On the isoperimetric inequality and surface diffusion flow for multiply winding curves

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Lisbon WADE seminar (at Zoom)
23rd June 2020

Joint work with Shinya Okabe (Tohoku University)

Organization

1. Introduction (2 pages)
2. Isoperimetric Inequality (6 pages)
3. Surface Diffusion Flow (6 pages)

1. Introduction (2 pages)
    - Main characters
    - Outline

Main characters

Multiply winding curves:

- Immersed closed curves γ in $\mathbf{R}^2$ of rotation number ≥ 2.

Isoperimetric Inequality (Iso Ineq):

- Classical geometric inequality. For a closed plane curve γ,

$$\mathrm{Length}(\gamma)^2 \geq 4\pi \mathrm{Area}(\gamma).$$

Surface Diffusion Flow (SDF):

- 4th order geometric evolution equation. For closed plane curves,

$$V = -\partial_{ss}\kappa,$$

where V normal velocity, s arclength, κ curvature.

Outline

Stationary solutions:

- **SDF**'s stationary solution satisfies $0 = \partial_{ss}\kappa$.
- Since the curve is closed, curvature is constant.
- Must be a circle C_N, of an arbitrary rotation number $N \geq 1$.

Stability ($N = 1$):

- A singly winding circle C_1 is "dynamically stable".
- **Iso Ineq** $L^2 \geq 4\pi A$ comes into play in a variational proof.

Stability ($N \geq 2$):

- Multiply-winding circles C_N are "not stable".
- Lacking is **Iso Ineq** of the form $L^2 \geq 4\pi NA$ (equality for C_N).

Main results:

- **Iso Ineq**: $L^2 \geq 4\pi NA$ under rotational symmetry.
- **SDF**: Stability of C_N ($N \geq 2$) for rotationally symmetric perturbations.

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Section 2: Isoperimetric Inequality

1. Isoperimetric Inequality (6 pages)
    - Basic definitions
    - Isoperimetric ratio
    - Rotational symmetry
    - Main theorem I: Isoperimetric Inequality
    - Idea of the proof

Basic definitions

Let $\gamma : \mathbf{S}^1 \to \mathbf{R}^2$, where $\mathbf{S}^1 := \mathbf{R}/\mathbf{Z}$, be smooth and regular $|\partial_x \gamma(x)| > 0$.

- **Length**:

$$L(\gamma) := \int_\gamma ds,$$

where s denotes the arclength parameter.

- **Signed area**: counterclockwise = positive.

$$A(\gamma) := -\frac{1}{2} \int_\gamma \gamma \cdot \nu ds,$$

where $\nu := R_{\frac{\pi}{2}} \partial_s \gamma$ and R_θ: θ-rotation matrix.

- **Rotation number**:

$$N(\gamma) := \frac{1}{2\pi} \int_\gamma \kappa ds \in \mathbf{Z},$$

where $\kappa = \partial_s^2 \gamma \cdot \nu$.

Isoperimetric ratio

Isoperimetric ratio:

$$I(\gamma) := \begin{cases} \dfrac{L(\gamma)^2}{4\pi A(\gamma)} & (A(\gamma) > 0), \\ \infty & (A(\gamma) \leq 0). \end{cases}$$

Remark:

- In general, $I \geq 1$ holds.
- For N-times covered circle C_N, we have $I(C_N) \geq N$.
- If γ_R consists of two circles of radii 1 and $R \geq 1$, then

$$I(\gamma_R) = \frac{(2\pi + 2\pi R)^2}{4\pi(\pi + \pi R^2)} = 1 + \frac{2R}{1 + R^2}.$$

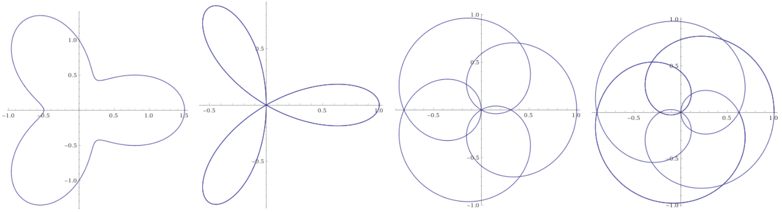
  The value $I(\gamma_R)$ decreases from 2 to 1 as $R : 1 \to \infty$.

Goal: Find a class X for which $\inf_X I = N$ holds. (N: rotation number.)

Rotational symmetry

Class $A(n,m)$:

- $A(n,m) := \{\gamma \in \mathrm{Sym}(m) \mid N(\gamma) = n\}$.



$A(1,3)$ $A(2,3)$ $A(4,3)$ $A(5,3)$

m**-th rotational symmetry**:

- $\gamma \in \mathrm{Sym}(m) \iff \exists i \in \{1,\dots,m\}$ such that $\gamma \in \mathrm{Sym}(m,i)$.
- $\gamma \in \mathrm{Sym}(m,i) \iff \gamma(x+\frac{1}{m}) = R_{\frac{2\pi i}{m}}\gamma(x)$ holds for every $x \in \mathbf{S}^1$.

Remark: The index i is characterized by n, m.

- $\gamma \in A(n,m) \implies \gamma \in \mathrm{Sym}(m, i_{n,m})$, where $i_{n,m} := n + m - m\left\lceil \frac{n}{m} \right\rceil$.
- The index $i_{n,m}$ is a unique element of $\{1,\dots,m\} \cap (n + m\mathbf{Z})$.

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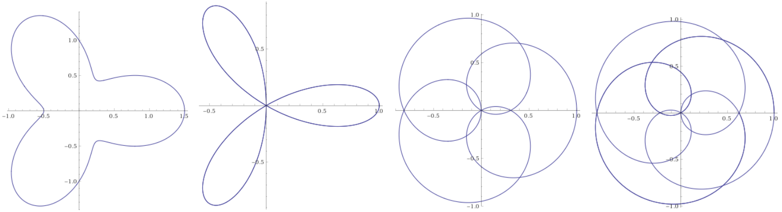

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Main theorem I: Isoperimetric Inequality

Theorem 1 (M.-Okabe)

Let $n \in \mathbf{Z}$ and $m \in \mathbf{Z}_{>0}$. Recall $A(n,m) := \{\gamma \in \mathrm{Sym}(m) \mid N(\gamma) = n\}$. Then

$$\inf_{\gamma \in A(n,m)} I(\gamma) = i_{n,m}.$$

The infimum is attained iff $1 \leq n \leq m$ ($\Leftrightarrow i_{n,m} = n$) and γ is an n-circle.

Corollary 2 (Isoperimetric Inequality)

If $1 \leq n \leq m$, then $I(\gamma) \geq n$ for $\gamma \in A(n,m)$. Equality only by an n-circle.

Remark:

- Corollary 2 has been known if γ is in addition locally convex. [Epstein-Gage'87] ($1 \leq n \leq m/2$), [Chou'03], [Süssmann'11], [Wang-Li-Chao'17].

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If $1 \leq n \leq m$, *then* $I(\gamma) \geq n$ *for* $\gamma \in A(n,m)$. *Equality only by an* n*-circle.*

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Idea of the proof

Direct method for closed curves?:

- Take a min seq $\{\gamma_j\} \subset A(n,m)$ such that $I(\gamma_j) \to \inf_{A(n,m)} I$.
- By compactness, up to subseq, $\gamma_{j'} \to \exists\bar{\gamma}$ in certain first order sense.
- If $\bar{\gamma} \in A(n,m)$, then $\bar{\gamma}$ attains $\inf$ by lower semicontinuity of I.
- BUT $N[\bar{\gamma}] = n$ may not hold in general! N is of second order.

Change the strategy:

- Just look at one period of $\gamma \in A(n,m)$.
- Direct method for open curves.

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Free boundary problem for open curves:

- $X_\theta := \{\gamma \in \mathrm{Lip}([0,1];\mathbf{R}^2) \mid \text{(Boundary Condition)}\}$.
- (BC) $\frac{\gamma(0)}{|\gamma(0)|} = (1,0)$, $\frac{\gamma(1)}{|\gamma(1)|} = (\sin\theta, \cos\theta)$, and $|\gamma(0)| = |\gamma(1)| > 0$.
- All zeroth order. Direct method applicable.

Theorem 3

For $\theta \in (0, 2\pi]$, $\min_{X_\theta} I(\gamma) = \theta/2\pi$. Equality only by a circular arc of angle θ.

Original inequality:

- One period $\gamma|_m$ of $\gamma \in A(n,m)$ lives in X_θ for $\theta := \dfrac{2\pi i_{n,m}}{m}$.
- Since $I(\gamma) = mI(\gamma|_m)$, we get $I(\gamma) \geq m \cdot \theta/2\pi = i_{n,m}$.

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Section 3: Surface Diffusion Flow

3. Surface Diffusion Flow (6 pages)
   - Quick review
   - Global existence: Singly winding case $(N = 1)$
   - Global existence: Multiply winding case $(N \geq 2)$
   - Main theorem II: Global existence for SDF
   - Sketch of the proof

Quick review

Surface Diffusion Flow: Given a smooth initial data $\gamma_0 : \mathbf{S}^1 \to \mathbf{R}^2$, consider

$$\begin{cases} \partial_t \gamma = (-\partial_s^2 \kappa)\nu & \text{on } \mathbf{S}^1 \times [0,T), \\ \gamma(\cdot, 0) = \gamma_0, \end{cases}$$

where $\gamma : \mathbf{S}^1 \times [0,T) \to \mathbf{R}^2$ is a family of immersed curves.

- $T \in (0, \infty]$: maximal existence time. ($T > 0$ by parabolicity.)

Problem: Which initial curve γ_0 admits a global solution ($T = \infty$)?

Basic facts[1]:

- Along the flow, $\frac{d}{dt}L \leq 0$ and $\frac{d}{dt}A = 0$. Hence, I is non-increasing.
- If $T = \infty$, then γ converges to an $N(\gamma_0)$-circle as $t \to \infty$.
- $\exists$ initial curve γ_0 with finite time blowup ($T < \infty$).
- If $T < \infty$, then L^2-blowup of curvature $\int \kappa^2 ds \gtrsim (T-t)^{-1/4}$.
- Convexity and embeddedness are not necessarily preserved.

[1] Cf. Giga-Ito'98,'99, Dziuk-Kuwert-Schätzle'02, Chou'03, Wheeler (arXiv:2004.08494).

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Global existence: Singly winding case ($N = 1$)

Finite time blowup:

- Example: $N(\gamma_0) = 1$ and $A(\gamma_0) \leq 0$.
  - If $T = \infty$, then the solution would converge to a counterclockwise circle. However this is impossible by the area-preserving property.

Global existence:

- If γ_0 is close to a circle, then $T = \infty$.
  - [Elliott-Garcke'97], [Escher-Mayer-Simonett'98], [Wheeler'13]

Major open problems: (not addressed in this talk)

- Finite time blowup for γ_0 embedded? (or $A(\gamma_0) > 0$?)
- Giga's conjecture: If $\gamma(\cdot, t)$ embedded for all $t \in [0, T)$, then $T = \infty$?
- Chou's conjecture: Concerning classification of "Type I" singularity.

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Finite time blowup:

- Example: $I(\gamma_0) < N(\gamma_0)$ [Chou '03].
- The above occurs even if γ_0 is close to an N-circle.

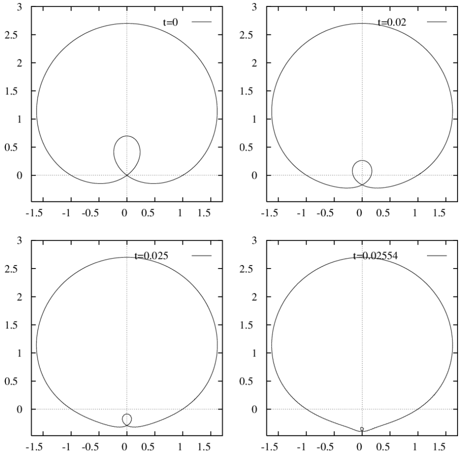

FIG. 1. *The limaçon* $r(\theta) = 1 + 1.7\sin(\theta)$.

[Escher-Mayer-Simonett '98]

Global existence: Multiply winding case ($N \geq 2$)

Global existence:

- Symmetric global solutions are known numerically.

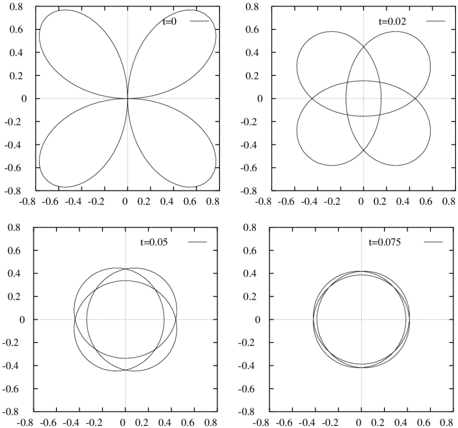

FIG. 2. *The rose* $r(\theta) = \sin(2\theta)$.

[Escher-Mayer-Simonett '98]

Main theorem II: Global existence for SDF

Theorem 4 (M.-Okabe)

Let $1 \leq n \leq m$. If $\gamma_0 \in A(n,m)$, and if γ_0 is "H^2-close" to an n-circle, then γ_0 admits a global solution to SDF, i.e., $T = \infty$.

Remark:

- The proof crucially relies on our isoperimetric inequality $I(\gamma(t)) \geq n$.
- Key point: No a priori convexity along SDF, but our isoperimetric inequality does not assume convexity!

Sketch of the proof:

- **Goal**: Prove L^2-boundedness of curvature $\Longrightarrow T = \infty$.
- Wheeler's estimate: Let $K_n^* := \frac{2\pi}{3}(\sqrt{1+3\pi n^2} - \sqrt{3\pi n^2})$. Then, as long as

$$K_{\mathrm{osc}}(\gamma(t)) := \frac{1}{L(\gamma(t))} \int_{\gamma(t)} (\kappa - \bar{\kappa})^2 ds \leq 2K_n^*,$$

  the curvature oscillation is more precisely controlled:

$$K_{\mathrm{osc}}(\gamma(t)) \leq K_{\mathrm{osc}}(\gamma_0) + 4\pi^2 n^2 \log \frac{L(\gamma_0)^2}{L(\gamma(t))^2}.$$

- By our isoperimetric inequality "$L(\gamma(t))^2 \geq 4\pi n A(\gamma(t))$",

$$\frac{L(\gamma_0)^2}{L(\gamma(t))^2} \text{“}\leq\text{”} \frac{L(\gamma_0)^2}{4\pi n A(\gamma(t))} = \frac{L(\gamma_0)^2}{4\pi n A(\gamma_0)} = \frac{I(\gamma_0)}{n}.$$

- If $K_{\mathrm{osc}}(\gamma_0) \leq K_n^*$ and $\frac{I(\gamma_0)}{n} \leq \exp(\frac{K_n^*}{8\pi^2 n^2})$, then

$$\sup_{t\in[0,T)} K_{\mathrm{osc}}(\gamma(t)) \leq \frac{3}{2} K_n^*.$$

Summary and future directions

Summary:

- Isoperimetric Inequality for rotationally symmetric curves:

$$1 \leq n \leq m, \quad \gamma \in A(n,m) \quad \Longrightarrow L^2 \geq 4\pi n A.$$

- Surface Diffusion Flow admits rotationally symmetric global solutions:

$$1 \leq n \leq m, \quad \gamma_0 \in A(n,m), \quad \gamma_0 \text{ nearly circular} \quad \Longrightarrow T = \infty.$$

Future directions:

- Iso Ineq: How about curved ambient spaces?
- SDF: More precise understanding of singularities.

– Thank you very much!