|\Omega^+|} \right)^{p/n} \frac{\left(\int_0^{\psi_p} t^{n-1} \Psi_p(t)^r dt \right)^{pq/n(q-r)}}{\left(\int_0^{\psi_p} t^{n-1} \Psi_p(t)^q dt \right)^{pr/n(q-r)}}$$

Setting

$$f(s) = \left(\frac{\int_0^{\psi_p} t^{n-1} \Psi_p(t)^s dt}{\int_0^{\psi_p} t^{n-1} dt} \right)^{1/s} = \left(\frac{n}{\psi_p^n} \int_0^{\psi_p} t^{n-1} \Psi_p(t)^s dt \right)^{1/s},$$

and recalling that $|\Omega^+| \leq |\Omega|/2$, we get

$$\mu_{1,p}(\Omega) \geq \alpha \left(\frac{2\omega_n}{|\Omega|} \right)^{\frac{p}{n}} \left(\frac{f(r)}{f(q)} \right)^{\frac{pqr}{n(q-r)}} \psi_p^p.$$

Step 2: eigenvalue estimate

$$\mu_{1,p}(\Omega) \geq \alpha \left(\frac{2\omega_n}{|\Omega|}\right)^{\frac{p}{n}} \left(\frac{f(r)}{f(q)}\right)^{\frac{pqr}{n(q-r)}} \psi_p^p.$$

It is easy to check that

$$\sup_{0<r<q} \left(\frac{f(r)}{f(q)}\right)^{pqr/n(q-r)} = 1$$

$$\Downarrow$$

$$\mu_{1,p}(\Omega) \geq 2^{p/n} \alpha \frac{\psi_p^p}{\left(\frac{|\Omega|}{\omega_n}\right)^{p/n}}$$

$$\alpha = \left(\frac{K_n(\Omega)}{n\omega_n^{1/n}}\right)^p$$

Comparison with Avinyo - Mora estimates

For any $p \geq 2$, our estimate

$$\mu_{1,p}(\Omega) \geq 2^{p/n} \frac{K_n(\Omega)^p}{|\Omega|^{p/n}} \left(\frac{\psi_p}{n}\right)^p$$

improves the following bound contained in [Avinyo and Mora '98]

$$\mu_{1,p}(\Omega) \geq 2^{p/n} \frac{K_n(\Omega)^p}{|\Omega|^{p/n}} \left(\frac{n}{p(n-1)}\right)^p$$

To this aim it suffices to verify that

$$\psi_p > \frac{n^2}{p(n-1)}, \qquad p \geq 2, \quad n \geq 2.$$

Indeed

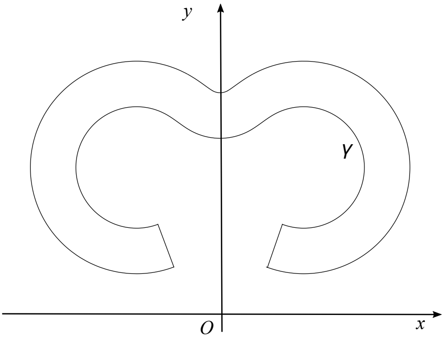
$$\left.\begin{array}{ll} \psi_p \geq \frac{2}{p} j_{n/2-1,1} \quad \forall p \geq 2 & \text{(Lindqvist '90)} \\ j^2_{n/2-1,1} > \frac{n}{2}\left(\frac{n}{2}+4\right) & \text{(Lorch '93)} \end{array}\right. \Rightarrow \psi_p > \frac{n^2}{p(n-1)}, \quad p \geq 2, \; n \geq 2.$$

A special class of non-convex planar domains

Let $\gamma(s) = (x(s), y(s))$, $s \in [0, L]$, be a smooth, simple curve, such that

$$x(L-s) = -x(s), \quad y(L-s) = y(s), \quad s \in \left[0, \frac{L}{2}\right].$$

Let us consider the annular domain D consisting of the points on one side of γ, within a suitable distance δ from γ.



When $\mu_{1,2}(D)$ has an odd eigenfunction?

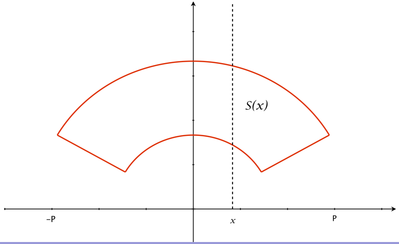
Proposition

Suppose that γ may be realized as the graph of a function. We denote by $\Pi_x(D) = (-P, P)$ the projection of D onto the x-axis. Let $S(x)$ denote the vertical cross sections of D, i.e., $S(x) = \{(\widetilde{x}, \widetilde{y}) \in D : \widetilde{x} = x\}$, and define $S = \max_{x \in [0,P)} |S(x)|$. If

$$S^2 < P^2 \frac{\int_D \sin^2\left(\frac{\pi}{2P}x\right) dxdy}{\int_D \cos^2\left(\frac{\pi}{2P}x\right) dxdy}, \tag{3}$$

then

$$\mu_{1,2}(D) = \mu_{1,2}^{odd}(D).$$



When $\mu_{1,2}(D)$ has an odd eigenfunction?

Proposition

$$\mu_{1,2}(D) = \mu_{1,2}^{odd}(D)$$

if one of the following alternatives holds:

1) $k(s) \geq 0$ for all $s \in [0, L]$ and

$$\max_{s\in[0,L]} \delta^2(2+\delta k(s))^2 < \frac{L^2}{\pi^2} \frac{\int_0^L \left(\int_0^\delta \cos^2\left(\frac{\pi}{L}s\right)(1+rk(s))dr\right) ds}{\int_0^L \left(\int_0^\delta \sin^2\left(\frac{\pi}{L}s\right)\frac{1}{1+rk(s)}dr\right) ds};$$

2) $k(s) < 0$ for all $s \in [0, L]$ and

$$\max_{s\in[0,L]} \frac{4\delta^2}{(1+\delta k(s))^2} < \frac{L^2}{\pi^2} \frac{\int_0^L \left(\int_0^\delta \cos^2\left(\frac{\pi}{L}s\right)(1+rk(s))dr\right) ds}{\int_0^L \left(\int_0^\delta \sin^2\left(\frac{\pi}{L}s\right)\frac{1}{1+rk(s)}dr\right) ds};$$

When $\mu_{1,2}(D)$ has an odd eigenfunction?

Proposition

$$\mu_{1,2}(D) = \mu_{1,2}^{odd}(D)$$

if one of the following alternatives holds:

3) $k(s)$ changes its sign in $[0, L]$, and

$$\max\left\{\max_{s\in[0,L]} \delta^2(2+\delta k(s))^2, \max_{s\in[0,L]} \frac{4\delta^2}{(1+\delta k(s))^2}\right\}$$
$$< \frac{L^2}{\pi^2} \frac{\int_0^L \left(\int_0^\delta \cos^2\left(\frac{\pi}{L}s\right)(1+rk(s))dr\right) ds}{\int_0^L \left(\int_0^\delta \sin^2\left(\frac{\pi}{L}s\right)\frac{1}{1+rk(s)}dr\right) ds}.$$

A special class of non-convex planar domains

Theorem

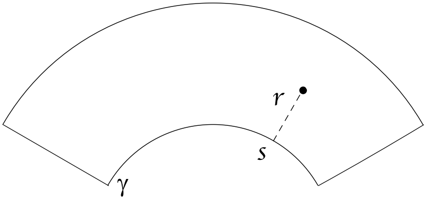
Suppose that the curvature $k(s)$ of γ is concave in $[0, L]$ and let $\delta > 0$ be such that $1 + k(s)\delta > 0$ in $[0, L]$. If D is simply connected and one of the previous geometric conditions is fulfilled, then

$$\mu_{1,2}(D) \geq B\frac{\pi^2}{L^2},$$

where $B = \left[1 + \max_{s\in[0,L]} |k(s)|\,\delta\right]^{-2}$, equality holding if γ is a segment.

Main steps of the proof

- We divide D into thin slices D_i parallel to γ where the considered eigenfunction has zero mean value (adaptation of the slicing technique introduced by Payne and Weinberger).
- We construct a Fermi coordinate system (r, s) whereby points (x, y) in D are determined by specifying the distance $r = \text{dist}_\gamma(x, y)$ to the curve γ, and the arc length s of the point on γ nearest to (x, y).



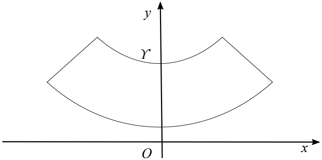
In this way, D is mapped into the rectangle $[0, L] \times [0, \delta]$.

- Since the slices D_i are arbitrarily thin, we are led to a one-dimensional problem, defined in the interval $[0, L]$, which is easier to handle with.

A simple example

Let $0 < a \leq \operatorname{arcsinh}\left(\frac{1}{\sqrt{3}}\right)$ and let us consider the arch of catenary

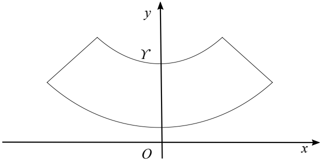
$$\begin{cases} x(s) & = \operatorname{arcsinh}(s - \sinh a) \\ y(s) & = \sqrt{1 + (s - \sinh a)^2} \end{cases}, \quad s \in [0, L]$$



A simple example

Let $0 < a \leq \operatorname{arcsinh}\left(\frac{1}{\sqrt{3}}\right)$ and let us consider the arch of catenary

$$\begin{cases} x(s) &= \operatorname{arcsinh}(s - \sinh a) \\ y(s) &= \sqrt{1 + (s - \sinh a)^2} \end{cases}, \quad s \in [0, L]$$



If δ is small enough (so that (3) is satisfied), then

$$\mu_{1,2}(D) = \mu_{1,2}^{odd}(D) \geq \frac{\pi^2}{4(1+\delta)\sinh^2 a}$$

Annular sector

Let $R > 0$ and consider the annular sector

$$D = \{\rho e^{i\theta} : \; R < \rho < R + \delta, \; \frac{\pi}{2} - \tilde{\theta} < \theta < \frac{\pi}{2} + \tilde{\theta}\},$$

with $\tilde{\theta} \in (0, \pi)$.
If δ satisfies

$$(2R\delta + \delta^2)\left(\log\left(\frac{R+\delta}{R}\right)\right) < \frac{2\tilde{\theta}^2 R^2}{\pi^2},$$

then

$$\mu_{1,2}(D) = \mu_{1,2}^{odd}(D) \geq \frac{\pi^2}{4\tilde{\theta}^2 (R+\delta)^2}$$

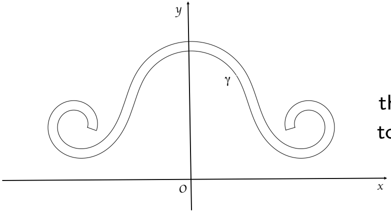
Handlebar moustache

We begin by considering the following concave function on the interval $[0, 1.6]$:

$$k(s) = \begin{cases} 50s - 25 & \text{if } 0 \leq s \leq 0.6, \\ 5 & \text{if } 0.6 \leq s \leq 1, \\ -50s + 55 & \text{if } 1 \leq s \leq 1.6. \end{cases}$$

Up to a rotation and a translation, there exists a unique curve $\gamma(s) = (x(s), y(s))$ (parametrized with respect to its arc length) having curvature $k(s)$:

$$\gamma(s) = \left(\int_0^s \cos\left(\int_0^u k(t)dt \right) du, \int_0^s \sin\left(\int_0^u k(t)dt \right) du \right), \quad 0 \leq s \leq 1.6.$$

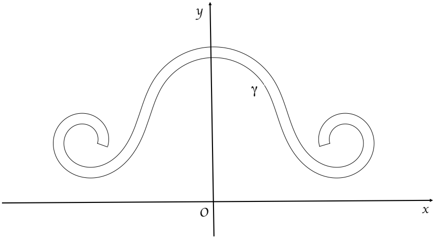


By rotating and translating so that γ is symmetric with respect to the y-axis, we may build D as in figure.

Handlebar moustache

If δ is small enough (and we explicitly know how much), it holds that

$$\mu_{1,2}(D) = \mu_{1,2}^{odd}(D) \geq \frac{\pi^2}{2.56(1+5\delta)^2}.$$



A generalization

As before, let $\gamma(s) = (x(s), y(s)),\ s \in [0, L]$, be a smooth, simple curve, such that

$$x(L-s) = -x(s), \quad y(L-s) = y(s), \quad s \in \left[0, \frac{L}{2}\right].$$

Let us consider the annular domain D consisting of the points on one side of γ, within a suitable non-constant distance $\delta(s) \geq 0$ from γ. Using the normal vector to $\gamma(s)$ obtained by rotating $\gamma'(s)$ clockwise by $\frac{\pi}{2}$, we may describe D as follows

$$D = \left\{(x(s) + ry'(s), y(s) - rx'(s)) :\ 0 \leq s \leq L,\ 0 \leq r \leq \delta(s)\right\}$$

A generalization

As before, let $\gamma(s) = (x(s), y(s)),\ s \in [0, L]$, be a smooth, simple curve, such that

$$x(L-s) = -x(s), \quad y(L-s) = y(s), \quad s \in \left[0, \frac{L}{2}\right].$$

Let us consider the annular domain D consisting of the points on one side of γ, within a suitable non-constant distance $\delta(s) \geq 0$ from γ. Using the normal vector to $\gamma(s)$ obtained by rotating $\gamma'(s)$ clockwise by $\frac{\pi}{2}$, we may describe D as follows

$$D = \left\{(x(s) + ry'(s), y(s) - rx'(s)) :\ 0 \leq s \leq L,\ 0 \leq r \leq \delta(s)\right\}$$

The needed assumption is $\delta(s) \cdot k(s)$ to be concave.

THANK YOU FOR YOUR ATTENTION