Rational and algebraic links and knots-quivers correspondence

Marko Stošić

1CAMGSD, Departamento de Matemática, Instituto Superior Técnico, Portugal
2Mathematical Institute SANU, Belgrade, Serbia

TQFT online seminar, 26.6.2020.
Colored HOMFLY–PT polynomials:

Symmetric ($S^r$)-colored HOMFLY–PT polynomials are 2-variable invariants of knots:

$$P_r(K)(a, q).$$

For $a = q^N$ they are ($sl(N), S^r$) quantum polynomial invariants:

$$P(a = q^N, q) = P^{sl(N), S^r}(q).$$
Colored HOMFLY–PT polynomials:

Symmetric ($S^r$)-colored HOMFLY–PT polynomials are 2-variable invariants of knots:

$$P_r(K)(a, q).$$

For $a = q^N$ they are ($sl(N), S^r$) quantum polynomial invariants:

$$P(a = q^N, q) = P^{sl(N), S^r}(q).$$

Already interesting is the ”bottom row”: the coefficient of the lowest nonzero power of $a$ appearing in $P_r(a, q)$

$$P_r^-(q) = \lim_{a \to 0} a^\# P_r(a, q)$$
LMOV conjecture

Generating function of all symmetric-colored HOMFLY-PT polynomials of a given knot $K$ is:

$$P(x, a, q) := \sum_{r \geq 0} P_r(a, q)x^r = \exp \left( \sum_{n, r \geq 1} \frac{1}{n} f_r(a^n, q^n) x^{rn} \right),$$

$$f_r(a, q) = \sum_{i,j} \frac{N_{r,i,j} a^i q^j}{q - q^{-1}}.$$
Generating function of all symmetric-colored HOMFLY-PT polynomials of a given knot $K$ is:

$$P(x, a, q) := \sum_{r \geq 0} P_r(a, q)x^r = \exp \left( \sum_{n, r \geq 1} \frac{1}{n} f_r(a^n, q^n)x^{rn} \right),$$

$$f_r(a, q) = \sum_{i, j} \frac{N_{r, i, j}a^i q^j}{q - q^{-1}}.$$
Generating function of all symmetric-colored HOMFLY-PT polynomials of a given knot $K$ is:

$$P(x, a, q) := \sum_{r \geq 0} P_r(a, q)x^r = \exp \left( \sum_{n, r \geq 1} \frac{1}{n} f_r(a^n, q^n)x^{rn} \right),$$

$$f_r(a, q) = \sum_{i,j} N_{r,i,j} a^i q^j \frac{q^j}{q - q^{-1}}.$$  

One can easily get: $N_{r,i,j} \in \mathbb{Q}$

**LMOV conjecture:** $N_{r,i,j} \in \mathbb{Z}$
LMOV conjecture

Generating function of all symmetric-colored HOMFLY-PT polynomials of a given knot \( K \) is:

\[
P(x, a, q) := \sum_{r \geq 0} P_r(a, q)x^r = \exp \left( \sum_{n, r \geq 1} \frac{1}{n} f_r(a^n, q^n)x^{rn} \right),
\]

\[
f_r(a, q) = \sum_{i, j} N_{r, i, j} a^i q^j.
\]

One can easily get: \( N_{r, i, j} \in \mathbb{Q} \)

**LMOV conjecture:** \( N_{r, i, j} \in \mathbb{Z} \)!

\( N_{r, i, j} \) are BPS numbers. They represent (super)-dimensions of certain homological groups. Physically, they "count" particles of certain type (therefore are integers).
Quivers are oriented graphs, possibly with loops and multiple edges.

$Q_0 = \{1, \ldots, m\}$ – set of vertices.

$Q_1$ the set of edges $\{\alpha : i \rightarrow j\}$. 
Quivers are oriented graphs, possibly with loops and multiple edges. 

$Q_0 = \{1, \ldots, m\}$ – set of vertices.

$Q_1$ the set of edges $\{\alpha : i \rightarrow j\}$.

Let $d = (d_1, \ldots, d_m) \in \mathbb{N}^m$ be a dimension vector.

We are interested in moduli space of representations of $Q$ with the dimension vector $d$:

$$M_d = \left\{ R(\alpha) : \mathbb{C}^{d_i} \rightarrow \mathbb{C}^{d_j} \mid \text{for all } \alpha : i \rightarrow j \in Q_1 \right\} \mod G,$$

where $G = \prod_i GL(d_i, \mathbb{C})$. 
$C$ is a matrix of a quiver with $m$ vertices.

\[ P_C(x_1, \ldots, x_m) := \sum_{d_1, \ldots, d_m} \frac{(-q)^{\sum_{i,j=1}^m C_{i,j}d_i d_j}}{(q^2; q^2)_{d_1} \cdots (q^2; q^2)_{d_m}} x_1^{d_1} \cdots x_m^{d_m}. \]

$q$-Pochhammer symbol \((q^2; q^2)_n := \prod_{i=1}^n (1 - q^{2i}).\)
Quivers and motivic generating functions

$C$ is a matrix of a quiver with $m$ vertices.

$$P_C(x_1, \ldots, x_m) := \sum_{d_1, \ldots, d_m} \frac{(-q)^{\sum_{i,j=1}^m C_{i,j} d_i d_j}}{(q^2; q^2)_{d_1} \cdots (q^2; q^2)_{d_m}} x_1^{d_1} \cdots x_m^{d_m}.$$ 

q-Pochhammer symbol \((q^2; q^2)_n := \prod_{i=1}^n (1 - q^{2i}).\)

Motivic (quantum) Donaldson-Thomas invariants $\Omega_{d_1, \ldots, d_m; j}$ of a symmetric quiver $Q$:

$$P_C = \prod_{(d_1, \ldots, d_m) \neq 0} \prod_{j \in \mathbb{Z}} \prod_{k \geq 0} \left(1 - (x_1^{d_1} \cdots x_m^{d_m}) q^{j+2k+1}\right) (-1)^{j+1} \Omega_{d_1, \ldots, d_m; j}.$$
$C$ is a matrix of a quiver with $m$ vertices.

$$P_C(x_1, \ldots, x_m) := \sum_{d_1, \ldots, d_m} \frac{(-q)^{\sum_{i,j=1}^m C_{i,j}d_id_j}}{(q^2; q^2)_{d_1} \cdots (q^2; q^2)_{d_m}} x_1^{d_1} \cdots x_m^{d_m}.$$ 

q-Pochhammer symbol $(q^2; q^2)_n := \prod_{i=1}^n (1 - q^{2i})$.

Motivic (quantum) Donaldson-Thomas invariants $\Omega_{d_1, \ldots, d_m; j}$ of a symmetric quiver $Q$:

$$P_C = \prod_{(d_1, \ldots, d_m) \neq 0} \prod_{j \in \mathbb{Z}} \prod_{k \geq 0} \left(1 - (x_1^{d_1} \cdots x_m^{d_m}) q^{j+2k+1}\right)^{(-1)^{j+1} \Omega_{d_1, \ldots, d_m; j}}.$$

**Theorem (Kontsevich-Soibelman, Efimov)**

$\Omega_{d_1, \ldots, d_m; j}$ are nonnegative integers.
Knots–quivers correspondence

[P. Kucharski, M. Reineke, P. Sułkowski, M.S., Phys. Rev. D 2017]

New relationship between HOMFLY–PT / BPS invariants of knots and motivic Donaldson-Thomas invariants for quivers

Figure: Trefoil knot and the corresponding quiver.

The generating series of HOMFLY-PT invariants of a knot matches the motivic generating series of a quiver, after setting $x_i \rightarrow x$. 
<table>
<thead>
<tr>
<th><strong>Knots</strong></th>
<th><strong>Quivers</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Generators of HOMFLY homology</td>
<td>Number of vertices</td>
</tr>
<tr>
<td>Homological degrees, framing</td>
<td>Number of loops</td>
</tr>
<tr>
<td>Colored HOMFLY-PT</td>
<td>Motivic generating series</td>
</tr>
<tr>
<td>LMOV invariants</td>
<td>Motivic DT-invariants</td>
</tr>
<tr>
<td>Classical LMOV invariants</td>
<td>Numerical DT-invariants</td>
</tr>
<tr>
<td>Algebra of BPS states</td>
<td>Cohom. Hall Algebra</td>
</tr>
</tbody>
</table>
BPS/LMOV invariants of knots are refined through motivic DT invariants of a corresponding quiver, and so

**Theorem**

*For all knots for which there exists a corresponding quiver, the LMOV conjecture holds.*
Open questions

• Find quivers for (large) classes of knots

• How to find a quiver for a given knot directly (geometrically, topologically...)? Other, better definition?

• The (non)uniqueness of a quiver:
  – What is the smallest possible size of the quiver?
  – Among the ones with the same size – discrete group action?

• What is so special for quivers that correspond to knots? (Combinatorial identities for binomial coefficients, and extended integrality/divisibility hold precisely for them.)
Application 2 – Lattice paths counting

Figure: A lattice path under the line $y = \frac{1}{4}x$, and a shaded area between the path and the line.

$$y_P(x) = \sum_{k=0}^{\infty} \sum_{\pi \in k\text{-paths}} x^k = \sum_{k=0}^{\infty} c_k(1)x^k,$$

$$y_{qP}(x) = \sum_{k=0}^{\infty} \sum_{\pi \in k\text{-paths}} q^{\text{area}(\pi)}x^k = \sum_{k=0}^{\infty} c_k(q)x^k.$$
Surprisingly, colored HOMFLY–PT polynomials are closely related to the purely combinatorial problem of counting lattice paths under lines with rational slope.

Proposition (M. Panfil, P. Sulkowski, M.S., 2018)

Let \( r \) and \( s \) be mutually prime. Let \( K = T_f = -\frac{rs}{r, s} \) be the \( rs \)-framed \((r, s)\)-torus knot. Then the coefficients \( a_n \) of the series governing the growth of the generating series of colored HOMFLY-PT polynomial of \( K \), are equal to the number of directed lattice path from \((0, 0)\) to \((sn, rn)\) under the line \( y = \left(\frac{r}{s}\right)x \).
Surprisingly, colored HOMFLY–PT polynomials are closely related to the purely combinatorial problem of counting lattice paths under lines with rational slope.

**Proposition (M. Panfil, P. Sulkowski, M.S., 2018)**

*Let $r$ and $s$ be mutually prime.*

*Let $K = T_{r,s}^{f=-rs}$ be the $rs$-framed $(r, s)$-torus knot.*

*Then the coefficients $a_n$ of the series governing the growth of the generating series of colored HOMFLY-PT polynomial of $K$, are equal to the number of directed lattice path from $(0, 0)$ to $(sn, rn)$ under the line $y = (r/s)x$.***
Knots and quivers – results

Knots-quivers correspondence naturally suggests a particular refinement of the numbers $a_n$.

\[
\begin{align*}
(2/3)n &= \sum_{i+j=n} (7i + 5j + 1)(5i + 5j + 1) \\
&= n \sum_{i=0}^{1} (5n + 2i)(5n + 1n - i) \\
&= \sum_{i=1}^{1} (5n + 2i)(5n + 1n - i) \\
&= (5n + 2)(5n + 1) \\
&= 25n^2 + 15n + 2.
\end{align*}
\] (rediscovered Duchon formula)
Knots and quivers – results

Knots-quivers correspondence naturally suggests a particular refinement of the numbers $a_n$. For example, for $2/3$ slope, after computing the relevant invariants (for the bottom row) of the $T_{2,3}$ knot (trefoil), the corresponding quiver is:

$$
\begin{bmatrix}
7 & 5 \\
5 & 5 \\
5 & 5
\end{bmatrix}.
$$

(rediscovered Duchon formula)
Knots and quivers – results

Knots-quivers correspondence naturally suggests a particular refinement of the numbers $a_n$. For example, for 2/3 slope, after computing the relevant invariants (for the bottom row) of the $T_{2,3}$ knot (trefoil), the corresponding quiver is:

$$\begin{bmatrix} 7 & 5 \\ 5 & 5 \end{bmatrix}.$$ 

$$a_n^{(2/3)} = \sum_{i+j=n} \frac{1}{7i + 5j + 1} \binom{7i + 5j + 1}{i} \binom{5i + 5j + 1}{j}$$

$$= \sum_{i=0}^{n} \frac{1}{5n + i + 1} \binom{5n + 2i}{i} \binom{5n + 1}{n - i}.$$ 

(rediscovered Duchon formula)
All this also rediscovers some binomial identities, like e.g.

\[
\binom{5n}{2n} = \sum_{i=0}^{n} \frac{5n}{5n+2i} \binom{5n+2i}{i} \binom{5n}{n-i}
\]

Comes from counting of paths under line with slope 2/3.
All this also rediscovers some binomial identities, like e.g.

\[ \binom{5n}{2n} = \sum_{i=0}^{n} \frac{5n}{5n+2i} \binom{5n+2i}{i} \binom{5n}{n-i} \]

Comes from counting of paths under line with slope 2/3.

One can obtain such identities precisely for the quivers that correspond to (torus) knots.
Proposition

The generating function \( y_{qP}(x) \) of lattice paths under the line of the slope \( r/s \), weighted by the area between this line and a given path, is equal to

\[
y_{qP}(x) = \sum_{k=0}^{\infty} \sum_{\pi \in \text{k-paths}} q^{\text{area}(\pi)} x^k = \frac{P_C(q^2 x_1, \ldots, q^2 x_m)}{P_C(x_1, \ldots, x_m)} \bigg|_{x_i = x q^{-1}}.
\]

For the line of the slope \( r/s \), the quiver in question is defined by the matrix \( C \) that encodes extremal invariants of left-handed \((r, s)\) torus knot in framing \( rs \).
Paths under the line with slope $3/4$

$$C^{(3,4)} = \begin{bmatrix}
7  & 7  & 7  & 7  & 7 \\
7  & 9  & 8  & 9  & 9 \\
7  & 8  & 9  & 9  & 10 \\
7  & 9  & 9  & 11 & 11 \\
7  & 9  & 10 & 11 & 13
\end{bmatrix}$$

$$\#\text{paths} = \sum_{l_1+\cdots+l_5=n} A_{(3,4)}(l_1,l_2,l_3,l_4,l_5) \times$$

$$\times \frac{1}{7l_1+7l_2+7l_3+7l_4+7l_5+1} \left( \frac{7l_1+7l_2+7l_3+7l_4+7l_5+1}{l_1} \right) \times$$

$$\times \frac{1}{7l_1+9l_2+8l_3+9l_4+9l_5+1} \left( \frac{7l_1+9l_2+8l_3+9l_4+9l_5+1}{l_2} \right) \times$$

$$\times \frac{1}{7l_1+8l_2+9l_3+9l_4+10l_5+1} \left( \frac{7l_1+8l_2+9l_3+9l_4+10l_5+1}{l_3} \right) \times$$

$$\times \frac{1}{7l_1+9l_2+9l_3+11l_4+11l_5+1} \left( \frac{7l_1+9l_2+9l_3+11l_4+11l_5+1}{l_4} \right) \times$$

$$\times \frac{1}{7l_1+9l_2+10l_3+11l_4+13l_5+1} \left( \frac{7l_1+9l_2+10l_3+11l_4+13l_5+1}{l_5} \right).$$
\[ A_{(3,4)}(l_1, l_2, l_3, l_4, l_5) = 1 + 28 l_1 + 294 l_1^2 + 1372 l_1^3 + 2401 l_1^4 + 33 l_2 + 693 l_1 l_2 + 4851 l_1^2 l_2 + 11319 l_1^3 l_2 + 407 l_1^2 + 5698 l_1 l_2^2 +
+ 19943 l_1^2 l_2^2 + 2223 l_2^2 + 15561 l_1 l_2^2 + 4536 l_2^2 + 34 l_3 + 714 l_1 l_3 + 4998 l_1^2 l_3 + 11662 l_1^2 l_3 + 838 l_2 l_3 + 11732 l_2 l_3 +
+ 41062 l_1 l_2 l_3 + 6860 l_2 l_3 + 48020 l_1 l_2 l_3 + 18648 l_2 l_3^2 + 431 l_3 + 6034 l_1 l_3 + 21119 l_1^2 l_3 + 7051 l_1 l_3 + 49357 l_1 l_2 l_3^2 +
+ 28728 l_2^2 l_3^2 + 2414 l_3 + 1689 l_1 l_3^2 + 1965 l_2 l_3^2 + 5040 l_3^2 + 36 l_4 + 756 l_1 l_4 + 5292 l_2 l_4 + 12348 l_3^3 l_4 + 887 l_2 l_4 +
+ 12418 l_1 l_2 l_4 + 43463 l_1^2 l_2 l_4 + 7258 l_2 l_4 + 5080 l_1^2 l_2 l_4 + 1971 l_2 l_4 + 21294 l_3^3 l_4 + 482 l_4^2 + 674 l_2 l_4^2 + 23618 l_3^2 l_4^2 +
+ 912 l_3 l_4 + 1276 l_1 l_3 l_4 + 4468 l_2^2 l_3 l_4 + 14914 l_2 l_3 l_4 + 104398 l_1 l_2 l_3 l_4 + 60372 l_1^2 l_2 l_3 l_4 + 7656 l_3^2 l_4 + 53592 l_1 l_3^2 l_4 + 62307 l_2 l_3^2 l_4 +
+ 7879 l_2^2 l_4 + 5515 l_1 l_2 l_4^2 + 32067 l_2^2 l_4 + 8086 l_1^2 l_3 l_4 + 56602 l_1 l_3 l_4^2 + 23688 l_2^3 l_3 l_4 + 6237 l_4^3 + 65772 l_1 l_2 l_3 l_4 + 37 l_5 + 777 l_1 l_5 +
+ 33705 l_3^2 l_4 + 2844 l_4^3 + 1900 l_1^3 l_5^2 + 2312 l_1^2 l_4^3 + 5439 l_1^2 l_5 + 1260 l_1 l_5 + 912 l_2 l_5 + 1276 l_1 l_2 l_5 + 4468 l_1^2 l_2 l_5 +
+ 7465 l_2 l_5 + 5225 l_1^2 l_5 + 2028 l_2 l_5 + 695 l_1^2 l_5 + 3650 l_1 l_5 + 901 l_1 l_5 + 6307 l_1 l_4 l_5 + 3950 l_4 l_5 + 25074 l_2 l_5 +
+ 938 l_3 l_5 + 1313 l_1 l_3 l_5 + 4596 l_2^2 l_3 l_5 + 1534 l_2 l_3 l_5 + 10739 l_1 l_2 l_3 l_5 + 6248 l_2^2 l_3 l_5 + 787 l_3^2 l_5 + 3083 l_5^3 + 2158 l_1 l_5^3 +
+ 5513 l_1 l_3 l_5 + 6410 l_2 l_3 l_5 + 21910 l_3 l_5 + 991 l_4 l_5 + 13874 l_1 l_4 l_5 + 4855 l_1^2 l_4 l_5 + 1620 l_2 l_4 l_5 + 7326 l_1 l_4 l_5 + 75068 l_3 l_4 l_5 +
+ 11342 l_1 l_3 l_5 + 6596 l_2 l_3 l_5 + 16632 l_3 l_4 l_5 + 11642 l_1 l_3 l_4 l_5 + 13529 l_2 l_3 l_4 l_5 + 69335 l_3 l_4 l_5 + 25690 l_3 l_5^3 +
+ 8771 l_4 l_5 + 61397 l_1 l_4 l_5 + 7131 l_2 l_4 l_5 + 73066 l_3 l_4 l_5 + 25641 l_3 l_4 l_5 + 509 l_5^2 + 7126 l_1 l_5^3 + 27027 l_4 l_5^3 +
+ 2494 l_1 l_5^2 + 8325 l_2 l_5^2 + 58275 l_1 l_5^2 + 3384 l_2 l_5^2 + 8546 l_3 l_5^2 + 5982 l_1 l_5^2 + 6930 l_5^4.
Schröder paths

Figure: An example of a Schröder path of length 6.
Schröder paths and full colored HOMFLY-PT

Quiver corresponding to the full colored HOMFLY-PT invariants of knots in framing $f = 1$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

This corresponds to counting paths under the diagonal line $y = x$. 
Quiver corresponding to the full colored HOMFLY-PT invariants of knots in framing \( f = 1 \)

\[
C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}
\]

This corresponds to counting paths under the diagonal line \( y = x \). In this case we take specializations:

\[
x_1 \rightarrow x, \quad x_2 \rightarrow ax
\]
Schröder paths and full colored HOMFLY-PT

Quiver corresponding to the full colored HOMFLY-PT invariants of knots in framing $f = 1$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

This corresponds to counting paths under the diagonal line $y = x$. In this case we take specializations:

$$x_1 \rightarrow x, \quad x_2 \rightarrow ax$$

Then from the quiver generating function of $C$ we get

$$y(x, a, q) = 1 + (q + a)x + (q^2 + q^4 + (2q + q^3)a + a^2)x^2 + \ldots$$

with the height of a path measured by the power of $x$ and the number of diagonal steps measured by the power of $a$. 
Figure: All 6 Schröder paths of height 2 represented by the quadratic term \( q^2 + q^4 + (2q + q^3)a + a^2 \) of the generating function.
Consequence 2 – Some divisibilities (integrality)

If $p$ is prime, then:

$$p \mid \binom{3p-1}{p-1} - 1.$$
Consequence 2 – Some divisibilities (integrality)

If $p$ is prime, then:

$p | \binom{3p-1}{p-1} - 1.$

$p^2 | \binom{3p-1}{p-1} - 1.$
If $p$ is prime, then:

\[ p \mid \binom{3p-1}{p-1} - 1. \]

\[ p^2 \mid \binom{3p-1}{p-1} - 1. \]

\[ p^3 \mid 2 \left( \binom{3p-1}{p-1} - 1 \right). \]
Consequence 2 – Some divisibilities (integrality)

If $p$ is prime, then:

\[ p \mid \binom{3p-1}{p-1} - 1. \]

\[ p^2 \mid \binom{3p-1}{p-1} - 1. \]

\[ p^3 \mid 2 \left( \binom{3p-1}{p-1} - 1 \right). \]

If $r \in \mathbb{N}$, then

\[ r^2 \mid \sum_{d \mid r} \mu \left( \frac{r}{d} \right) \binom{3d-1}{d-1}. \]

\[ \mu(n) = \begin{cases} (-1)^k, & n = p_1 p_2 \cdots p_k, \\ 0, & p^2 \mid n \end{cases} \]
Consequence 2 – Some divisibilities (integrality)

If \( p \) is prime, then:

\[ p \mid \binom{3p-1}{p-1} - 1. \]

\[ p^2 \mid \binom{3p-1}{p-1} - 1. \]

\[ p^3 \mid 2 \left( \binom{3p-1}{p-1} - 1 \right). \]

If \( r \in \mathbb{N} \), then

\[ r^2 \mid \sum_{d \mid r} \mu \left( \frac{r}{d} \right) \binom{3d-1}{d-1}. \]

\[ \mu(n) = \begin{cases} (-1)^k, & n = p_1 p_2 \cdots p_k, \\ 0, & p^2 \mid n \end{cases} \]

Corresponds to the fact that DT invariants are non-negative integers (in this case of the quiver of the framed unknot — one vertex, \( m \)-loop quiver)
Quivers for rational knots

[M.S., P. Wedrich, IMRN 2019].

\[ \frac{p}{q} = [a_1, \ldots, a_r] = a_r + \frac{1}{a_{r-1} + \frac{1}{a_{r-2} + \ldots}} \]

Rational tangle encoded by [2, 3, 1]
Rational knots

\[ T(\ ) := \quad , \quad R(\ ) := \quad \]
Rational knots

\[ T(\text{ }) := \quad , \quad R(\text{ }) := \]

\[ UP : \quad , \quad OP : \quad , \quad RI : \quad \]
Skein theory

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\xymatrix{\cdots & k & k \\
& l & k \\
k & & l
\end{array} \\
\sum_{h=0}^{l-1} (-q)^{h-l} = k \geq l
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
\xymatrix{\cdots & k & k \\
& l & k \\
k & & l
\end{array} \\
\sum_{h=0}^{k-1} (-q)^{h-k} = k \leq l
\end{array}
\end{align*}
\]
Basic webs and twist rules

\[
UP[j, k] = \begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \\
\downarrow \uparrow
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \\
\downarrow \uparrow
\end{array}
\end{array}, \quad OP[j, k] = \begin{array}{c}
\begin{array}{c}
\downarrow \uparrow \\
\uparrow \downarrow
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\downarrow \uparrow \\
\uparrow \downarrow
\end{array}
\end{array}, \quad RI[j, k] = \begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \\
\downarrow \uparrow
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\uparrow \downarrow \\
\downarrow \uparrow
\end{array}
\end{array}
\]

1. \(TUP[j, k] = \sum_{h=k}^{j} (-q)^{h-j} q^{k^2} \binom{h}{k} + UP[j, h]\)
2. \(RUP[j, k] = \sum_{h=0}^{k} (-q)^{h-j} a^{h-j} q^{-2kh+k^2+j^2} \binom{j-h}{k-h} + OP[j, h]\)
3. \(TOP[j, k] = \sum_{h=k}^{j} (-q)^{h} a^{k} q^{k^2-2jk} \binom{h}{k} + RI[j, h]\)
4. \(ROP[j, k] = \sum_{h=0}^{k} (-q)^{h-j} a^{k-j} q^{2h(j-k)+(k-j)^2} \binom{j-h}{k-h} + UP[j, h]\)
5. \(TRI[j, k] = \sum_{h=k}^{j} (-q)^{h} a^{h} q^{k^2-2jh} \binom{h}{k} + OP[j, h]\)
6. \(RRI[j, k] = \sum_{h=0}^{k} (-q)^{h} q^{h(2j-2k)+k^2-j^2} \binom{j-h}{k-h} + RI[j, h]\)
Theorem

Let $K$ be a rational knot and let $Q_K$ be the corresponding quiver. Then, the vertices of $Q_K$ are in bijection with generators of the reduced HOMFLY-PT homology of $K$, such that the $(a, q, t)$-trigrading of the $i^{th}$ generator is given by $(a_i, -Q_{i,i} - q_i, -Q_{i,i})$ where $Q_{i,i}$ denotes the number of loops at the $i^{th}$ vertex of $Q_K$. 
A rational knot $K_{p/q}$ can be presented as the closure of

$$T^{a_r} R^{a_{r-1}} T^{a_{r-2}} \ldots T^{a_3} R^{a_2} T^{a_1} K_0$$

$$p/q = a_r + \cfrac{1}{a_{r-1} + \cfrac{1}{a_{r-2} + \cfrac{1}{\cdots + \cfrac{1}{a_3 + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}}}$$
$K_{p/q}$ rational knot can be presented as the closure of

$$T^{a_r} R^{a_{r-1}} T^{a_{r-2}} \ldots T^{a_3} R^{a_2} T^{a_1} K_0$$

$$R^{a_2} T^{a_1} K_0$$

$$p/q = a_r + \frac{1}{a_{r-1} + \frac{1}{a_{r-2} + \frac{1}{\ldots + \frac{1}{a_3 + \frac{1}{a_1 a_2 + 1}}}}}$$
A rational knot $K_{p/q}$ can be presented as the closure of

$$T^{a_r} R^{a_{r-1}} T^{a_{r-2}} \cdots T^{a_3} R^{a_2} T^{a_1} K_0$$

or equivalently,

$$T^{a_3} R^{a_2} T^{a_1} K_0$$

where $p/q = a_r + \frac{1}{a_{r-1} + \frac{1}{a_{r-2} + \frac{1}{a_1 a_2 + 1 \cdots + \frac{1}{a_1 a_2 a_3 + a_1 + a_3}}}}.$
Extend the correspondence to 4-ended tangles:

Ask that knots-quivers correspondence hold for any link obtained by closing off the four ends of a tangle.
Extend the correspondence to 4-ended tangles:

Ask that knots-quivers correspondence hold for any link obtained by closing off the four ends of a tangle.

The rational tangles satisfy tangle-quiver correspondence.

For $p/q$ rational tangle we can write the generating series of the colored HOMFLY-PT invariants as a summation with $p + q$ variables ($p$ ”active”, $q$ ”inactive”).
For rational tangles, we had Top and Right twists.

Tangle addition operation:

\[ +: \left( \tau_1, \tau_2 \right) \rightarrow \tau_1 \tau_2 \]
For rational tangles, we had Top and Right twists.

Tangle addition operation:

\[ + : \left( \tau_1, \tau_2 \right) \rightarrow \tau_1 \tau_2 \]

Addition of rational tangles give algebraic tangles. Closures of such tangles are called arborescent links, or algebraic links (in the sense of Conway).
Tangle addition

For rational tangles, we had Top and Right twists.

Tangle addition operation:

\[ + : \left( \tau_1, \tau_2 \right) \rightarrow \tau_1 \tau_2 \]

Addition of rational tangles give algebraic tangles. Closures of such tangles are called arborescent links, or algebraic links (in the sense of Conway).

IN MEMORIAM: John H. Conway (1937-2020)
4-ended tangle family $\mathcal{QT}_4$

**Theorem (M.S., P. Wedrich, 2020)**

There exists a family $\mathcal{QT}_4$ of 4-ended framed oriented tangles with the following properties:

- $\mathcal{QT}_4$ contains the trivial 2-strand tangle.
- $\mathcal{QT}_4$ is closed under diffeomorphisms of $(B^3, \partial B^3, \{4 \text{ pts}\})$.
- $\mathcal{QT}_4$ is closed under Conway's tangle addition, the binary operation of gluing two 4-ended tangles at pairs of boundary points as follows:

$$\tau_1, \tau_2 \rightarrow \tau_1 \tau_2$$

- The appropriate analogue of the knots-quivers correspondence holds for any link obtained by closing off a tangle in $\mathcal{QT}_4$. 
We will consider six refined types of tangles $\tau \in \mathcal{T}_4$, which encode boundary types and connectivity between boundary points:

- $UP_{par}$
- $OP_{ud}$
- $RI_{par}$
- $UP_{cr}$
- $OP_{lr}$
- $RI_{cr}$

For example, an $UP_{par}$ tangle has one strand directed from the $SW$ to the $NW$ boundary point and the other strand directed from the $SE$ to the $NE$ boundary point, and possibly additional closed components.
Tangle addition: binary addition operation on 4-ended tangles, which is given by gluing along pairs of adjacent boundary points, provided the orientations are compatible there.

\[
+ : \left( \tau_1, \tau_2 \right) \rightarrow \tau_1 \tau_2
\]

Main result is the following theorem:

**Theorem**

Let \( \tau_1, \tau_2 \in QT_4 \) with orientations such that \( \tau_1 + \tau_2 \) is defined. Then \( \tau_1 + \tau_2 \in QT_4 \).
Due to different types of tangles and orientations, there are five cases to check. In each of them the gluing formula for the HOMFLY-PT skein generating functions is established.
Conclusion: Algebraic links satisfy knots-quivers correspondence. This includes Montesinos links, pretzel knots, etc.
Conclusion: Algebraic links satisfy knots-quivers correspondence. This includes Montesinos links, pretzel knots, etc..

What is the minimal size of the corresponding quiver?
Conclusion: Algebraic links satisfy knots-quivers correspondence. This includes Montesinos links, pretzel knots, etc..

What is the minimal size of the corresponding quiver?

The size of the quiver produced by algorithm has an upper bound (for addition of two tangles the bound is bilinear in the sizes of the quivers of individual tangles), but in particular cases it seems that it can be lowered even further.
Thank you for your attention!