From Racks to Pointed Hopf Algebras

LisMath Seminar

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Universidade de Lisboa

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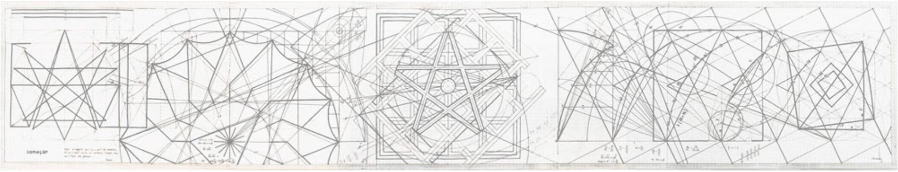

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Can we decide whether two knots are equivalent? Can we classify knots?

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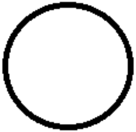

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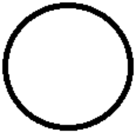

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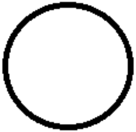

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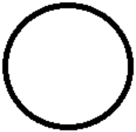

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An *arc* is a portion of a knot diagram which runs from one undercrossing to the next.

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Theorem (Reidemeister, 1927)

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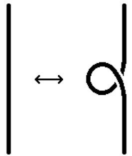

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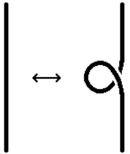

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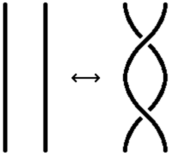

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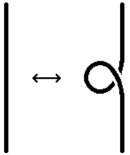

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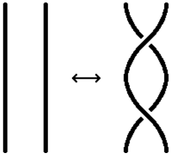

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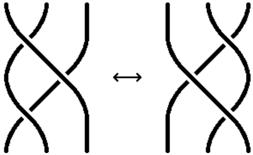

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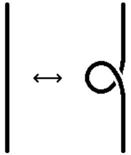

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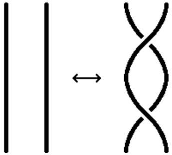

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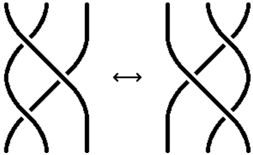

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We are going after a knot invariant.

Quandles

Definition

Let X be a set and let $*$ be a binary operation on X. The pair $(X, *)$ is said to be a *quandle* if, for each $i, j, k \in X$,

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3. $(i * j) * k = (i * k) * (j * k)$ (self-distributivity).

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2. For each $n \geq 3$, $(R_n, *)$ denotes the quandle whose underlying set is $\mathbb{Z}_n$ and whose operation is $i * j = 2j - i \mod n$, $\forall i, j \in \mathbb{Z}_n$. This is called the *dihedral quandle of order* n;

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3. Let G be a group and let $*$ be the binary operation on G given by $a * b = bab^{-1}$, $\forall a, b \in G$, where the juxtaposition on the right-hand side denotes group multiplication. Then, the pair $(G, *)$ is a quandle;

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4. Let G be a group, let $s : G \to G$ be a group automorphism and let $*$ be the binary operation on G given by $a * b = bs(ab^{-1})$, $\forall a, b \in G$. Then, the pair $(G, *)$ is a quandle. Quandles obtained in this way are called *twisted homogeneous crossed sets*;

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5. Let A be an abelian group, let $s : A \to A$ be a group automorphism and let $*$ be the binary operation on A given by $a * b = b + s(a - b)$, $\forall a, b \in A$. Then, the pair $(A, *)$ is a quandle, which is usually denoted by (A, s). Quandles obtained in this way are called *affine crossed sets*;

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6. Let $P \subset \mathbb{R}^3$ be a regular polyhedron centered at the origin with vertices $X = \{1, ..., n\}$. For $1 \leq i \leq n$, let $T_i : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation by $2\pi/r$ fixing i and the origin (r is the number of edges ending in each vertex; look from i to the origin and rotate counterclockwise). Let $*$ be the binary operation on X given by $i * j = T_j(i)$. Then $(X, *)$ is a quandle.

Quandles

$*$	1	2	3	4
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Table 2: Tetrahedron quandle;

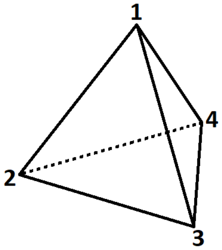

Figure 7: Tetrahedron.

Quandles

$*$	1	2	3	4	5	6
1	1	1	6	5	3	4
2	2	2	5	6	4	3
3	5	6	3	3	2	1
4	6	5	4	4	1	2
5	4	3	1	2	5	5
6	3	4	2	1	6	6

Table 3: Octahedron quandle;

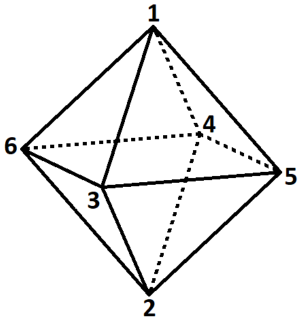

Figure 8: Octahedron.

$*$	1	2	3	4	5	6	7	8
1	1	4	2	3	1	3	4	2
2	3	2	4	1	4	2	1	3
3	4	1	3	2	2	4	3	1
4	2	3	1	4	3	1	2	4
5	5	8	6	7	5	7	8	6
6	7	6	8	5	8	6	5	7
7	8	5	7	6	6	8	7	5
8	6	7	5	8	7	5	6	8

Table 4: Cube quandle;

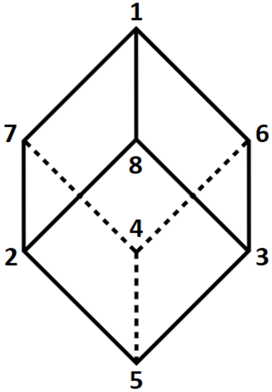

Figure 9: Cube.

A Knot Invariant

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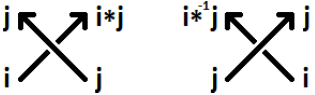

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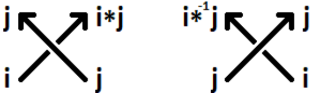

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Let $Q = (X, *)$ be a finite quandle and let K be an oriented knot diagram. The *quandle counting invariant* $|\mathrm{Hom}(K, Q)|$ is the total number of quandle colorings of K by Q.

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Let Q be a finite quandle and let K be an oriented knot diagram. Then, the quandle counting invariant $|\mathrm{Hom}(K, Q)|$ is invariant under the Reidemeister moves and under planar orientation-preserving homeomorphisms.

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We analyse each Reidemeister move separately:

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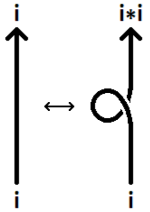

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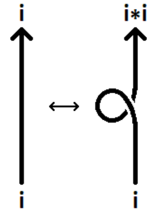

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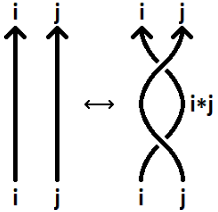

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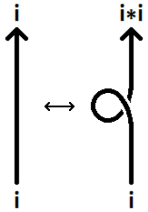

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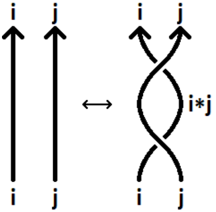

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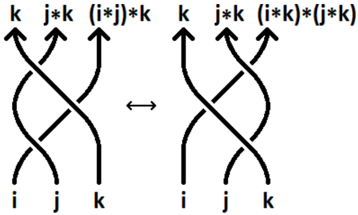

Figure 13: Type III.

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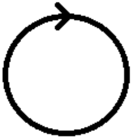

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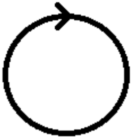

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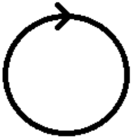

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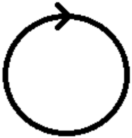

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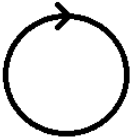

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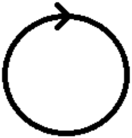

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$*$	${\color{red}\bullet}$	${\color{green}\bullet}$	${\color{blue}\bullet}$
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${\color{green}\bullet}$	${\color{blue}\bullet}$	${\color{green}\bullet}$	${\color{red}\bullet}$
${\color{blue}\bullet}$	${\color{green}\bullet}$	${\color{red}\bullet}$	${\color{blue}\bullet}$

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A Knot Invariant

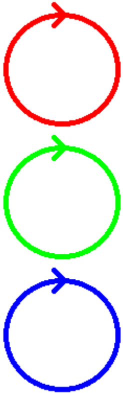

Figure 17: $|\mathrm{Hom}(K_0, Q)| = 3$;

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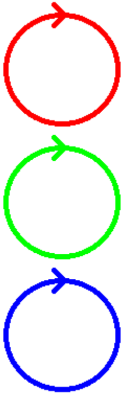

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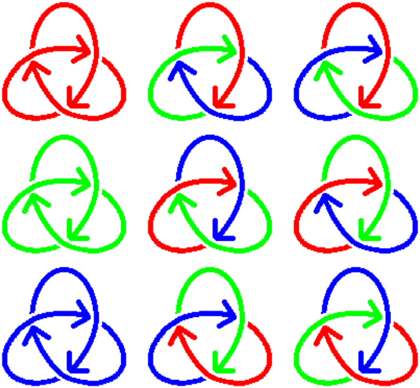
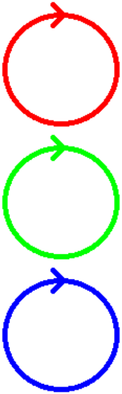


Figure 18: $|\mathrm{Hom}(K_1, Q)| = 9$.

A Knot Invariant



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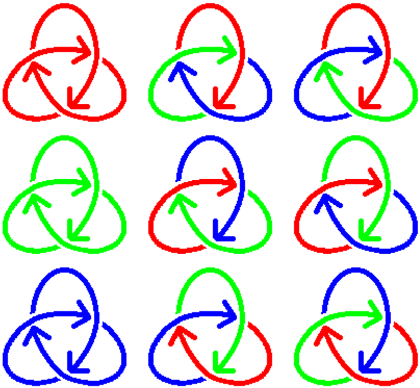

Figure 18: $|\mathrm{Hom}(K_1, Q)| = 9$.

Since $3 \neq 9$, we conclude that the unknot and the trefoil are not equivalent.

A Knot Invariant

Figure 19: $|\mathrm{Hom}(K_2, Q)| = 3$;

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We cannot decide whether the unknot and the figure-eight are equivalent.

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Is there a class of finite quandles particularly efficient in distinguishing knots?

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We cannot decide whether the unknot and the figure-eight are equivalent.

Is there a class of finite quandles particularly efficient in distinguishing knots?

Can we determine all finite quandles?

Racks

Definition

Let X be a set and let $*$ be a binary operation on X. The pair $(X, *)$ is said to be a *rack* if, for each $i, j, k \in X$,

1. $\exists! \, x \in X : x * i = j$;
2. $(i * j) * k = (i * k) * (j * k)$.

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Examples of racks:

1. Every quandle $Q = (X, *)$ is a rack;
2. For each $n \in \mathbb{N}$, $(C_n, *)$ denotes the rack whose underlying set is $\mathbb{Z}_n$ and whose operation is $i * j = i + 1 \mod n$, $\forall i, j \in \mathbb{Z}_n$. This is called the *cyclic rack of order n*;

Definition

Let $(X, *)$ and $(Y, *')$ be two racks. A map $f : X \to Y$ is said to be a *rack homomorphism* if $f(i * j) = f(i) *' f(j)$, $\forall i, j \in X$.

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A rack X is said to be *simple* if it is not the trivial rack and for every surjective rack homomorphism $f : X \to Y$ either $|Y| = 1$ or $|Y| = |X|$.

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Simple racks are the building blocks for finite racks.

Definition

A *decomposition* of a rack $(X, *)$ is a nontrivial partition $X = Y \cup Z$ such that $(Y, *)$ and $(Z, *)$ are both subracks of $(X, *)$. A rack $(X, *)$ is said to be *decomposable* if it admits a decomposition and *indecomposable* otherwise.

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$*$	Y	Z
Y	Y	
Z		Z

Table 6: A decomposable rack.

Classifying Racks and Quandles

Definition

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$*$	Y	Z
Y	Y	Y
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Table 6: A decomposable rack.

Proposition

Every rack is the disjoint union of indecomposable subracks.

Proposition

Let $(X, *)$ be a rack, let S be a non-empty set, let $\alpha : X \times X \to \mathrm{Fun}(S \times S, S)$ be a function, so that for each $i, j \in X$ and $s, t \in S$ we have $\alpha_{i,j}(s, t) \in S$, and let $\alpha_{i,j}(t) : S \to S$ denote the function given by $\alpha_{i,j}(t)(s) = \alpha_{i,j}(s, t)$. Then, $(X \times S, *')$ is a rack with respect to $(i, s) *' (j, t) = (i * j, \alpha_{i,j}(s, t))$ if and only if, for each $i, j, k \in X$ and $s, t, u \in S$, the following conditions hold:

1. $\alpha_{i,j}(t)$ is a bijection;
2. $\alpha_{i*j,k}(\alpha_{i,j}(s, t), u) = \alpha_{i*k,j*k}(\alpha_{i,k}(s, u), \alpha_{j,k}(t, u))$.

Classifying Racks and Quandles

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Definition

If the conditions in the previous proposition hold, α is said to be a *dynamical cocycle* and $X \times S$ is said to be an *extension of X by S*. This extension shall be denoted by $X \times_\alpha S$.

Definition

Let X and Y be racks and let $f : X \to Y$ be a rack homomorphism. Given an element $i \in Y$, the set $F_i := f^{-1}(i)$ is said to be a *fiber of f*.

Classifying Racks and Quandles

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Let X and Y be racks and let $f : X \to Y$ be a surjective rack homomorphism. If X is indecomposable, then Y is indecomposable and $|F_i| = |F_j|, \forall i, j \in Y$.

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Proposition

Let $(X, *)$ and $(Y, *')$ be racks and let $f : X \to Y$ be a surjective rack homomorphism such that all the fibers of f have the same cardinality. Then, X is an extension $X = Y \times_\alpha S$.

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Take a set S such that $|S| = |F_i|$. Set a bijection $g_i : F_i \to S,\ \forall i \in Y$. Let $\alpha : Y \times Y \to \mathrm{Fun}(S \times S \to S)$ be given by $\alpha_{ij}(s, t) = g_{i*'j}(g_i^{-1}(s) * g_j^{-1}(t))$. Then, $T : X \to Y \times_\alpha S$ given by $T(x) = (f(x), g_{f(x)}(x))$ is an isomorphism.

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Corollary

Every indecomposable rack is the extension of a simple rack.

Classifying Racks and Quandles

Theorem (Andruskiewitsch and Graña, 2003)

Let X be a simple rack. Then, one and only one of the following holds:

- $|X| = p$, where p is a prime, and X is the cyclic rack of order p, $(C_p, *)$;
- $|X| = p^t$, where p is a prime, and X is an affine crossed set $(\mathbb{F}_p^t, T)$;
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We conclude that $\mathbf{Inn_X}$ is a normal subgroup of $\mathbf{Aut_X}$.

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Definition

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Theorem (Equivalent Definition of Quandle)

Let X be a set and let $\mu_i : X \to X$ be a permutation assigned to each $i \in X$. Then, the expression $j * i := \mu_i(j), \forall j \in X$, yields a quandle structure if and only if $\mu_{\mu_i(j)} = \mu_i \mu_j \mu_i^{-1}$ and $\mu_i(i) = i, \forall i, j \in X$. This quandle structure is uniquely determined by the set of n permutations.

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Examples

$*$	1	2	3	4	5	6	7	8
1	1	4	2	3	1	3	4	2
2	3	2	4	1	4	2	1	3
3	4	1	3	2	2	4	3	1
4	2	3	1	4	3	1	2	4
5	5	8	6	7	5	7	8	6
6	7	6	8	5	8	6	5	7
7	8	5	7	6	6	8	7	5
8	6	7	5	8	7	5	6	8

Table 7: A decomposable quandle;

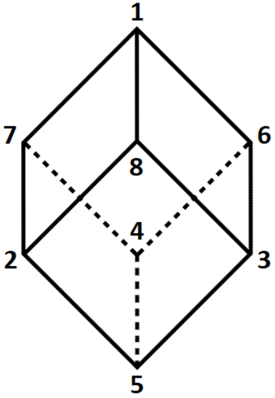

Figure 21: Cube.

Examples

$*$	1	2	3	4	5	6	7	8
1	1	4	2	3	1	3	4	2
2	3	2	4	1	4	2	1	3
3	4	1	3	2	2	4	3	1
4	2	3	1	4	3	1	2	4
5	5	8	6	7	5	7	8	6
6	7	6	8	5	8	6	5	7
7	8	5	7	6	6	8	7	5
8	6	7	5	8	7	5	6	8

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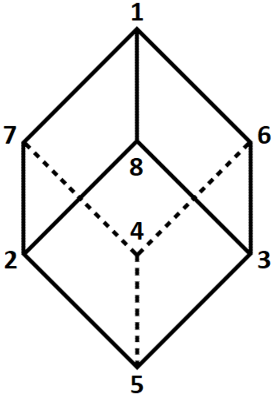

Figure 21: Cube.

The cube quandle is the disjoint union of two tetrahedron quandles.

Examples

*	1	2	3	4	5	6
1	1	1	6	5	3	4
2	2	2	5	6	4	3
3	5	6	3	3	2	1
4	6	5	4	4	1	2
5	4	3	1	2	5	5
6	3	4	2	1	6	6

Table 8: An indecomposible but not simple quandle;

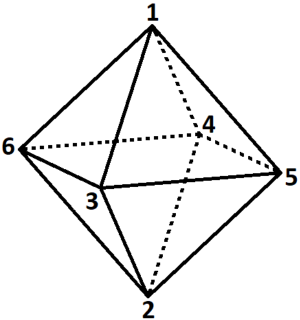

Figure 22: Octahedron.

$*$	1	2	3	4	5	6
1	1	1	6	5	3	4
2	2	2	5	6	4	3
3	5	6	3	3	2	1
4	6	5	4	4	1	2
5	4	3	1	2	5	5
6	3	4	2	1	6	6

Table 8: An indecomposible but not simple quandle;

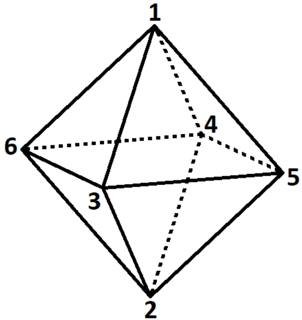

Figure 22: Octahedron.

The octahedron quandle is an extension of $(R_3, *)$ by a set S with 2 elements.

Examples

$*$	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

Table 9: A simple quandle $(X, *)$;

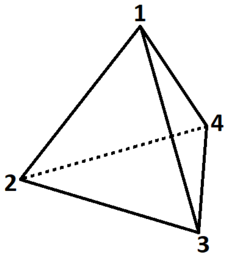

Figure 23: Tetrahedron.

Examples

$*$	1	2	3	4
1	1	4	2	3
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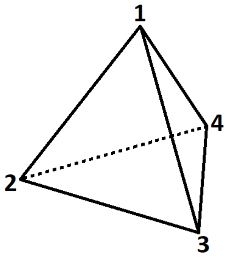

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Let $f : X \to Y$ be a quandle homomorphism onto a certain quandle $(Y, *')$.

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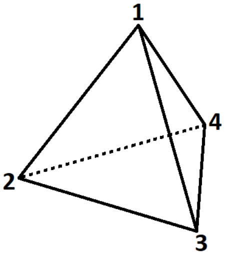

Figure 23: Tetrahedron.

Let $f : X \to Y$ be a quandle homomorphism onto a certain quandle $(Y, *')$. Assume that $f(2) = f(1)$.

Examples

$*$	1	2	3	4
1	1	4	2	3
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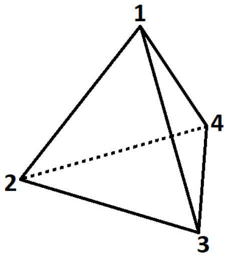

Figure 23: Tetrahedron.

Let $f : X \to Y$ be a quandle homomorphism onto a certain quandle $(Y, *')$. Assume that $f(2) = f(1)$. Then, we conclude that $(X, *)$ is simple, because:

$$f(3) = f(2 * 1) = f(2) *' f(1) = f(1) = f(1) *' f(2) = f(1 * 2) = f(4).$$

Examples

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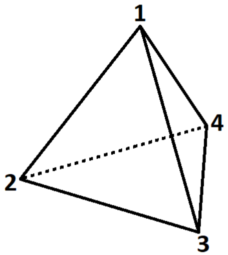

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The tetrahedron quandle is the affine crossed set $(\mathbb{F}_2^2, T)$, where $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Applications

Definition

Let X be a non-empty set and let $S : X \times X \to X \times X$ be a bijective map. The pair (X, S) is said to be a *braided set* if, for each $x, y, z \in X$:

$$(id \times S)(S \times id)(id \times S)(x, y, z) = (S \times id)(id \times S)(S \times id)(x, y, z).$$

This equation is called the *braid equation* or the *Yang-Baxter equation*.

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Proposition

Let $(X, *)$ be a rack and let $S : X \times X \to X \times X$ be the map given by $S(x, y) = (y * x, x), \forall x, y \in X$. Then, the pair (X, S) is a braided set.

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The map S is clearly bijective. For each $x, y, z \in X$, it is easy to see that:

$(id \times S)(S \times id)(id \times S)(x, y, z) = ((z * y) * x, y * x, x)$;

$(S \times id)(id \times S)(S \times id)(x, y, z) = ((z * x) * (y * x), y * x, x)$.

Thank you for your attention!

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