NOTES ON GIT AND SYMPLECTIC STABILITY

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Abstract. These notes are extracted from the various excellent surveys on the subject in the bibliography. They present the basic notions to construct GIT quotients and GIT stability, symplectic quotients and symplectic stability, the relation in between by the Kempf-Ness theorem and ideas about maximal unstability and different degrees of unstability. The construction of the projective space and the Grassmannian variety as GIT or symplectic quotients, and the study of the classical problem of classifying configurations of points on the projective line are the basic examples through which the introduced notions are visualized.

1. Geometric Invariant Theory

Let G be a reductive complex Lie group acting on an algebraic variety X. The purpose of Geometric Invariant Theory (abbreviated GIT) is to provide a way to define a quotient of X by the action of G with an algebro-geometric structure.

In the case when the variety X is affine there is a simpler solution which dates back to Hilbert's 14^{th} problem. Let $A(X)$ denote the coordinate ring of the affine variety X. Nagata proved that if G is reductive, the ring of invariants $A(X)^G$ is finitely generated, hence is the coordinate ring of an affine variety, therefore we can define the quotient X/G as the affine variety associated to the ring $A(X)^G$.

When taking the quotient of a projective variety by a group G, there are some issues which have to be taken into account. First one has to do with the separatedness of the quotient space and will led us to the definition of S-equivalence, or equivalence of orbits under the action of G. Here it is a simple example which shows this even in an affine case:

Example 1.1. *Consider the action*

$$\sigma : \mathbb{C}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$(\lambda, (x, y)) \longmapsto (\lambda x, \lambda^{-1} y)$$

whose orbits are represented in Figure 1. The orbits are the hyperboles $xy =$ constant, plus three special orbits, the x-axis, the y-axis and the origin. Observe that the origin is in the closure of the x-axis and the y-axis.

The coordinate ring of $\mathbb{C}^2$ is $\mathbb{C}[X, Y]$, and the ring of invariants is $\mathbb{C}[X, Y]^{\mathbb{C}^*} \simeq \mathbb{C}[XY] \simeq \mathbb{C}[Z]$. So, the ring of invariants does not distinguish between the three special orbits, and identifies them in a unique single point in the quotient space.

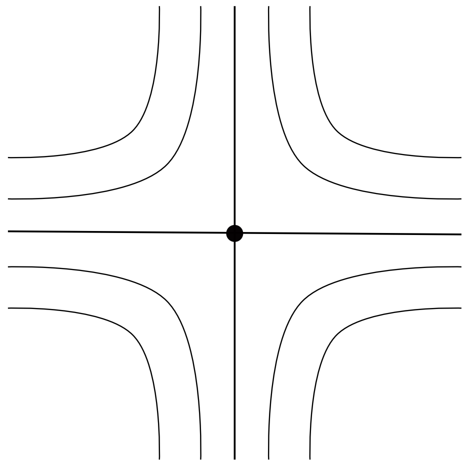

FIGURE 1. Orbits of the action in Example 1.1

Hence, the orbit space (the space where each point corresponds to an orbit) would be non separated, but the quotient space whose ring of functions is $\mathbb{C}[X,Y]^{\mathbb{C}^*} \simeq \mathbb{C}[Z]$ *is the affine line, which is separated.*

Once we now how to take quotients of the affine varieties, let us deal with the projective case. We can guess that, as projective varieties are given by gluing affine pieces, we can take the quotient of each affine piece and then glue them. As we want these pieces to be respected by the action of G we want them to be G-invariant, hence we are looking for subsets of the form

$$X_f = \{x \in X \mid f(x) \neq 0\}$$

which are G-invariant or, equivalently, looking for $f \in \mathbb{C}[X_0, \ldots, X_n]$ G-invariant.

Now, as the following example shows, note that the action of G on X projective does not determine an action on the graded ring $\mathbb{C}[X_0, \ldots, X_n]$ (or a quotient of it).

Example 1.2. *Let* $\mathbb{C}^*$ *act on* $\mathbb{P}^1_{\mathbb{C}}$ *trivially, i.e. given* $g \in \mathbb{C}^*$, $g \cdot [x_0 : x_1] = [x_0 : x_1]$. *This action is compatible with the trivial action of* $\mathbb{C}^*$ *on* $\mathbb{C}[X_0, X_1]$ *which acts as* $g \cdot f = f$, *but it is also compatible with the action* $g \cdot f = gf$ *which multiplies each homogeneous polynomial by the corresponding scalar.*

Hence, we have to linearize the action of G to $\mathbb{C}^{n+1}$ (i.e. the affine cone of X), meaning to give an action on $\mathbb{C}^{n+1}$ which is the former action of G when restricted to X. Once we have this linearization, we can consider the action on the (graded) coordinate ring of X, as we did in the affine case. We are seeking affine pieces defined as the complement of the vanishing locus of a G-invariant polynomial, then those points (or orbits) contained on the vanishing locus of ALL the G-invariant polynomials cannot appear at any of the affine pieces, hence they cannot be in our quotient. This motivates the following:

Definition 1.3. *A point $x \in X$ is called* GIT semistable *if there exists a G-invariant homogeneous polynomial f of degree ≥ 1, such that $s(x) \neq 0$. If, moreover, the orbit of x is closed, it is called* GIT polystable *and, if furthermore, this closed orbit has the same dimension as G (i.e. if x has finite stabilizer), then x is called a* GIT stable *point. We say that a closed point of X is* GIT unstable *if it is not GIT semistable.*

In the previous definition, the idea of semistable points are those which are separated by homogeneous polynomials, and the stable ones are those which are infinitesimally separated by homogeneous polynomials. Indeed, in Example 1.1, all the orbits $xy = a,\ a \neq 0$, are separated, even infinitesimally, by the homogeneous polynomial XY (the differential of the function XY along the transverse direction of the orbits is non zero) whereas for the three orbits identified (the two axes and the origin), while they are separated of the other orbits by the polynomial XY, none of them is infinitesimally separated from the rest. Hence the orbits $xy = a,\ a \neq 0$ are the stable ones and the three orbits will define the same point in the quotient, we will be defined to be equivalent (we will technically say that they are *S-equivalent*), being the three of them semistable but not stable and the origin being polystable (the unique closed orbit in the S-equivalence class).

Note that in this example there are not unstable points, as it will occur in every affine example (in affine cases all points are at least semistable because the constants are always G-invariant functions).

Remark 1.4. *In general, we consider X embedded in a projective space by the ample line bundle $\mathcal{O}_X(1)$,*

$$X \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_X(1))^\vee) = \mathbb{P}(V)\ .$$

We can see a section $s \in H^0(\mathcal{O}_X(m))$ as a homogeneous polynomial of degree m in V. Then, the GIT unstable points are those for which, for all $m > 0$, all G-invariant homogeneous polynomials vanish at the point. This way, the notion of GIT stability depends on the embedding and the linearization (i.e. it depends on a line bundle and a lifting of the action to the total space of this line bundle).

Mumford developed its Geometric Invariant Theory to give a meaningful geometric structure to the quotient X/G. It turns out that for the semistable orbits we can give a good solution to our quotient problem. Here we state the technical definition of a good quotient and the central result of Mumford's GIT.

Definition 1.5. *Let X be a projective variety endowed with a G-action. A* good quotient *is a scheme M with a G-invariant morphism $p : X \longrightarrow M$ such that*

1. *p is surjective and affine.*
2. *$p_*(\mathcal{O}_X^G) = \mathcal{O}_M$, where $\mathcal{O}_X^G$ is the sheaf of G-invariant functions on X.*
3. *If Z is a closed G-invariant subset of X, then $p(Z)$ is closed in M. Furthermore, if Z_1 and Z_2 are two closed G-invariant subsets of X with $Z_1 \cap Z_2 = \emptyset$, then $p(Z_1) \cap p(Z_2) = \emptyset$.*

Theorem 1.6. [Mu, Proposition 1.9, Theorem 1.10] *Let X^{ss} (respectively, X^s) be the subset of GIT semistable points (respectively, GIT stable). Both X^{ss} and X^s are open subsets. There is a good quotient $X^{ss} \longrightarrow X^{ss}/\!\!/G$ (where closed points are in one-to-one correspondence to the orbits of GIT polystable points), the image $X^s/\!\!/G$ of X^s is open, $X/\!\!/G$ is projective, and the restriction $X^s \to X^s/\!\!/G$ is a geometric quotient.*

Remark 1.7. *The use of double slash $/\!\!/$ in the quotient means that we make two identifications: one is the identification of the points of each orbit; the other one is the identification of S-equivalent orbits.*

Remark 1.8. *Two orbits which have non empty intersection will be called S-equivalent and will define the same point in the quotient. Geometric Invariant Theory proves that there is only one closed orbit on each equivalence class (the orbit which is called polystable). The points of the moduli space are in correspondence with these distinguished closed orbits, so the moduli space we obtain classifies polystable points, or points modulo S-equivalence.*

Next we start to analyze the first main example, the construction of the projective space as a GIT quotient.

Example 1.9. *Let $\mathbb{C}^* \curvearrowright \mathbb{C}^{n+1}$ be the scalar action, i.e. $w\cdot(z_0,\dots,z_n) = (wz_0,\dots,wz_n)$, $w \in \mathbb{C}^*$, $(z_0,\dots,z_n) \in \mathbb{C}^{n+1}$. Note that the only invariant functions will be the constants, hence to have more invariant functions and then a richer quotient space when applying GIT we will consider invariant sections of the lifted action by characters (called sometimes in the literature semi-invariants).*

Let $\mathbb{C}^{n+1}\times\mathbb{C}$ be the trivial line bundle on it and consider different linearizations of the action given by characters

$$\chi_p : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$

$$\lambda \longmapsto \lambda^{-p}$$

such that the linearized action is

$$\mathbb{C}^* \times (\mathbb{C}^{n+1}\times\mathbb{C}) \longrightarrow \mathbb{C}^{n+1}\times\mathbb{C}$$

$$(\lambda,((z_0,\dots,z_n),w)) \longmapsto ((\lambda z_0,\dots,\lambda z_n),\lambda^{-p}w)$$

If $p>0$ the invariant sections are given by the homogeneous polynomials

$$f(X_0,\dots,X_n,W) = g_{mp}(X_0,\dots,X_n)\cdot W^m\ ,$$

where g_{mp} is a degree $m\cdot p$ homogeneous polynomial on $(X_0,\dots,X_n)$. The origin will be the unique unstable point (all G-invariant homogeneous polynomials do vanish simultaneously just at the origin). The semistable locus (indeed the stable locus, given *that all rays are closed orbits in the semistable locus with maximal dimension) will be* $X^{ss} = \mathbb{C}^{n+1} - \{0\}$ *and the quotient, by Theorem 1.6, will be a projective variety which we do represent by*

$$X^{ss} /\!\!/ \mathbb{C}^* = \mathbb{P}^n_{\mathbb{C}} \,.$$

.

If $p < 0$ *note that there are no invariant sections, hence all orbits are unstable and the quotient is empty.*

If $p = 0$ *the only invariant functions are the constants (it corresponds to the trivial character), hence we cannot separate any of the orbits from the others and obtain a single point as a quotient.*

This example shows how GIT stability of the orbits depends essentially on the different choice of linearization, giving completely different GIT quotients for different linearizations.

To determine whether an orbit is GIT stable or unstable we have to calculate invariant sections or functions. This calculation is very complicate and dates back to Hilbert. One of Mumford's major achievements was to give a very simple numerical criterion to determine GIT stability, called in the literature the *Hilbert-Mumford criterion*.

It can be proved that a point x is GIT semistable if $0 \notin \overline{G \cdot \hat{x}}$, where $\hat{x}$ lies over x in the affine cone. Intuitively one direction is clear. Recall that the GIT unstable points are those for which, for all $m > 0$, all G-invariant homogeneous polynomials vanish at that point. As all homogeneous polynomials (in particular the G-invariant ones) vanish at zero, the points which contain zero in the closure of their orbits will be GIT unstable. The converse can be seen in [Ne, Proposition 4.7] or [Mu, Proposition 2.2].

The essence of the Hilbert-Mumford criterion is that GIT stability for the group can be checked through 1-parameter subgroups

$$\rho : \mathbb{C}^* \to G$$

stating that we can reach every point in the closure of an orbit through these 1-parameter subgroups, hence a point is GIT (semi)stable for the action of G if and only if it is for the action of every 1-parameter subgroup. Then, with the observation of the previous paragraph, GIT stability measures whether 0 belongs to the closure of the lifted orbit or not, belonging which can be checked through 1-dimensional path which are the 1-parameter subgroups.

Theorem 1.10. *Let* $\hat{x}$ *be a point in the affine cone over* X*, lying over* $x \in X$*.*

- x *is* semistable *if for all* 1*-parameter subgroups* ρ, $\exists \lim_{t \to 0} \rho(t) \cdot \hat{x} \neq 0$ *or* $\lim_{t \to 0} \rho(t) \cdot \hat{x} = \infty$.
- x *is* polystable *if it is semistable and the orbit of* $\hat{x}$ *is closed.*
- x *is* stable *if for all* 1*-parameter subgroups* ρ, $\lim_{t \to 0} \rho(t) \cdot \hat{x} = \infty$ *(then the stabilizer of* x *is finite).*
- x *is* unstable *if there exists a* 1*-parameter subgroup* ρ *such that* $\lim_{t \to 0} \rho(t) \cdot \hat{x} = 0$.

Given ρ a 1-parameter subgroup of G, and given $x \in X$, we can define $\Phi : \mathbb{C}^* \longrightarrow X$ by $\Phi(t) = \rho(t) \cdot x$. We say $\lim_{t\to 0}\rho(t) \cdot x = \infty$ if Φ cannot be extended to a map $\widetilde{\Phi} : \mathbb{C} \longrightarrow X$. If Φ can be extended, we write $\lim_{t\to 0}\rho(t) \cdot x = x_0$. The point x_0 is, clearly, a fixed point or the action of $\mathbb{C}^*$ on X induced by ρ. Thus, $\mathbb{C}^*$ acts on the fiber of the line bundle over x_0, say, with weight ρ_x. One defines the numerical function

$$\mu(x\rho) := \rho_x \ .$$

We will call this number ρ_x the *weight* of the action of ρ over x.

The 1-parameter subgroups induce a linear action of $\mathbb{C}^*$ in the total space of the line bundle, which we think as $\mathbb{C}^{n+1}$ for an n-dimensional projective variety X. By a result of Borel, any such action can be diagonalised such that there exists a basis $e_0, \ldots, e_n$ of $\mathbb{C}^{n+1}$ with

$$\rho(t) \cdot \hat{x} = t^{\rho_i} \hat{x}_i e_i \ .$$

Taking into account this, the previous definition of $\mu(\rho, x)$ can be restated as

$$\mu(\rho, x) = \min\{\rho_i \ : \ \hat{x}_i \neq 0\} \ .$$

Being defined $\mu(x, \rho)$ we are ready to state the Hilbert-Mumford numerical criterion of GIT stability:

Theorem 1.11 (**Hilbert-Mumford numerical criterion**). [Mu, Theorem 2.1], [Ne, Theorem 4.9] *With the previous notations:*

- x *is* semistable *if for all* 1*-parameter subgroups* ρ, $\mu(x, \rho) \leq 0$.
- x *is* polystable *if* x *is semistable and for all* 1*-parameter subgroups* ρ *such that* $\mu(x, \rho) = 0$, $\exists g \in G$ *with* $x_0 = g \cdot x$.
- x *is* stable *if for all* 1*-parameter subgroups* ρ, $\mu(x, \rho) < 0$.
- x *is* unstable *if there exists a* 1*-parameter subgroup* ρ *such that* $\mu(x, \rho) > 0$.

Example 1.12. *In example 1.9 we can easily check the GIT stability of the orbits by using the numerical Hilbert-Mumford criterion. In this case there is essentially one* 1*-parameter subgroup up to rescaling, hence we can directly calculate the minimal relevant weight for the action of the group.*

For all p*, the action of* $\mathbb{C}^*$ *can be extended to the origin in* $\mathbb{C}^{n+1}$ *which is a fixed point for the action. On the fiber over the origin, the action is given by multiplying by* $\cdot\lambda^{-p}$*, hence for all points* $x \neq 0$ *the minimal relevant exponent is* $\rho_x = -p$ *for this "unique"* 1*-parameter subgroup we are allowed to consider. Therefore, by the Hilbert-Mumford criterion, if* $p < 0$ *all points* $x \neq 0$ *are GIT unstable and if* $p > 0$ *all points* $x \neq 0$ *are stable. When* $p < 0$ *it is also clear that the origin is unstable because the weight is* $-p$ *which is positive. But when* $p > 0$ *we can "choose another"* 1*-parameter subgroup (for example with* $\cdot\lambda^{2p}$*) to obtain a positive weight too, giving the unstability for the point. Essentially, the origin is a fixed point for the action and there is no possible linearization making it stable.*

If $p = 0$ we have $\rho_x = 0$ for all points x, all orbits are semistable and S-equivalent and the only polystable orbit is the origin, because it is the limit $x_0 = 0$ not contained in any other orbit but a fixed point.

The next example is the fundamental one: the moduli space of binary forms or configurations of n points in the projective line. It is originally due to Hilbert and it is the starting point for this theory.

Example 1.13. *Let N be an integer and consider the set of all homogeneous polynomials of degree N in two variables with coefficients in $\mathbb{C}$, $V_N = \{f(X,Y) = a_0Y^n + a_1XY^{n-1} + a_2X^2Y^{n-2} + \cdots + a_{n-1}X^{n-1}Y + a_nX^n \mid a_i \in \mathbb{C}\}$. Let $\mathbb{P}(V_N)$ be its projectivization. The zeroes of an element $\overline{f} \in \mathbb{P}(V_N)$ define n points in $\mathbb{P}^1_{\mathbb{C}}$ counted with multiplicity, up to action of the group $G = SL(2,\mathbb{C})$:*

$$\begin{array}{ccc} SL(2,\mathbb{C}) \times \mathbb{P}(V_N) & \longrightarrow & \mathbb{P}(V_N) \\ (g,\overline{f}) & \longmapsto & \overline{f}(g^{-1}(X,Y)) \end{array} .$$

The orbit space $\mathbb{P}(V_N)/G$ is not a variety, because it is not Hausdorff. To see this, let $\overline{f}$ and $\overline{g}$ be represented by $f = X^n$ and $g = X^n + X^{n-1}Y$ respectively. The orbits of these two elements are disjoint because f has the root $[0:1]$ counted with multiplicity n and g has the root $[0:1]$ counted with multiplicity $n-1$ and the simple root $[1:-1]$. Let $h_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ be a curve of elements in $SL(2,\mathbb{C})$ and define

$$g_t := h_t \cdot g = g(h_t^{-1}(X,Y)) = g(t^{-1}X, tY) = t^{-n}X^n + t^{-n+2}X^{n-1}Y .$$

For each t, g_t defines an element in $\mathbb{P}(V_N)$ which can be represented (by rescalling) by $\overline{g_t} = X^n + t^2X^{n-1}Y$. Then, note that when t goes to 0, $\overline{g_t}$ tends to $X^n = \overline{f}$, therefore $\overline{f}$ lies in the closure of the orbit of $\overline{g}$ and the orbit space is not Hausdorff.

In order to construct a GIT quotient we are going to apply the Hilbert-Mumford criterion. The 1-parameter subgroups of $SL(2,\mathbb{C})$ can be diagonalized to be represented by a diagonal matrix as

$$\rho_k(t) = \begin{pmatrix} t^{-k} & 0 \\ 0 & t^k \end{pmatrix}$$

such that if we write $f(X,Y) = \sum_{i=0}^n a_iX^iY^{n-i}$ the action of ρ is given by

$$\rho_k(t)(f) = f(\rho_k(t)^{-1} \cdot (X,Y)) = f(t^kX, t^{-k}Y) = \sum_{i=0}^n a_i t^{k(2i-n)} X^iY^{n-i} .$$

The limit $f_0 = \lim_{t\to 0} \rho_k(t) \cdot f$ is equal to the monomial $a_{i_0}X^{i_0}Y^{n-i_0}$, where i_0 is the minimum index such that $a_i = 0$. For example, if $f = XY^4 + X^3Y^2 + X^5$, then $\rho_k(t)(f) = t^{-3k}XY^4 + t^kX^3Y^2 + t^{5k}X^5 \sim XY^4 + t^{4k}X^3Y^2 + t^{8k}X^5$, (when considering the projectivization) which tends to $XY^4 = f_0$ when t goes to 0.

Note that the weight ρ_k acts on the fiber of f_0 is $\rho_{k,f} = k(2i_0 - n)$ (in the example $\rho_{k,f} = -3k$). The Hilbert-Mumfurd criterion states that a point $\overline{f}$ is unstable if there exists a 1-parameter subgroup such that this weight is positive. Also observe that, up to conjugation in $SL(2,\mathbb{C})$ (or change of homogeneous coordinates $[X : Y]$) all 1-parameter subgroups are of the form diagonal form ρ_k hence all are classified by the exponent k. Therefore, f is unstable if, after a change of coordinates $f(X,Y) = \sum_{i=0}^{n} a_i X^i Y^{n-i}$, there exists a 1-parameter subgroup ρ_k such that $k(2i_0-n) > 0$ where i_0 is the minimum index such that $a_{i_0} \neq 0$. Given that $k(2i_0 - n) > 0 \Leftrightarrow i_0 > \frac{n}{2}$, it is equivalent to say that f is unstable if and only if f has a root of multiplicity greater that $n/2$.

In the example $f = XY^4 + X^3Y^2 + X^5$, the weight is $\rho_{k,f} = -3k < 0$, and the lifted orbit $\rho_k(t)(f) = t^{-3k}XY^4 + t^kX^3Y^2 + t^{5k}X^5$ tends to infinity when t goes to 0, hence this 1-parameter subgroup does not destabilize the point f. Indeed, it will occur the same with all 1-parameter subgroups as it is easy to check, because f has no root of multiplicity ≥ 3. However, the point $g = X^3Y^2$ will be acted by ρ_k as $\rho_k(t)(g) = t^kX^3Y^2$, which goes to 0 when t goes to 0. Hence, 0 is in the closure of the lifted orbit and the weight is $k > 0$, then the point is GIT unstable. Indeed g has a root with multiplicity 3 (in these coordinates the root is $[1 : 0]$).

Observe that, if n is odd, we cannot have $i_0 = \frac{n}{2}$, hence we cannot have strictly semistable points and all the GIT semistable points will be GIT stable.

If n is even, we can observe the S-equivalence phenomenon. Let $n = 4$ and consider the points $f = X^2Y^2 + X^3Y + X^4$ and $g = X^2Y^2$. By the same argument we used to show that the orbit space is not Hausdorff it is clear that f (with 2 roots equal and the other two different) and g (with roots equal pairwise) do not lie in the same orbit but g lies in the closure of the orbit of f. Hence the 2 points are S-equivalent. To determine which one is the only polystable orbit within this equivalente class we can use a 1-parameter subgroup of type ρ_2 which acts on the fiber of the limit point (common to f and g and indeed equal to g) with weight zero to conclude that g is the polystable orbit.

Remark 1.14. *The moduli space of configurations of n points in the projective line is the same that the moduli space of n-gons if we consider the isomorphism $\mathbb{P}^1_{\mathbb{C}} \simeq S^2$ and see points in $\mathbb{P}^1_{\mathbb{C}}$ as length unit vectors. A configuration of points will be unstable if there is a point with multiplicity more than half the points, the same way a polygon will be unstable if there is any of the vectors repeated more that half times. It can be shown that an unstable polygon does not close, in the sense that, after any change of coordinates by $SL(2,\mathbb{C})$, the sum of the vectors is not zero.*

2. Symplectic stability

In this section we will sketch the symplectic reduction procedure, giving the other side of the stability picture. The Kempf-Ness theorem will be the link in between the two of them. Let us begin by reviewing the basics about symplectic geometry.

Let (X, ω) be a symplectic manifold, where X is a smooth manifold and $\omega \in \Omega^2(X)$ is a closed non-degenerate two form (called a symplectic form). Two symplectic varieties (X_1, ω_1) and (X_2, ω_2) are *symplectomorphic* if there exists an diffeomorphism $\varphi : X_1 \to X_2$ such that $\varphi^* \omega_2 = \omega_1$. By Darboux's theorem every symplectic manifold is locally symplectomorphic to $\mathbb{R}^{2n}$ equipped with the standard symplectic 2-form $\sum_{i=1}^n dq_i \wedge dp_i$.

Given a symplectic manifold (X, ω), let $\mathrm{Symp}(X, \omega) \subset \mathrm{Diff}(X)$ be the group of symplectomorphism and let $\mathrm{Vect}^s(X) \subset \mathrm{Vect}(X)$ be the Lie subalgebra of symplectic vector fields $v \in \mathrm{Vect}(X)$ such that $\mathcal{L}_v \omega = d(\iota_v \omega)0$. Given a smooth function $H \in \mathcal{C}^\infty(X, \mathbb{R})$, it defines a symplectic vector field ξ_H by $\iota_{\xi_H} \omega = dH$. Observe that the image of $\mathcal{C}^\infty(X, \mathbb{R})$ lies in the subalgebra $\mathrm{Vect}^s(X)$ of symplectic vector fields. In local Darboux coordinates, ξ_H is given by

$$\xi_H = \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \ ,$$

from which we can see, by remembering the Hamilton equations, how symplectic geometry gives the natural framework for mechanics. We call $d\mathcal{C}^\infty(X, \mathbb{R}) = \mathrm{Vect}^H(X)$ the Hamiltonian vector fields. Given that d has kernel the constants, the Lie algebra of the hamiltonian automorphisms is $\mathcal{C}^\infty(X, \mathbb{R})/\mathbb{R}$.

Let K be a compact connected Lie group acting on a symplectic manifold (X, ω). We say that the action is symplectic if it preserves the symplectic form, i.e. $k_X \in \mathrm{Symp}(X, \omega)$, $\forall k \in K$. We say that the action is *hamiltonian* if the map $\mathfrak{k} \to \mathrm{Vect}(X)$ (which sends an element $\xi \in \mathfrak{k} = Lie(K)$ to the corresponding vector field in X) lifts, equivariantly by the action of K, to a hamiltonian vector field ξ_H, $H \in \mathcal{C}^\infty(X, \mathbb{R})$ such that $\iota_{\xi_H} \omega = dH$. In this case, we can define a moment map

$$\mu : X \to \mathfrak{k}^*$$

by the condition $\iota_\xi \omega = -d\langle \mu, \xi \rangle$, $\forall \xi \in \mathfrak{k}$. Given that the Lie algebra of the hamiltonian automorphisms is $\mathcal{C}^\infty(X, \mathbb{R})/\mathbb{R}$ we can choose each element ξ_H up to a constant; hence the lifting condition means that we choose these constants in such a way that μ is K-equivariant (by the coadjoint action on the right hand side). Therefore, given a hamiltonian K-action, the moment map is unique up to the addition of a central of $\mathfrak{k}^*$.

In the following, let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a projective variety with an action of a compact connected Lie group K, whose complexified group is G (which is, hence, reductive). For symplicity consider that $G \subset GL(n+1, \mathbb{C})$ and $K \subset U(n+1)$. Suppose that K acts on $\mathbb{P}^n_{\mathbb{C}}$ by preserving the almost-complex structure J and the Fubiny-Study metric g, hence K preserves the natural symplectic structure $\omega = g(\cdot, J\cdot)$. In this case there is a natural moment map which, for $K = U(n)$ and identifying the Lie algebra $\mathfrak{u}(n)$ with its dual via the inner product $(A, B) = \mathrm{trace}(A^* B)$, is given by $\mu : \mathbb{P}^n_{\mathbb{C}} \to \mathfrak{u}(n)^*$,

$$\mu(z) = \frac{i}{2} z z^* \tag{1}$$

up to addition of a central element which in this case is a constant. When we have a diagonal action on a product of symplectic varieties it can be proved that the moment map is the sum of the respective moment maps.

Remark 2.1. *The different moment maps for a given action correspond with the different polarizarions and linearizations of the action from the Geometric Invariant Theory side. If the symplectic form ω is integral, meaning that its cohomology class lies in $H^2(X,\mathbb{Z})/torsion \leq H^2(X,\mathbb{R})$, then $2\pi i\omega$ is the curvature of an hermitian line bundle L with unitary connection and the isometries of L preserving the connection cover the hamiltonian authomorphisms on X.*

In the projective case, the cohomology class is integral, hence we can develop this prequantization *to restrict to a discrete number of different moment maps, associated to the GIT linearizations.*

In the symplectic setting we state the following notion of stability.

Definition 2.2. *Let (X,ω) be a projective variety with the symplectic form coming from the Fubini-Studi metric, endowed with a hamiltonian K-action. Let μ be a moment map for this action. Let x be a point of X and let us denote by $G\cdot x$ its orbit by the complexified group $G=K^{\mathbb{C}}$.*

- *x is* μ-semistable *if $\overline{G\cdot x}\cap\mu^{-1}(0)\neq\emptyset$.*
- *x is* μ-polystable *if $G\cdot x\cap\mu^{-1}(0)\neq\emptyset$.*
- *x is* μ-stable *if x is μ-polystable and, in addition , the stabilizer of x under G is finite.*
- *x is* μ-unstable *if $\overline{G\cdot x}\cap\mu^{-1}(0)=\emptyset$*

The notions of GIT stability and μ-stability will be equivalent by the Kempf-Ness theorem.

Theorem 2.3 (**Kempf-Ness Theorem**)**.** [KN] *Let (X,ω) be a projective variety with the symplectic form coming from the Fubini-Studi metric, endowed with a hamiltonian K-action. Let μ be a moment map for this action. A G-orbit contains a zero of the moment map if and only if it is GIT polystable. A G-orbit is GIT semistable if and only if its closure contains a zero of the moment map and this zero lies in the unique GIT polystable orbit in the closure of the original orbit.*

We will make some considerations to sketch the proof of the Kempf-Ness theorem.

Let $(X,L=\mathcal{O}_X(1))$ be a projective polarized variety and choose an hermitian metric on L inducing a connection with curvature $2\pi i\omega$. Lift a point $x\in X$ to $\hat{x}\in L_x^{-1}$ and consider the functional norm $\|\hat{x}\|$. If $X\subset\mathbb{P}(H^0(L)^*)$ and we consider a metric in $H^0(L)^*$, it induces a metric in the total space of L^{-1} where $\|\hat{x}\|$ is the norm in the vector space where the affine cone $\hat{X}$ lives.

For each $\hat{x}$, define the Kempf-Ness function

$$\psi_{\hat{x}}:\mathfrak{k}\to\mathbb{R},\quad \xi\mapsto\frac{\log\|\exp(i\xi)\hat{x}\|^2}{2}\ .$$

The 1-parameter subgroups encoding GIT stability by the Hilbert-Mumford criterion can be thought as different directions in the G-orbit, hence different elements of the Lie algebra $Lie(G) = \mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. To study how this function varies alog 1-parameter subgroups we calculate

$$\partial_\lambda \psi_{\hat{x}}(\xi) = \frac{d}{dt}|_{t=0} \frac{\log \| \exp(i(\xi + t\lambda))\hat{x}\|^2}{2} =$$

$$\frac{\langle i\lambda \exp(i\xi)\tilde{x}, \exp(i\xi)\hat{x}\rangle}{\langle \exp(i\xi)\hat{x}, \exp(i\xi)\hat{x}\rangle} = 2\mu((\exp i\xi)x)(\lambda) \ ,$$

which can be expressed by saying that the Kempf-Ness function is an integral of the moment map. If we calculate the second derivative

$$\partial_\nu \partial_\lambda \psi_{\tilde{x}}(\xi) = 2\langle \mathcal{L}_{J\nu}\mu((\exp i\xi)x), \lambda\rangle =$$

$$(\omega(\lambda, J\nu))(\exp(i\xi)x) = g(\lambda, \nu)(\exp(i\xi)x)$$

which is non negative, since g is a Riemannian metric.

Hence, the Kempf-Ness function is convex, attaining a minimum at the zeroes of the function $\mu((\exp i\xi)x)$ which are the zeroes of the moment map. This way, x is μ-polystable if and only if $\psi_{\hat{x}}$ attains a minimum. If the Kempf-Ness function is bounded from below it does not necessarily attain a minimum but, if it does asymptotically, it means that the closure of the G-orbit of the point contains a zero of the moment map and the point is μ-semistable. The unstable points will be those for which the Kempf-Ness function is not bounded from below or, equivalently, the orbit under the complexified group does not intersect the zeroes of the moment map.

The GIT unstable points are those x for which $0 \in \overline{G \cdot \hat{x}}$, where $\hat{x}$ lies over x in the affine cone. From the definition of the Kempf-Ness function $\psi_{\hat{x}}$ in terms of the logarithm, $0 \in \overline{G \cdot \hat{x}}$ will be equivalent to $\psi_{\hat{x}}$ not to be bounded by below, which is equivalent to the μ-unstability of x.

From this, we can define the symplectic quotient:

Theorem 2.4 (**Meyer, Marsden-Weinstein**). *Let (X, ω) be a symplectic manifold endowed with a hamiltonian action of a compact connected Lie group K. If μ is a moment map for this action, and K acts freely and properly on $\mu^{-1}(0)$, the quotient $(\mu^{-1}(0)/K, \omega_0)$ is a smooth symplectic manifold with $i^*\omega = p^*\omega_0$, where $i : \mu^{-1}(0) \to X$ is the inclusion and $p : \mu^{-1}(0) \to \mu^{-1}(0)/K$ the projection, respectively.*

By the Kempf-Ness theorem, we will have the following bijection relating the GIT and the symplectic quotients, which is indeed an isomorphism:

$$\mu^{-1}(0)/K \simeq X^{ps}/G = X^{ss} /\!\!/ G \ .$$

Next, we will calculate the moment map for the examples studied from the algebraic setting and check that the Kempf-Ness theorem holds in these cases.

Example 2.5. *Let us go back to Example 1.9. The compact group in this case is $K = U(1) \subset GL(1,\mathbb{C}) = \mathbb{C}^*$. In this case the different moment maps are given by (c.f. (1) and* [Wo]*)*

$$\mu : \mathbb{C}^{n+1} \longrightarrow \mathfrak{k}^* = \mathfrak{u}(1)^* \simeq \mathbb{R}$$

$$(z_0,\dots,z_n) \longmapsto \tfrac{i}{2}(|z_0|^2 + \cdots |z_n|^2 - a)$$

where a comes from a central element of $\mathfrak{k}^* = \mathfrak{u}^*$, which in this case is any real number. If we add the condition that the lifted action of $\mathfrak{u}$ descends to an action of the group $K = U(1)$ on the trivial line bundle we have that $a \in \mathbb{Z}$. The different $a \in \mathbb{Z}$ correspond to the integers p of the different characters in Example 1.9.

If $a < 0$ there are no $\mathbb{C}^*$-orbits intersecting $\mu^{-1}(0)$, not even in the closure, hence all points are μ-unstable as well as they were GIT unstable.

If $a = 0$, the origin in $\mathbb{C}^{n+1}$ is μ-polystable because its orbit intersects $\mu^{-1}(0)$ and all the other orbits are μ-semistable but not μ-polystable because their closures intersect $\mu^{-1}(0)$. The origin in in the closure of all orbits, hence it is the unique polystable point in the unique S-equivalence class. Therefore, the symplectic quotient is again a single point.

If $a > 0$, the origin is μ-unstable because its orbit does not intersect $\mu^{-1}(0)$. All the other orbits intersect $\mu^{-1}(0)$ at some $(z_0,\dots,z_n)$ such that $\sum_{i=0}^n |z_i|^2 = a$, hence all rays are μ-polystable (indeed μ-stable) and the quotient is the expected projective space $\mathbb{P}^n_{\mathbb{C}}$.

Example 2.6. *Now we recall the classification of configurations of n points in $\mathbb{P}^1_{\mathbb{C}}$, as in Example 1.13. Identify each $f \in \mathbb{P}(V_n)$ with the set of its n zeroes counted with multiplicity and, by the isomorphism $\mathbb{P}^1_{\mathbb{C}} \simeq S^2$, identify them with n vectors in the unit sphere. The compact group now is $S0(3,\mathbb{R}) \subset SL(2,\mathbb{C})$, acting diagonally on $(S^2)^n$ by rotations. The Lie algebra of $SO(3,\mathbb{R})$ is $\mathfrak{so}(3,\mathbb{R}) \simeq \mathbb{R}^3$ and the moment map in this case is just the sum of the inclusions of each vector in $\mathbb{R}^3$, hence given by (c.f.* [Wo]*)*

$$\mu : (S^2)^n \longrightarrow \mathfrak{su}(2)^* \simeq \mathbb{R}^3 \quad .$$

$$(v_1,\dots,v_n) \longmapsto v_1 + \cdots v_n$$

Then, a configuration of points will be μ-semistable if and only if the associated n-tuple of vectors $(v_1,\dots,v_n)$ (up to action of the complexified group $SL(2,\mathbb{C})$), verify $\sum_{i=1}^n v_i = 0$, which is the equivalent to say that a "polygon closes", identifying this problem with the moduli space of polygons.

Since the Kempf-Ness theorem asserts that μ-stability is equal to GIT stability, this means that a configuration of n points in $\mathbb{P}^1_{\mathbb{C}}$ can be moved, by an element of $SL(2,\mathbb{C})$, such that the corresponding n-tuple of vectors in S^2 (counted with multiplicity) have *center of mass the origin, if and only if there is no point with multiplicity greater than half the total, which means that the point is semistable.*

In the case n is even, we can have a point with multiplicity exactly half the total (recall that this meant the point is GIT semistable but not stable). The polynomials $f = X^2Y^2 + XY^3 + Y^4$ and $g = X^2Y^2$ verify that $g \in \overline{SL(2,\mathbb{C}) \cdot f}$. The polynomial g defines a configuration with only two points each of the same multiplicity equal to half the total, say $[1:0]$ and $[0:1]$, which in S^2 can be thought as the vectors $(0,0,1)$ and $(0,0,-1)$. Then, g is the only polystable orbit in the closure of the orbit of f which defines a (degenerate) configuration of vectors in the unit sphere with center of mass the origin, therefore $\mu^{-1}(0) \cap G \cdot g \neq \emptyset$ and $\mu^{-1}(0) \cap \overline{G \cdot f} \neq \emptyset$ but $\mu^{-1}(0) \cap G \cdot f = \emptyset$, meaning that f is μ-semistable but not μ-polystable and g is μ-polystable. By visualizing polygons, this situation in general corresponds to the degenerate polygon with $n/2$ vectors equal to v and the other $n/2$ equal to $-v$ lying on a line, which only can appear for n even. This limit point corresponds to the polystable orbit with stabilizer $\mathbb{C}^*$.

Example 2.7. *We will obtain the Grassmannian as a GIT quotient and as a symplectic quotient.*

Let $SL(r,\mathbb{C}) \curvearrowright \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $r < n$, be the action such that $A \cdot g^{-1}$ for $A \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $g \in SL(r,\mathbb{C})$, and linearize the induced action on the projectivized vector space $\mathbb{P}(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n))$ to the tautological line bundle by the natural way,

$$g \cdot ([A], \lambda) = \lambda A \cdot g^{-1} \ ,$$

where λ is an element of the fiber of the tautological line bundle lying over $[A]$. The points of the Grassmannian of r-planes in $\mathbb{C}^n$ will correspond to injective homomorphisms from $\mathbb{C}^r$ to $\mathbb{C}^n$, up to change of basis. This change of basis is encoded by considering the projectivized $\mathbb{P}(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n))$ (two linear maps differing by multiplication of a scalar define the same r-plane) and by the action of $SL(r,\mathbb{C})$ (changes of frame with determinant 1).

Hence, let us prove that $[A] \in \mathbb{P}(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n))$ is GIT stable if and only if $A \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ has rank r.

If $\mathrm{rk}\, A < r$, pick a basis $\{v_1, \ldots, v_r\}$ of $\mathbb{C}^r$ such that $v_1 \in \mathrm{Ker}\, A$. Choose a 1-parameter subgroup ρ adapted to the basis such that it has the diagonal form

$$\begin{pmatrix} t^{r-1} & 0 & \cdots & 0 \\ 0 & t^{-1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t^{-1} \end{pmatrix}$$

Then,

$$A \cdot \rho^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \cdot \begin{pmatrix} t^{1-r} & 0 & \cdots & 0 \\ 0 & t^{1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t^{1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & t \cdot * & \cdots & t \cdot * \\ \vdots & \vdots & & \vdots \\ 0 & t \cdot * & \cdots & t \cdot * \end{pmatrix} = t \cdot A \ ,$$

hence ρ fixes $[A] \in \mathbb{P}(\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n))$ and acts on the fiber $\mathbb{C}\cdot A$ as $\cdot t$, this is with weight $\rho_A = 1 > 0$, therefore $[A]$ is GIT unstable.

Conversely, if A has full rank, up to action of $SL(2,\mathbb{C})$ there exists a splitting $\mathbb{C}^n \simeq \mathbb{C}^r \oplus \mathbb{C}^{n-r}$ where A is the inclusion of the first factor in this splitting. Given ρ a 1-parameter subgroup of $SL(2,\mathbb{C})$, we can assume that we can choose a basis which both diagonalizes ρ and agrees with the splitting. Then, ρ is

$$\begin{pmatrix} t^{\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & t^{\lambda_2} & 0 & \cdots & 0 \\ 0 & 0 & t^{\lambda_3} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t^{\lambda_r} \end{pmatrix}$$

and assume further that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, with $\sum_{i=1}^r \lambda_i = 0$. Note that, by rescalling in $\mathbb{P}(\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n))$, the action of ρ in $[A]$, i.e. $[A]\cdot\rho^{-1}$, is the same that $[A]\cdot\rho^{-1}\cdot t^{\lambda_1}$. Then, the diagonal of $\rho^{-1}\cdot\lambda_1$ is $(1,\ldots,1,t^{-\lambda_i+\lambda_1},\ldots,t^{-\lambda_r+\lambda_1})$, where all $-\lambda_i+\lambda_1 > 0$ (if ρ is not trivial). When we take the limit $t\to 0$, A tends to A_0 where A_0 represents the inclusion of $\mathbb{C}^p$ as the first p vectors of the basis in $\mathbb{C}^n$ (p is the number of 1's in the diagonal of $\rho^{-1}\cdot\lambda_1$, equal to the number of exponents λ_1 in ρ). Finally, the weight of ρ in the fiber over the limit point A_0 is $-\lambda_1 < 0$, and $[A]$ is GIT stable.

Equivalently, from the symplectic point of view, we have the action of the unitary group $U(r)\subset GL(r,\mathbb{C})$ acting on $\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n)$ the same way. By considering the inner product $(A,B)=\mathrm{trace}(A^*B)$ which identifies $\mathfrak{u}^*(n)$ with $\mathfrak{u}(n)$, a moment map for the action is (c.f. (1))

$$\mu : \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n) \longrightarrow \mathfrak{u}(r)^*$$

$$A \longmapsto \tfrac{i}{2}(A^*A - Id)$$

Hence, $\mu^{-1}(0)$ are those matrices such that, up to action of $GL(r,\mathbb{C})$, verify $A^*A = Id$, which is to say that a linear map is congruent by $GL(r,\mathbb{C})$ to an isometric embedding if and only if it is injective.

In general, we could have added a central element (in this case a scalar) to the moment map to get $\mu(A)=\frac{i}{2}(A^*A-\tau\cdot Id)$. If $\tau > 0$ we obtain the same result. If $\tau = 0$ the quotient is a single point and if $\tau < 0$ all points are μ-unstable. This corresponds to different linearizations in the GIT problem.

3. Maximal unstability

After studying the relation between GIT stability and symplectic stabilily by the Kempf-Ness theorem, in this section we will focus on the unstable locus. We will classify the unstable points by degrees of unstability and will check that this notion agrees when considered from both points of view.

The moment map $\mu : X \to \mathfrak{k}^*$ is invariant by the adjoint action of the compact group K but not by the action of its complexified group $G = K^{\mathbb{C}}$. If we choose an inner product $\langle \cdot, \cdot \rangle$ in $\mathfrak{k}$, invariant by K, we can identify $\mathfrak{k}^*$ with $\mathfrak{k}$ and define the function $\|\mu\| : X \to \mathbb{R}$ by $\|\mu(x)\| = \langle \mu(x), \mu(x) \rangle$, to which we will refer as the *moment map square*. Recall that the Kempf-Ness function is an integral of the moment map. The μ-unstable points are those x for which $\mu(g \cdot x)$ does not achieve zero as a limit point, for $g \in G$, hence the Kempf-Ness function for these points is unbounded.

Define the function $\Omega_x(g) = \|\mu(g \cdot x)\|$, $g \in G$. The function Ω_x is a Morse-Bott function and it takes some infimum value $m_x \geq 0$ at the critical set. The idea is that there exists a direction of maximal descense for the negative gradient flow of the Kempf-Ness function, directions thought as cosets in G/K, minimizing the moment map square, i.e. the function Ω_x (c.f. [Ki] and [GRS]). Then, the G-orbit of a μ-unstable point x does not achieve $\Omega_x^{-1}(0)$ but it achieves, in their closure, $\Omega_x^{-1}(m_x)$ for some positive number m (c.f. Moment limit theorem [GRS, Theorem 6.4] and Generalized Kempf Existence Theorem [GRS, Theorem 11.1]). Of course, for the μ-semistable ones this infimum m_x is zero.

From the algebraic point of view, recall that a point x is GIT unstable if there exists a 1-parameter subgroup ρ such that the weight ρ_x is positive (recall that the number ρ_x is the weight ρ is acting with on the fiber of the fixed limit point of $\rho(t)$ when t goes to zero). Having chosen the inner product $\langle \cdot, \cdot \rangle$ in $\mathfrak{k}$, it extends uniquely to an inner product in G. Considering the 1-parameter subgroups as directions given by elements in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, it makes sense to define the norm $\|\rho\|$ of a 1-parameter subgroup and define the function $\Phi_x(\rho) = \rho_x / \|\rho\|$. If x is GIT unstable, there exists ρ such that $\Phi_x(\rho) > 0$. The result in [Ke] asserts that the supremum of the function Φ_x is attained at some unique ρ (up to conjugation by the parabolic subgroup of G defined by ρ), hence there exist a unique 1-parameter subgroup maximizing the Hilbert-Mumford criterion, or giving the maximal way to destabilize a GIT unstable point. The norm in the denominator serves to calibrate this maximal degree of unstability when rescalling (i.e. multiplying the exponents of the 1-parameter subgroups by a scalar).

The principal result in [GRS] (c.f. [GRS, Theorem 13.1]) shows that, for x an unstable point,

$$\sup_{\rho \in \mathfrak{g}} \Phi_x(\rho) = \sup_{\rho \in \mathfrak{g}} \frac{\rho_x}{\|\rho\|} = m_x = \inf_{g \in G} \Omega_x = \inf_{g \in G} \|\mu(g \cdot x)\| \ ,$$

this is, the weight of the 1-parameter subgroup which maximally destabilizes a GIT unstable point x (after normalization) is the infimum of the moment map square over the G-orbit of a μ-unstable point.

Example 3.1. *Let us go back to Example 1.13, the configurations of points in $\mathbb{P}^1_{\mathbb{C}}$. The group $SL(2, \mathbb{C})$ is simple, then there is only one invariant inner product up to multiplying by a scalar, say the Killing norm. Then, we can choose $\langle \cdot, \cdot \rangle$ such that* *the associated norm verifies*

$$\left\| \begin{pmatrix} t^{-k} & 0 \\ 0 & t^{k} \end{pmatrix} \right\| = k \ .$$

We did calculate in Example 1.13 that the weight of a 1*-parameter subgroup* ρ_k *which has exponents* $-k$ *and* k *in its diagonal form is* $\rho_f = k(2i_0 - n)$ *where, recall that* i_0 *is the maximum number of points in* $\mathbb{P}^1_{\mathbb{C}}$ *which are equal. It is clear that*

$$\sup_{\rho\in\mathfrak{g}} \Phi_f(\rho) = \sup_{\rho\in\mathfrak{g}} \frac{\mu(f,\rho)}{\|\rho\|} = \frac{k(2i_0-n)}{k} = 2i_0 - n$$

which is a positive number if f *is unstable.*

Now, from the symplectic point of view, recall that we associate to each point in $\mathbb{P}^1_{\mathbb{C}}$ *a vector in* S^2 *and the moment map is given by* $\mu(x) = v_1 + \cdots v_n \in \mathbb{R}^3$*, after identifying* $\mathfrak{so}(3,\mathbb{R})^* \simeq \mathbb{R}^3$*. The norm chosen in* $\mathfrak{so}(3,\mathbb{R})$ *can be identified with the usual norm in* $\mathbb{R}^3$*.*

Suppose that x *is an unstable configuration, hence it defines* $i_0 > \frac{n}{2}$ *identical vectors in* S^2*. By changing the coordinates in* $\mathbb{P}^1_{\mathbb{C}}$*, we can consider that the configuration is given by a binary form*

$$f = a_{n-i_0}X^{n-i_0}Y^{i_0} + a_{n-i_0+1}X^{n-i_0-1}Y^{i_0+1} + \cdots + a_{n-1}XY^{n-1} + a_nY^n \ ,$$

which we can move in its G*-orbit by elements* $g_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ *to obtain*

$$g_t \cdot f = f g_t^{-1} = f(t^{-1}X, tY) =$$

$$t^{2i_0-n}a_{n-i_0}X^{n-i_0}Y^{i_0} + t^{2i_0-n+2}a_{n-i_0+1}X^{n-i_0-1}Y^{i_0+1} + \cdots + t^{n-2}a_{n-1}XY^{n-1} + t^n a_n Y^n \ .$$

We can multiply it by t^{-2i_0+n-2} *and still define the same form* $\overline{f}$ *in the projective space,*

$$a_{n-i_0}X^{n-i_0}Y^{i_0} + t^2 a_{n-i_0+1}X^{n-i_0-1}Y^{i_0+1} + \cdots + t^{2n-2i_0-2}a_{n-1}XY^{n-1} + t^{2n-2i_0}a_nY^n \ ,$$

which tends to $f_0 = a_{n-i_0}X^{n-i_0}Y^{i_0}$ *when* t *goes to* 0*. The zeroes of* $\overline{f_0}$ *are* $[1:0]$ *with multiplicity* i_0 *and* $[0:1]$ *with multiplicity* $n-i_0$ *and, when considering a isomorphism* $\mathbb{P}^1_{\mathbb{C}} \simeq S^2$ *we can associate the roots to the vectors* $(0,0,1)$ *and* $(0,0,-1)$ *in* S^2*. Hence, the calculation of the infimum of the moment mal square is*

$$\inf_{g\in G} \Omega_f = \inf_{g\in G} \|\mu(g\cdot f)\| \leq \inf_t \|\mu(g_t \cdot f)\| =$$

$$|\sum_{i_0}(0,0,1) + \sum_{n-i_0}(0,0,-1)| = |\sum_{2i_0-n}(0,0,1)| = 2i_0 - n = m_f \ ,$$

and it is clear that the value obtained is indeed the infimum because the best we can do in order to get the infimum, once we have i_0 *identical vectors in* S^2*, is to dispose the rest (after the action of* $SL(2,\mathbb{C})$*) in the opposite direction, which we did by the curve of elements* $g_t \in G$*.*

As we observe,

$$\sup_{\rho\in\mathfrak{g}} \Phi_f(\rho) = 2i_0 - n = \inf_{g\in G} \Omega_f$$

therefore there are different levels of unstability, indexed by the numbers $2i_0 - n$, corresponding to binary forms with different number of identical roots, or to vectors in S^2 which do not close to form a polygon because they have more than half, but different, number of identical vectors.

Example 3.2. *Now we recall Example 2.7.*

Let $A \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ of rank $m < r$, hence $[A]$ is GIT unstable. Following the argument in the example, there exists a basis $\{v_1, \dots, v_r\}$ of $\mathbb{C}^r$ such that $L\{v_1, \dots, v_{r-m}\} = \mathrm{Ker}\, A$. The different 1-parameter subgroups ρ, adapted to the basis in such a way they take the diagonal form are given by

$$\begin{pmatrix} t^{\lambda_1} & 0 & \cdots & 0 \\ 0 & t^{\lambda_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t^{\lambda_r} \end{pmatrix}$$

where we impose the convention $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. Then,

$$A\cdot\rho^{-1} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & * & \cdots & * \end{pmatrix} \cdot \begin{pmatrix} t^{-\lambda_1} & 0 & \cdots & 0 \\ 0 & t^{-\lambda_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t^{-\lambda_r} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & t^{-\lambda_{m+1}}\cdot * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & * & \cdots & t^{-\lambda_r}\cdot * \end{pmatrix}$$

Hence, we observe that the number $\mu([A], \rho)$ of the Hilbert-Mumford criterion, i.e. the minimal exponent multiplying a non-zero coordinate, is $-\lambda_{m+1}$. Therefore, in order to maximize this weight, keeping the condition of $\rho \in SL(r, \mathbb{C})$ hence all exponents sum 0, the maximal 1-parameter subgroups will be of the form

$$\begin{pmatrix} t^m & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & 0 \\ & & t^m & & & \\ \vdots & & & t^{m-r} & & \vdots \\ & & & & \ddots & \\ 0 & 0 & & \cdots & & t^{m-r} \end{pmatrix}$$

where the exponent m is repeated $r-m$ times and the exponent $m-r$ is repeated m times. Then it is clear that for this 1-parameter subgroups we have $\mu([A],\rho)=r-m$. Note that we could have achieved the same maximal result by multiplying the exponents m and $m-r$ by the same positive constant, hence up to rescalling the maximal weight will remain $r-m$. In other words

$$\sup_{\rho\in\mathfrak{g}}\Phi_{[A]}(\rho)=\sup_{\rho\in\mathfrak{g}}\frac{\mu([A],\rho)}{\|\rho\|}=r-m\ .$$

From the symplectic side, recall that the moment map was given by $\mu(A)=\frac{i}{2}(A^*A-Id)$. Having chosen the invariant product in $\mathfrak{u}(n)$ given by $\mathrm{trace}(A^*B)$, the moment map square is given by

$$\|\mu(A)\|=\mathrm{trace}((A^*A-Id)^*(A^*A-Id))=\mathrm{trace}(A^*A-Id)^2)$$

up to a constant (related with the rescalling of the norm discussed before from the GIT point of view). By an element of $SL(r,\mathbb{C})$ (or by change of basis) we can take suppose that A^*A is a matrix which has a diagonal block which is the idendity (of size the rank of A) and zeroes elsewhere. Therefore it is clear that

$$\inf_{g\in G}\Omega_A=\inf_{g\in G}\|\mu(g\cdot A)\|=$$

$$\mathrm{trace}\left(\left(\begin{pmatrix} 0 & 0 & & \cdots & & 0\\ 0 & \ddots & & & & 0\\ & & 0 & & & \\ \vdots & & & 1 & & \vdots\\ & & & & \ddots & \\ 0 & 0 & & \cdots & & 1\end{pmatrix}-Id\right)^2\right)=\mathrm{trace}\begin{pmatrix} 1 & 0 & & \cdots & & 0\\ 0 & \ddots & & & & 0\\ & & 1 & & & \\ \vdots & & & 0 & & \vdots\\ & & & & \ddots & \\ 0 & 0 & & \cdots & & 0\end{pmatrix}=r-m\ ,$$

which is the quantity $\sup_{\rho\in\mathfrak{g}}\Phi_{[A]}(\rho)$. Hence the different unstability levels are indexed by the complementary of the rank of A, being $m=r$ the case where the supremum and the infimum, respectively, achieve zero, as it has to be in the stable case.

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